# GENERALIZED CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS 

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#### Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P})=n$ iff $\mathcal{P} \subseteq \mathcal{Q}^{n}$ but $\mathcal{P} \nsubseteq \mathcal{Q}^{n-1}$. We investigate the generalized chromatic numbers of the well-known properties of graphs $\mathcal{I}_{k}, \mathcal{O}_{k}, \mathcal{W}_{k}, \mathcal{S}_{k}$ and $\mathcal{D}_{k}$.


Keywords: property of graphs, additive, hereditary, generalized chromatic number.
2000 Mathematics Subject Classification: 05C15.

## 1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$. A property of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$.

Throughout the text we will call a component of a graph that is a spanning supergraph of a path $P_{k}$ of order $k$ a $k$-component. Let $G$ be a graph and $V_{1} \subseteq V(G)$. We say that a vertex $v \in V(G)-V_{1}$ is adjacent to a $k$-component of $G\left[V_{1}\right]$ if $v$ is adjacent to a vertex of some $k$-component of $G\left[V_{1}\right]$.

Example. For a positive integer $k$ we define the following well-known properties:

$$
\begin{aligned}
\mathcal{O}= & \{G \in \mathcal{I}: E(G)=\emptyset\} \\
\mathcal{I}_{k}= & \left\{G \in \mathcal{I}: G \text { does not contain } K_{k+2}\right\}, \\
\mathcal{O}_{k}= & \{G \in \mathcal{I}: \text { each component of } G \text { has at most } k+1 \text { vertices }\}, \\
\mathcal{W}_{k}= & \{G \in \mathcal{I}: \text { each path in } G \text { has at most } k+1 \text { vertices }\} \\
\mathcal{S}_{k}= & \{G \in \mathcal{I}: \text { the maximum degree of } G \text { is at most } k\}, \\
\mathcal{T}_{k}= & \left\{G \in \mathcal{I}: G \text { contains no subgraph homeomorphic to } K_{k+2}\right. \text { or } \\
& \left.K_{\left\lfloor\frac{k+3}{2}\right\rfloor,\left\lceil\frac{k+3}{2}\right\rceil}\right\} \\
\mathcal{D}_{k}= & \{G \in \mathcal{I}: G \text { is } k \text {-degenerate, i.e., every subgraph of } G \text { has a vertex } \\
& \text { of degree at most } k\} .
\end{aligned}
$$

For every additive hereditary property $\mathcal{P} \neq \mathcal{I}$ there is a smallest integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the completeness of $\mathcal{P}$. Note that all the properties in the above example, except $\mathcal{O}$, are of completeness $k$. The set $\mathbf{F}(\mathcal{P})$ of minimal forbidden subgraphs is defined by $\{G \in \mathcal{I}: G \in \overline{\mathcal{P}}$ and $H \in \mathcal{P}$ for all $H \subset G\}$.

Let $\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}$ be arbitrary hereditary properties of graphs. A vertex $\left(\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}\right)$-partition of a graph $G$ is a partition $\left\{V_{1}, V_{2}, \ldots, V_{n}\right\}$ of $V(G)$ such that for each $i=1,2, \ldots, n$ the induced subgraph $G\left[V_{i}\right]$ has the property $\mathcal{Q}_{i}$. The property $\mathcal{R}=\mathcal{Q}_{1} \circ \mathcal{Q}_{2} \circ \cdots \circ \mathcal{Q}_{n}$ is defined as the set of all graphs having a vertex $\left(\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}\right)$-partition. It is easy to see that if $\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}$ are additive and hereditary, then $\mathcal{R}=\mathcal{Q}_{1} \circ \mathcal{Q}_{2} \circ \cdots \circ \mathcal{Q}_{n}$ is additive and hereditary too. If $\mathcal{Q}_{1}=\mathcal{Q}_{2}=\cdots=\mathcal{Q}_{n}=\mathcal{Q}$, then we write $\mathcal{Q}^{n}=\mathcal{Q}_{1} \circ \mathcal{Q}_{2} \circ \cdots \circ \mathcal{Q}_{n}$.

The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P})=$ $n$ iff $\mathcal{P} \subseteq \mathcal{Q}^{n}$ but $\mathcal{P} \nsubseteq \mathcal{Q}^{n-1}$.

As an example of the non-existence of $\chi_{\mathcal{Q}}(\mathcal{P})$ we have $\chi_{\mathcal{O}}\left(\mathcal{I}_{1}\right)$ since it is well known that there exist triangle-free graphs of arbitrary chromatic number. The following theorem, due to J. Nešetřil and V. Rödl (see [12]), implies that for some additive hereditary properties $\mathcal{P}$ we have that $\chi_{\mathcal{Q}}(\mathcal{P})$ exists if and only if $\chi_{\mathcal{Q}}(\mathcal{P})=1$. In particular, $\chi_{\mathcal{Q}}\left(\mathcal{I}_{k}\right)$ exists if and only $\chi_{\mathcal{Q}}\left(\mathcal{I}_{k}\right)=1$.

Theorem 1.1 [12]. Let $\mathbf{F}(\mathcal{P})$ be a finite set of 2 -connected graphs. Then for every graph $G \in \mathcal{P}$ there exists a graph $H \in \mathcal{P}$ such that for any partition $\left\{V_{1}, V_{2}\right\}$ of $V(H)$ there is an $i, i=1$ or $i=2$, for which $G \leq H\left[V_{i}\right]$.

Corollary 1.2. If $\mathbf{F}(\mathcal{P})$ is a finite set of 2-connected graphs, then for any additive hereditary property $\mathcal{Q}$ it follows that $\chi_{\mathcal{Q}}(\mathcal{P})$ exists if and only if $\mathcal{P} \subseteq \mathcal{Q}$.

The value of $\chi_{\mathcal{Q}}(\mathcal{P})$ is known for various choices of $\mathcal{P}$ and $\mathcal{Q}$. In the remainder of this section we mention some simple results, most of which are known or follow immediately from well-known results. See for example [2] and [5].

It is easy to see that $\mathcal{O}_{a+b+1} \subseteq \mathcal{O}_{a} \circ \mathcal{O}_{b}$ and $\mathcal{D}_{a+b+1} \subseteq \mathcal{D}_{a} \circ \mathcal{D}_{b}$ (see for example [9]), which implies that $\chi_{\mathcal{Q}}\left(\mathcal{O}_{k}\right)=\left\lceil\frac{k+1}{n+1}\right\rceil$ for any property $\mathcal{Q}$ of completeness $n$, and $\chi_{\mathcal{D}_{n}}(\mathcal{P})=\left\lceil\frac{k+1}{n+1}\right\rceil$ for any property $\mathcal{P}$ such that $\mathcal{O}_{k} \subseteq \mathcal{P} \subseteq \mathcal{D}_{k}$. Note that Corollary 1.2 implies that the latter equality does not extend to $c(\mathcal{P})=n$.
The well-known theorem of Lovász states:
Theorem 1.3 [10]. $\mathcal{S}_{a+b+1} \subseteq \mathcal{S}_{a} \circ \mathcal{S}_{b}$ for all $a, b \geq 0$.
This implies that $\chi_{\mathcal{S}_{n}}\left(\mathcal{S}_{k}\right)=\left\lceil\frac{k+1}{n+1}\right\rceil \cdot($ See [5].)
It is also easy to see that if $\mathcal{O}_{k} \subseteq \mathcal{P} \subseteq \mathcal{O}^{k+1}$, then $\chi \mathcal{I}_{n}(\mathcal{P})=\left\lceil\frac{k+1}{n+1}\right\rceil$.
The next result is interesting since it shows that the value of $\chi_{\mathcal{S}_{n}}\left(\mathcal{D}_{k}\right)$ is independent of $n$.

Theorem 1.4. For all $k$ and $n$ we have $\chi_{\mathcal{S}_{n}}\left(\mathcal{D}_{k}\right)=k+1$.
Proof. Since $\mathcal{D}_{k} \subseteq \mathcal{O}^{k+1} \subseteq \mathcal{S}_{n}^{k+1}$ we have the upper bound. We prove the lower bound by induction on $k$. The result is true for $k=1$ since $\mathcal{D}_{1} \nsubseteq \mathcal{S}_{n}$. Assume, therefore, that $\mathcal{D}_{k} \nsubseteq \mathcal{S}_{n}^{k}$ and let $H \in \mathcal{D}_{k}$ such that $H \notin \mathcal{S}_{n}^{k}$. Let $G=(n+1) H+K_{1}$. Since every subgraph of $(n+1) H$ has a vertex of degree at most $k$, every subgraph of $G$ has a vertex of degree at most $k+1$. Thus $G \in \mathcal{D}_{k+1}$.

Also, $G \notin \mathcal{S}_{n}^{k+1}$ : Suppose, to the contrary, that $\left\{V_{1}, V_{2}, \ldots, V_{k+1}\right\}$ is an $\mathcal{S}_{n}^{k+1}$-partition of $V(G)$. Let $v$ be the universal vertex of $G$ and suppose, without loss of generality, that $v \in V_{1}$. Since $G\left[V_{1}\right] \in \mathcal{S}_{n}$ it follows that $\left|V_{1}\right| \leq n+1$. Since there are $n+1$ copies of $H$ in $G$ we have that for some copy $F$ of $H, F \cap V_{1}=\emptyset$. This contradicts the fact that $H \notin \mathcal{S}_{n}^{k}$.

The lattice of (additive) hereditary properties is discussed in [1] - we use the supremum and infimum of properties in our next result without further discussion.

Theorem 1.5. Let $\mathcal{P}_{1}, \mathcal{P}_{2}$ and $\mathcal{Q}$ be additive hereditary properties such that $\chi_{\mathcal{Q}}\left(\mathcal{P}_{1}\right)$ and $\chi_{\mathcal{Q}}\left(\mathcal{P}_{2}\right)$ are finite. The following hold:
(i) $\chi_{\mathcal{Q}}\left(\mathcal{P}_{1} \cup \mathcal{P}_{2}\right)=\chi_{\mathcal{Q}}\left(\mathcal{P}_{1} \vee \mathcal{P}_{2}\right)=\max \left\{\chi_{\mathcal{Q}}\left(\mathcal{P}_{1}\right), \chi_{\mathcal{Q}}\left(\mathcal{P}_{2}\right)\right\}$.
(ii) $\chi_{\mathcal{Q}}\left(\mathcal{P}_{1} \cap \mathcal{P}_{2}\right) \leq \min \left\{\chi_{\mathcal{Q}}\left(\mathcal{P}_{1}\right), \chi_{\mathcal{Q}}\left(\mathcal{P}_{2}\right)\right\}$.
(iii) $\max \left\{\chi_{\mathcal{Q}}\left(\mathcal{P}_{1}\right), \chi_{\mathcal{Q}}\left(\mathcal{P}_{2}\right)\right\} \leq \chi_{\mathcal{Q}}\left(\mathcal{P}_{1} \circ \mathcal{P}_{2}\right) \leq \chi_{\mathcal{Q}}\left(\mathcal{P}_{1}\right)+\chi_{\mathcal{Q}}\left(\mathcal{P}_{2}\right)$.

We remark that the inequality in Theorem $1.5(\mathrm{ii})$ may be strict. For example $\chi_{\mathcal{O}}\left(\mathcal{T}_{3}\right)=4$ and $\chi_{\mathcal{O}}\left(\mathcal{I}_{1}\right)$ is infinite but $\chi_{\mathcal{O}}\left(\mathcal{T}_{3} \cap \mathcal{I}_{1}\right)=3$. (See [2].)

## 2. Results on $\mathcal{W}_{k}$

In this section we investigate the value of $\chi \mathcal{W}_{n}\left(\mathcal{W}_{k}\right)$. The problem of determining it has been discussed in (or is related to problems in) several papers (see for example [3], [4], [6], [7], [8] and [11]) and the following conjecture has been made in at least three of them:

Conjecture 2.1 [3], [6], [7]. $\mathcal{W}_{a+b+1} \subseteq \mathcal{W}_{a} \circ \mathcal{W}_{b}$ for all positive integers a and $b$.

This conjecture implies the following for $\chi \mathcal{W}_{n}\left(\mathcal{W}_{k}\right)$ :
Conjecture 2.2. For every $n, k \geq 1$, the following holds:

$$
\chi \mathcal{W}_{n}\left(\mathcal{W}_{k}\right)=\left\lceil\frac{k+1}{n+1}\right\rceil .
$$

In [6] the bound $\chi \mathcal{W}_{n}\left(\mathcal{W}_{k}\right) \leq\left\lfloor\frac{k-n+1}{2}\right\rfloor+2$ is proved. The following theorem will enable us to improve on this bound.

Theorem 2.3. $\mathcal{W}_{\left\lceil\frac{2 a}{3}\right\rceil+b+1} \subseteq \mathcal{W}_{a} \circ \mathcal{W}_{b}$ for all $a \geq 15$ and $b \geq 1$.
Proof. Consider any graph $G$ in $\mathcal{W}_{\left\lceil\frac{2 a}{3}\right\rceil+b+1}$. Take $V_{1}$ to be a maximal subset of $V(G)$ such that $G\left[V_{1}\right]$ is in $\mathcal{W}_{a}$. Let $V_{2}=V(G)-V_{1}$. Suppose that there is a path $P$ in $G\left[V_{2}\right]$ of length $b+1$ and let $v_{1}$ and $v_{2}$ denote the end-vertices of $P$. Since $V_{1}$ is maximal in $\mathcal{W}_{a}$ it follows that there is a path $P_{1}$ of length $a+1$ in $G\left[V_{1} \cup\left\{v_{1}\right\}\right]$ and a path $P_{2}$ of length $a+1$ in $G\left[V_{1} \cup\left\{v_{2}\right\}\right]$. Note that if either $v_{1}$ or $v_{2}$ is an end-vertex of $P_{1}$ or $P_{2}$ respectively, then in both cases we get a path of length at least $a+b+3$
in $G$, a contradiction. Therefore the vertices $v_{1}$ and $v_{2}$ are not end-vertices of their respective paths. Let $P_{11}$ and $P_{12}$ denote the paths on either side of $v_{1}$ such that $P_{11} \cup\left\{v_{1}\right\} \cup P_{12}=P_{1}$. Similarly, let $P_{21} \cup\left\{v_{2}\right\} \cup P_{22}=P_{2}$. Now suppose, without loss of generality, that $x=\left|E\left(P_{11}\right)\right|+1 \leq y=\left|E\left(P_{12}\right)\right|+1$, so that $x+y=a+1$.

It is easily seen that if $y \geq\left\lfloor\frac{2 a+2}{3}\right\rfloor+1$, then by simply taking the path $P_{12} \cup P$, we get a path of length at least $\left\lfloor\frac{2 a+2}{3}\right\rfloor+1+b+1 \geq \frac{2 a+2-2}{3}+b+2>$ $\left\lceil\frac{2 a}{3}\right\rceil+b+1$ in $G$, a contradiction. Therefore $\left\lceil\frac{a+1}{2}\right\rceil \leq y \leq\left\lfloor\frac{2 a+2^{3}}{3}\right\rfloor$. Moreover, each $P_{i j}, i, j \in\{1,2\}$ has length at least $\left\lfloor\frac{a-5}{3}\right\rfloor$, since $x=a+1-y \geq$ $a-\left\lfloor\frac{2 a+2}{3}\right\rfloor+1 \geq a-\frac{2 a+5}{3}=\frac{a-5}{3} \geq\left\lfloor\frac{a-5}{3}\right\rfloor$.

Note that $P_{11}$ and $P_{12}$ are neccessarily disjoint as are $P_{21}$ and $P_{22}$, and that $v_{1}$ and $v_{2}$ are not on any of these paths.
$P_{12}$ must intersect both $P_{21}$ and $P_{22}$ : Firstly, $P_{12}$ must intersect the longer of $P_{21}$ and $P_{22}$ since otherwise we get a too long path in $G$; containing the two longer paths and $P$. Furthermore, if $P_{12}$ does not intersect the shorter of $P_{21}$ and $P_{22}$, then we get a path of length at least $\left\lceil\frac{a+1}{2}\right\rceil+b+1+$ $\left\lfloor\frac{a-5}{3}\right\rfloor \geq \frac{a+1}{2}+\frac{a-7}{3}+b+1=\frac{5}{6}(a-1)>\left\lceil\frac{2 a}{3}\right\rceil+1+b($ since $a \geq 15)$ in $G$; containing $P_{12}, P$ and the shorter of $P_{21}$ and $P_{22}$, a contradiction. Similarly, the longer of $P_{21}$ and $P_{22}$ must intersect both $P_{11}$ and $P_{12}$.

Note that since $P_{11}$ and $P_{12}$ are disjoint and $P_{21}$ and $P_{22}$ are disjoint, $P_{2 i}, i \in\{1,2\}$ can only intersect one of $P_{11}$ and $P_{12}$ first and vice-versa.

Suppose that both $P_{21}$ and $P_{22}$ intersect $P_{12}$ first. Then we obtain a path of length at least $x+b+1+1+\left\lfloor\frac{y}{2}\right\rfloor \geq a+1-y+\frac{y-1}{2}+b+2 \geq$ $a-\frac{1}{2}\left\lfloor\frac{2 a+2}{3}\right\rfloor+\frac{5}{2}+b \geq a-\frac{1}{2}\left(\frac{2 a+2}{3}\right)+\frac{5}{2}+b=\frac{2 a}{3}+\frac{13}{6}+b>\left\lceil\frac{2 a}{3}\right\rceil+1+b$ in $G$; containing $P_{11}, P$, at least one edge of either $P_{21}$ or $P_{22}$ and at least a half of $P_{12}$, a contradiction.

Now, suppose that $P_{21}$ or $P_{22}$ intersects $P_{11}$ first, say $P_{21}$. Then we obtain a path of length at least $y+\left\lfloor\frac{x}{2}\right\rfloor+b+1+1=y+\left\lfloor\frac{1}{2}(a+1-y)\right\rfloor+b+2 \geq$ $y+\frac{a+1-y-1}{2}+b+2=\frac{y}{2}+\frac{a}{2}+b+2 \geq \frac{1}{2}\left\lceil\frac{a+1}{2}\right\rceil+\frac{a}{2}+b+2 \geq \frac{3 a+9}{4}+b>\left\lceil\frac{2 a}{3}\right\rceil+1+b$ in $G$; containing $P_{12}, P$, at least one edge of $P_{21}$ and at least a half of $P_{11}$, a contradiction.

Theorem 2.4. $\chi \mathcal{W}_{n}\left(\mathcal{W}_{k}\right) \leq\left\lceil\frac{3 k}{2 n+3}\right\rceil$ for all $n \geq 15$ and $k \geq 1$.
Proof. $\mathcal{W}_{c\left\lceil\frac{2 n+3}{3}\right\rceil} \subseteq \mathcal{W}_{n}^{c}$ for all positive integers $c$ and $n$ : the proof is by induction on $c$. The result holds for $c=1$. Suppose now that the result holds for $c$. Note that $\mathcal{W}_{(c+1)\left\lceil\frac{2 n+3}{3}\right\rceil}=\mathcal{W}_{\left\lceil\frac{2 n}{3}\right\rceil+1+c\left\lceil\frac{2 n+3}{3}\right\rceil}$ which by Theorem 2.3 is
contained in $\mathcal{W}_{n} \circ \mathcal{W}_{c\left\lceil\frac{2 n+3}{3}\right\rceil}$ which by the induction hypothesis is contained in $\mathcal{W}_{n} \circ \mathcal{W}_{n}^{c}=\mathcal{W}_{n}^{c+1}$.
Now, with $c=\left\lceil\frac{3 k}{2 n+3}\right\rceil$, since $k \leq\left\lceil\frac{3 k}{2 n+3}\right\rceil\left\lceil\frac{2 n+3}{3}\right\rceil$ we have that $\mathcal{W}_{k} \subseteq \mathcal{W}_{c\left\lceil\frac{2 n+3}{3}\right\rceil}$ $\subseteq \mathcal{W}_{n}^{c}$.

This result is close to the bound $\chi_{\mathcal{W}_{n}}\left(\mathcal{W}_{k}\right) \leq\left\lceil\frac{3(k-n)}{2 n+2}\right\rceil+1$ presented in [7] but our method of proof is completely different.

## 3. Results Relating $\mathcal{S}_{k}$ and $\mathcal{W}_{n}$

Theorem 3.1. For positive integers $n$ and $k$ we have that

$$
\left\lceil\frac{k+1}{n+1}\right\rceil \leq \chi \mathcal{W}_{n}\left(\mathcal{S}_{k}\right) \leq\left\lceil\frac{k+1}{2}\right\rceil
$$

Proof. The left inequality holds since $K_{k+1} \in \mathcal{S}_{k}$. The right inequality follows as a corollary to Theorem 1.3.

The first inequality in Theorem 3.1 may be strict, for example $\chi_{\mathcal{W}_{2}}\left(\mathcal{S}_{2}\right)=$ $2>\left\lceil\frac{2+1}{2+1}\right\rceil\left(\right.$ since $\left.\mathcal{S}_{2} \nsubseteq \mathcal{W}_{2}\right)$. Equality in both the inequalities may be achieved, for example, by Theorem 1.3 we have that $\chi_{\mathcal{S}_{n}}\left(\mathcal{S}_{k}\right)=\left\lceil\frac{k+1}{n+1}\right\rceil$ and therefore $\chi_{\mathcal{W}_{1}}\left(\mathcal{S}_{k}\right)=\left\lceil\frac{k+1}{2}\right\rceil$.

Note that whether or not the second inequality proved in Theorem 3.1 may be strict still remains an open problem.

We now start working towards bounds on $\chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{k}\right)$.
Theorem 3.2. $\mathcal{W}_{4} \subseteq \mathcal{S}_{2} \circ \mathcal{S}_{1}$.
Proof. Consider any graph $G$ in $\mathcal{W}_{4}$. Take $V_{1}$ to be a subset of $V(G)$ such that, in order of priority:
(i) $G\left[V_{1}\right]$ is in $\mathcal{S}_{2}$,
(ii) $G\left[V_{1}\right]$ contains a maximum number of 4-components,
(iii) $G\left[V_{1}\right]$ contains a maximum number of components isomorphic to $K_{3}$,
(iv) $G\left[V_{1}\right]$ contains a maximum number of 2-components and
(v) $G\left[V_{1}\right]$ contains a maximum number of isolated vertices.
(In other words, we consider all subsets $V$ of $V(G)$ such that $G[V] \in \mathcal{S}_{2}$. Amongst these we consider all subsets $V$ for which $G[V]$ has a maximum
number of 4-components. Amongst these we consider all subsets inducing a maximum number of components isomorphic to $K_{3}$ etc.)
Let $V_{2}=V(G)-V_{1}$. We will show that $G\left[V_{2}\right] \in \mathcal{S}_{1}$. Suppose, to the contrary, that $G\left[V_{2}\right] \notin \mathcal{S}_{1}$ and let $v$ be a vertex in $G\left[V_{2}\right]$ of degree at least two with $u$ and $w$ two of its neighbours in $G\left[V_{2}\right]$. Note that by choice of $V_{1}$ every component in $G\left[V_{1}\right]$ is a 4-component, $K_{3}, K_{2}$ or $K_{1}$.

Moreover, by (v) it follows that $u, v$ and $w$ each have at least one neighbour in $V_{1}$. Furthermore, $v$ is adjacent to a nontrivial component in $G\left[V_{1}\right]$ : If this is not the case, then we can replace the vertices in $V_{1}$ that are adjacent to $v$ with a 2 -component; still satisfying (i) through (iii) but contradicting (iv). Similarly, $u$ and $w$ are adjacent to nontrivial components in $G\left[V_{1}\right]$.

Suppose that $v$ is adjacent to a triangle in $G\left[V_{1}\right]$. Note that neighbours of both $u$ and $w$ in $V_{1}$ can only lie on this triangle, otherwise we obtain at least a $P_{6}$ in $G$. However then we obtain a $P_{6}$ in $G$; containing all three vertices of the triangle in $G\left[V_{1}\right]$ as well as the $P_{3}$ formed by $u, v$ and $w$. Thus $v$ cannot be adjacent to a triangle in $G\left[V_{1}\right]$.

Furthermore, $v$ cannot be adjacent to a 4 -component in $G\left[V_{1}\right]$. This case is analogous to the above case since a 4 -component will also contribute three vertices to give a $P_{6}$ in $G$. Moreover, neither $u$ nor $w$ are adjacent to 4-components or triangles in $G\left[V_{1}\right]$, since otherwise we obtain at least a $P_{6}$ in $G$.

Therefore $v$ must be adjacent to a $K_{2}$ in $G\left[V_{1}\right]$. Note that $u$ and $w$ must each have at least one neighbour on the $K_{2}$ adjacent to $v$ in $G\left[V_{1}\right]$, otherwise we obtain a $P_{6}$ in $G$. If $v$ is adjacent to both vertices on the $K_{2}$ in $G\left[V_{1}\right]$, then we can replace the components in $G\left[V_{1}\right]$ that are adjacent to $u, v$ and $w$ with a triangle; still satisfying (i) and (ii), but contradicting (iii).

Thus $v$ has only one neighbour on any $K_{2}$ in $G\left[V_{1}\right]$. If $u$ or $w$ is adjacent to the same vertex as $v$ on the $K_{2}$ adjacent to $v$ in $G\left[V_{1}\right]$, then once again we can replace the components in $G\left[V_{1}\right]$ that are adjacent to $u, v$ and $w$ with a triangle; still satisfying (i) and (ii), but contradicting (iii). Therefore, both $u$ and $w$ are adjacent to the vertex on the $K_{2}$ in $G\left[V_{1}\right]$ that is not adjacent to $v$. However, then we can replace the components in $G\left[V_{1}\right]$ that are adjacent to $u, v$ and $w$ with a 4 -component; containing the $K_{2}$ in $G\left[V_{1}\right]$ and the vertices $v$ and either $u$ or $w$; still satisfying (i), but contradicting (ii). Therefore $G\left[V_{2}\right] \in \mathcal{S}_{1}$.

Corollary 3.3. For all $n \geq 2$ and $k$, we have that $\chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{k}\right) \leq 2\left\lceil\frac{k+1}{5}\right\rceil$.

Proof. It is known that $\mathcal{W}_{4+k+1} \subseteq \mathcal{W}_{4} \circ \mathcal{W}_{k}$ (see [3]). Similar to the proof of Theorem 1.3 it follows that $\mathcal{W}_{k} \subseteq \mathcal{W}_{4}^{\left\lfloor\frac{k+1}{5}\right\rceil}$. The result now follows from Theorem 3.2.

The inequality in Corollary 3.3 may be strict, for example $\chi_{\mathcal{S}_{2}}\left(\mathcal{W}_{1}\right)=1<$ $2=2\left\lceil\frac{2}{5}\right\rceil$. Equality may also be obtained, for example $\chi_{\mathcal{S}_{2}}\left(\mathcal{W}_{2}\right)=2=2\left\lceil\frac{3}{5}\right\rceil$.

Having proved Corollary 3.3 we naturally ask: Can this bound be improved and if so under what conditions? Corollary 3.5 gives us an answer for $n \geq 5$ and Theorem 3.7 for $n \geq 9$.

Theorem 3.4. For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 5$, the following holds: $\mathcal{W}_{k} \subseteq \mathcal{Q}^{\left\lceil\frac{k}{3}\right\rceil} \circ \mathcal{O}$.

Proof. Let $c=\left\lceil\frac{k}{3}\right\rceil$. Consider any graph $G$ in $\mathcal{W}_{k}$. Take $V_{1}$ to be a subset of $V(G)$ such that, in order of priority:
(i) $G\left[V_{1}\right]$ is in $\mathcal{Q}$,
(ii) $G\left[V_{1}\right]$ contains a maximum number of 6 -components,
(iii) $G\left[V_{1}\right]$ contains a maximum number of 4 -components,
(iv) $G\left[V_{1}\right]$ contains a maximum number of 2-components and
(v) $G\left[V_{1}\right]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take $V_{i}$ to be a subset of $V(G)-\bigcup_{j=1}^{i-1} V_{j}$ such that for each $i, G\left[V_{i}\right]$ satisfies the above list. Let $S=V(G)-\bigcup_{j=1}^{c} V_{j}$. We will show that $G[S] \in \mathcal{O}$. Suppose, to the contrary, that $G[S] \notin \mathcal{O}$ and let $v$ be a vertex in $G[S]$ of degree at least one and $u$ be a neighbour of $v$ in $G[S]$. Suppose that $v$ is not an end-vertex of a $P_{4}$ in $G\left[V_{c} \cup\{v\}\right]$ and that $u$ is not an end-vertex of a $P_{5}$ in $G\left[V_{c} \cup\{u\}\right]$. Note that for every $i$, the choice of $V_{i}$ gives that every component in $G\left[V_{i}\right]$ is a 6 -component, a 4 -component, $K_{2}$ or $K_{1}$. Moreover, by (v) it follows that $u$ and $v$ have at least one neighbour in $V_{c}$ each and by (iv) both $u$ and $v$ are adjacent to nontrivial components in $G\left[V_{c}\right]$. Since $v$ is not an end-vertex of a $P_{4}$ in $G\left[V_{c} \cup\{v\}\right]$ it follows that $v$ is not adjacent to a 6 -component or a 4 -component in $G\left[V_{c}\right]$. Similarly, $u$ is not adjacent to a 6 -component in $G\left[V_{c}\right]$.

Suppose that $u$ is adjacent to a 4 -component in $G\left[V_{c}\right]$. Then, since $v$ is not adjacent to a 4 -component in $G\left[V_{c}\right]$, we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u$ and $v$ with a 6 -component; still satisfying (i) since $K_{6} \in \mathcal{Q}$, but contradicting (ii), since neither $u$ nor $v$ is adjacent to a 6 -component in $G\left[V_{c}\right]$.

Therefore $u$ is adjacent to a 2 -component in $G\left[V_{c}\right]$. However, then we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u$ and $v$ with a 4component; satisfying (i) and (ii) but contradicting (iii).

Therefore, $u$ is an end-vertex of a $P_{5}$ in $G\left[V_{c} \cup\{u\}\right]$ or $v$ is an end-vertex of a $P_{4}$ in $G\left[V_{c} \cup\{v\}\right]$. In both cases it follows that there is a path $P$ of length four in $G\left[S \cup V_{c}\right]$. Let $x$ be the end-vertex of $P$ in $V_{c}$ and $y$ the neighbour of $x$ on $P$. By repeating this argument it follows that $x$ is an end-vertex of a $P_{4}$ in $G\left[V_{c-1} \cup\{x\}\right]$ or $y$ is an end-vertex of a $P_{5}$ in $G\left[V_{c-1} \cup\{y\}\right]$. Continuing in this way we obtain a path of length at least $3 c+1 \geq k+1$ in $G$, a contradiction. Therefore, $G[S] \in \mathcal{O}$.

Corollary 3.5. For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 5$, the following holds: $\quad \chi_{\mathcal{Q}}\left(\mathcal{W}_{k}\right) \leq\left\lceil\frac{k}{3}\right\rceil+1$.

The inequality in Corollary 3.5 may be strict, for example we have that $\chi_{\mathcal{I}_{5}}\left(\mathcal{W}_{k}\right)=\left\lceil\frac{k+1}{6}\right\rceil<\frac{k+3}{3} \leq\left\lceil\frac{k}{3}\right\rceil+1$ with $K_{6} \in \mathcal{I}_{5}$ and $K_{7} \notin \mathcal{I}_{5}$. Equality can also be obtained: In Theorem 3.8 (still to follow) we prove that $\chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{k}\right) \geq$ $\left\lfloor\log _{2}(k+2)\right\rfloor$ thus for all $n$ we have that $\chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{6}\right) \geq 3$ and by Corollary 3.5 we have $\chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{6}\right) \leq\left\lceil\frac{6}{3}\right\rceil+1=3$.

Theorem 3.6. For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 9$, the following holds: $\mathcal{W}_{k} \subseteq \mathcal{Q}^{\left\lceil\frac{k-1}{4}\right\rceil} \circ \mathcal{S}_{1}$.

Proof. Let $c=\left\lceil\frac{k-1}{4}\right\rceil$. Consider any graph $G$ in $\mathcal{W}_{k}$. Take $V_{1}$ to be a subset of $V(G)$ such that, in order of priority:
(i) $G\left[V_{1}\right]$ is in $\mathcal{Q}$,
(ii) $G\left[V_{1}\right]$ contains a maximum number of 10 -components,
(iii) $G\left[V_{1}\right]$ contains a maximum number of 8 -components,
(iv) $G\left[V_{1}\right]$ contains a maximum number of 6 -components,
(v) $G\left[V_{1}\right]$ contains a maximum number of 4 -components,
(vi) $G\left[V_{1}\right]$ contains a maximum number of 2 -components and
(vii) $G\left[V_{1}\right]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take $V_{i}$ to be a subset of $V(G)-\bigcup_{j=1}^{i-1} V_{j}$ such that for each $i, G\left[V_{i}\right]$ satisfies the above list. Let $S=V(G)-\bigcup_{j=1}^{c} V_{j}$. We will show that $G[S] \in \mathcal{S}_{1}$. Suppose, to the contrary, that $G[S] \notin \mathcal{S}_{1}$ and let $v$ be a vertex in $G[S]$ of degree at least two with $u$ and $w$ neighbours of $v$ in $G[S]$.

Suppose that $u$ is not an end-vertex of a $P_{7}$ in $G\left[V_{c} \cup\{u\}\right]$ and that $v$ is not an end-vertex of a $P_{6}$ in $G\left[V_{c} \cup\{v\}\right]$ and that $w$ is not an end-vertex of a $P_{5}$ in $G\left[V_{c} \cup\{w\}\right]$. Note that for every $i$, the choice of $V_{i}$ gives that every component in $G\left[V_{i}\right]$ is a 10 -component, an 8 -component, a 6 -component, a 4 -component, $K_{2}$ or $K_{1}$. Moreover, by (vii) it follows that $u, v$ and $w$ have at least one neighbour in $V_{c}$ each and by (vi) each of $u, v$ and $w$ is adjacent to a nontrivial component in $G\left[V_{c}\right]$. Since $u$ is not an end-vertex of a $P_{7}$ in $G\left[V_{c} \cup\{u\}\right]$ it follows that $u$ is not adjacent to a 10-component in $G\left[V_{c}\right]$. Similarly, $v$ is not adjacent to a 10 -component or an 8- component in $G\left[V_{c}\right]$ and $w$ is not adjacent to a 10-component, an 8- component or a 6 -component in $G\left[V_{c}\right]$.

Suppose that $u$ is adjacent to an 8 -component in $G\left[V_{c}\right]$. Then, since neither $v$ nor $w$ are adjacent to an 8-component in $G\left[V_{c}\right]$, we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u, v$ and $w$ with a 10-component; still satisfying (i) but contradicting (ii).

Suppose that $v$ is adjacent to a 6 -component in $G\left[V_{c}\right]$. Then, since $w$ is not adjacent to a 6 -component in $G\left[V_{c}\right]$, we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u, v$ and $w$ with an 8-component; still satisfying (i) and (ii) but contradicting (iii), since none of $u, v$ and $w$ is adjacent to a 10-component or an 8-component in $G\left[V_{c}\right]$, Similarly, $u$ is not adjacent to a 6 -component in $G\left[V_{c}\right]$.

Suppose that $v$ is adjacent to a 4 -component in $G\left[V_{c}\right]$. Note that since $u$, $v$ and $w$ are not adjacent to 6 -components in $G\left[V_{c}\right]$ it follows that neither $u$ nor $w$ is adjacent to a 4-component in $G\left[V_{c}\right]$ - otherwise we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u, v$ and $w$ with a 6 -component; satisfying (i) through (iii) but contradicting (iv). Therefore, since $u$ and $w$ are not adjacent to 4 -components in $G\left[V_{c}\right]$, we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u, v$ and $w$ with a 6 -component; containing three vertices of the 4-component, $u$ and its neighbour in $V_{c}$.

Therefore $v$ is adjacent to a 2 -component in $G\left[V_{c}\right]$. However, then we can replace the components in $G\left[V_{c}\right]$ that are adjacent to $u, v$ and $w$ with a 4 -component; satisfying (i) through (iv) but contradicting (v).

Therefore, $u$ is an end-vertex of a $P_{7}$ in $G\left[V_{c} \cup\{u\}\right]$ or $v$ is an end-vertex of a $P_{6}$ in $G\left[V_{c} \cup\{v\}\right]$ or $w$ is an end-vertex of a $P_{5}$ in $G\left[V_{c} \cup\{w\}\right]$. In each case it follows that there is a path $P$ of length 6 in $G\left[S \cup V_{c}\right]$. Let $z$ be the end-vertex of $P$ in $V_{c}, y$ the neighbour of $z$ on $P$ and $x$ the other neighbour of $y$ on $P$. By repeating the above argument it follows that $z$ is an end-vertex of a $P_{5}$ in $G\left[V_{c-1} \cup\{z\}\right]$ or $y$ is an end-vertex of a $P_{6}$ in $G\left[V_{c-1} \cup\{y\}\right]$ or $x$
is an end-vertex of a $P_{7}$ in $G\left[V_{c-1} \cup\{x\}\right]$. Continuing in this way we obtain a path of length at least $4 c+2 \geq k+1$ in $G$, a contradiction. Therefore, $G[S] \in \mathcal{S}_{1}$.

Theorem 3.7. For $n \geq 9$, the following holds:

$$
\left\lceil\frac{k+1}{n+1}\right\rceil \leq \chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{k}\right) \leq\left\lceil\frac{k-1}{4}\right\rceil+1 .
$$

Proof. The left inequality holds since $K_{k+1} \in \mathcal{W}_{k}$. The right inequality follows as a corollary of Theorem 3.6 since $K_{10} \in \mathcal{S}_{n}$ for each $n \geq 9$.

Our next result improves on the lower bound in Theorem 3.7 for large values of $n$.

Theorem 3.8 For all positive integers $k$ and $n, \chi \mathcal{S}_{n}\left(\mathcal{W}_{k}\right) \geq\left\lfloor\log _{2}(k+2)\right\rfloor$.
Proof. We first prove, by induction on $m$, that for all positive integers $m$ and $n, \mathcal{W}_{2^{m+1}-2} \nsubseteq \mathcal{S}_{n}^{m}$ : For the case where $m=1$ the result holds since $\mathcal{W}_{2} \nsubseteq \mathcal{S}_{n}$. Assume therefore that the result holds for $m-1$, thus there exists a graph $H$ such that $H \in \mathcal{W}_{2^{m}-2}$ and $H \notin \mathcal{S}_{n}^{m-1}$. Now let $G=(n+1) H+K_{1}$. Clearly $G \in \mathcal{W}_{2\left(2^{m}-2\right)+2}=\mathcal{W}_{2^{m+1}-2}$. As in the proof of Theorem 1.4 $G \notin \mathcal{S}_{n}^{m}$.

Now, let $k$ and $n$ be any positive integers. We have that $\mathcal{W}_{k} \supseteq$ $\mathcal{W}_{2\left\lfloor\log _{2}(k+2)\right\rfloor-2} \nsubseteq \mathcal{S}_{n}^{\left\lfloor\log _{2}(k+2)\right\rfloor-1}$ and the result follows.

Corollary 3.5 and Theorem 3.7 seem to suggest that for every $k$ and $m$ we can get $\mathcal{W}_{k} \subseteq \mathcal{S}_{n}^{\left\lceil\frac{k}{m}\right\rceil+1}$ for all $n$ sufficiently large. However, Theorem 3.8 implies that $\mathcal{W}_{6} \nsubseteq \mathcal{S}_{n}^{\left\lceil\frac{6}{6}\right\rceil+1}$ for all $n$ since $\chi_{\mathcal{S}_{n}}\left(\mathcal{W}_{6}\right) \geq\left\lfloor\log _{2}(8)\right\rfloor=3$. The method of proof in Theorem 3.6 does not extend. If we try to maximize with respect to 12 -components, 10 -components etc. the argument fails, and assuming that $k$ is large makes no difference.

## Acknowledgement

The authors would like to thank Michael Dorfling and an anonymous referee for their helpful suggestions.

## References

[1] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, A survey of hereditary properties of graphs, Discuss. Math. Graph Theory 17 (1997) 5-50.
[2] M. Borowiecki and P. Mihók, Hereditary properties of graphs, in: V.R. Kulli, ed., Advances in Graph Theory (Vishwa International Publication, Gulbarga, 1991) 41-68.
[3] I. Broere, M.J. Dorfling, J.E Dunbar and M. Frick, A path(ological) partition problem, Discuss. Math. Graph Theory 18 (1998) 113-125.
[4] I. Broere, P. Hajnal and P. Mihók, Partition problems and kernels of graphs, Discuss. Math. Graph Theory 17 (1997) 311-313.
[5] S.A. Burr and M.S. Jacobson, On inequalities involving vertex-partition parameters of graphs, Congr. Numer. 70 (1990) 159-170.
[6] G. Chartrand, D.P. Geller and S.T. Hedetniemi, A generalization of the chromatic number, Proc. Camb. Phil. Soc. 64 (1968) 265-271.
[7] M. Frick and F. Bullock, Detour chromatic numbers, manuscript.
[8] P. Hajnal, Graph partitions (in Hungarian), Thesis, supervised by L. Lovász (J.A. University, Szeged, 1984).
[9] T.R. Jensen and B. Toft, Graph colouring problems (Wiley-Interscience Publications, New York, 1995).
[10] L. Lovász, On decomposition of graphs, Studia Sci. Math. Hungar 1 (1966) 237-238; MR34\#1715.
[11] P. Mihók, Problem 4, p. 86 in: M. Borowiecki and Z. Skupień (eds), Graphs, Hypergraphs and Matroids (Zielona Góra, 1985).
[12] J. Nešetřil and V. Rödl, Partitions of vertices, Comment. Math. Univ. Carolinae 17 (1976) 85-95; MR54\#173.

Received 10 March 2001
Revised 3 December 2001

