*Discussiones Mathematicae*
*Graph Theory 22 (2002) 259–270*

# GENERALIZED CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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**Abstract**

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P}) = n$ iff $\mathcal{P} \subseteq \mathcal{Q}^n$ but $\mathcal{P} \not\subseteq \mathcal{Q}^{n-1}$. We investigate the generalized chromatic numbers of the well-known properties of graphs $\mathcal{I}_k$, $\mathcal{O}_k$, $\mathcal{W}_k$, $\mathcal{S}_k$ and $\mathcal{D}_k$.

**Keywords:** property of graphs, additive, hereditary, generalized chromatic number.

**2000 Mathematics Subject Classification:** 05C15.

## 1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$. A *property* of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$.

Throughout the text we will call a component of a graph that is a spanning supergraph of a path $P_k$ of order $k$ a *$k$-component*. Let $G$ be a graph and $V_1 \subseteq V(G)$. We say that a vertex $v \in V(G) - V_1$ *is adjacent to a $k$-component of* $G[V_1]$ if $v$ is adjacent to a vertex of some $k$-component of $G[V_1]$.

**Example.** For a positive integer $k$ we define the following well-known properties:

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,

$\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$,

$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of $G$ has at most $k+1$ vertices$\}$,

$\mathcal{W}_k = \{G \in \mathcal{I} :$ each path in $G$ has at most $k+1$ vertices$\}$,

$\mathcal{S}_k = \{G \in \mathcal{I} :$ the maximum degree of $G$ is at most $k\}$,

$\mathcal{T}_k = \{G \in \mathcal{I} : G$ contains no subgraph homeomorphic to $K_{k+2}$ or $K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}$,

$\mathcal{D}_k = \{G \in \mathcal{I} : G$ is $k$-degenerate, i.e., every subgraph of $G$ has a vertex of degree at most $k\}$.

For every additive hereditary property $\mathcal{P} \neq \mathcal{I}$ there is a smallest integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of $\mathcal{P}$. Note that all the properties in the above example, except $\mathcal{O}$, are of completeness $k$. The set $\mathbf{F}(\mathcal{P})$ of *minimal forbidden subgraphs* is defined by $\{G \in \mathcal{I} : G \in \overline{\mathcal{P}}$ and $H \in \mathcal{P}$ for all $H \subset G\}$.

Let $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ be arbitrary hereditary properties of graphs. A *vertex* $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$*-partition* of a graph $G$ is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the induced subgraph $G[V_i]$ has the property $\mathcal{Q}_i$. The property $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ is defined as the set of all graphs having a vertex $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n)$-partition. It is easy to see that if $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$ is additive and hereditary too. If $\mathcal{Q}_1 = \mathcal{Q}_2 = \cdots = \mathcal{Q}_n = \mathcal{Q}$, then we write $\mathcal{Q}^n = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$.

The generalized chromatic number $\chi_{\mathcal{Q}}(\mathcal{P})$ is defined as follows: $\chi_{\mathcal{Q}}(\mathcal{P}) = n$ iff $\mathcal{P} \subseteq \mathcal{Q}^n$ but $\mathcal{P} \not\subseteq \mathcal{Q}^{n-1}$.

As an example of the non-existence of $\chi_{\mathcal{Q}}(\mathcal{P})$ we have $\chi_{\mathcal{O}}(\mathcal{I}_1)$ since it is well known that there exist triangle-free graphs of arbitrary chromatic number. The following theorem, due to J. Nešetřil and V. Rödl (see [12]), implies that for some additive hereditary properties $\mathcal{P}$ we have that $\chi_{\mathcal{Q}}(\mathcal{P})$ exists if and only if $\chi_{\mathcal{Q}}(\mathcal{P}) = 1$. In particular, $\chi_{\mathcal{Q}}(\mathcal{I}_k)$ exists if and only $\chi_{\mathcal{Q}}(\mathcal{I}_k) = 1$.

**Theorem 1.1** [12]**.** *Let* $\mathbf{F}(\mathcal{P})$ *be a finite set of* 2*-connected graphs. Then for every graph* $G \in \mathcal{P}$ *there exists a graph* $H \in \mathcal{P}$ *such that for any partition* $\{V_1, V_2\}$ *of* $V(H)$ *there is an* $i$*,* $i = 1$ *or* $i = 2$*, for which* $G \leq H[V_i]$*.* ∎

**Corollary 1.2.** *If* $\mathbf{F}(\mathcal{P})$ *is a finite set of* 2*-connected graphs, then for any additive hereditary property* $\mathcal{Q}$ *it follows that* $\chi_{\mathcal{Q}}(\mathcal{P})$ *exists if and only if* $\mathcal{P} \subseteq \mathcal{Q}$. ■

The value of $\chi_{\mathcal{Q}}(\mathcal{P})$ is known for various choices of $\mathcal{P}$ and $\mathcal{Q}$. In the remainder of this section we mention some simple results, most of which are known or follow immediately from well-known results. See for example [2] and [5].

It is easy to see that $\mathcal{O}_{a+b+1} \subseteq \mathcal{O}_a \circ \mathcal{O}_b$ and $\mathcal{D}_{a+b+1} \subseteq \mathcal{D}_a \circ \mathcal{D}_b$ (see for example [9]), which implies that $\chi_{\mathcal{Q}}(\mathcal{O}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$ for any property $\mathcal{Q}$ of completeness $n$, and $\chi_{\mathcal{D}_n}(\mathcal{P}) = \left\lceil \frac{k+1}{n+1} \right\rceil$ for any property $\mathcal{P}$ such that $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{D}_k$. Note that Corollary 1.2 implies that the latter equality does not extend to $c(\mathcal{P}) = n$.

The well-known theorem of Lovász states:

**Theorem 1.3** [10]**.** $\mathcal{S}_{a+b+1} \subseteq \mathcal{S}_a \circ \mathcal{S}_b$ *for all* $a, b \geq 0$. ■

This implies that $\chi_{\mathcal{S}_n}(\mathcal{S}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil$. (See [5].)

It is also easy to see that if $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{O}^{k+1}$, then $\chi_{\mathcal{I}_n}(\mathcal{P}) = \left\lceil \frac{k+1}{n+1} \right\rceil$.

The next result is interesting since it shows that the value of $\chi_{\mathcal{S}_n}(\mathcal{D}_k)$ is independent of $n$.

**Theorem 1.4.** *For all* $k$ *and* $n$ *we have* $\chi_{\mathcal{S}_n}(\mathcal{D}_k) = k + 1$.

***Proof.*** Since $\mathcal{D}_k \subseteq \mathcal{O}^{k+1} \subseteq \mathcal{S}_n^{k+1}$ we have the upper bound. We prove the lower bound by induction on $k$. The result is true for $k = 1$ since $\mathcal{D}_1 \not\subseteq \mathcal{S}_n$. Assume, therefore, that $\mathcal{D}_k \not\subseteq \mathcal{S}_n^k$ and let $H \in \mathcal{D}_k$ such that $H \notin \mathcal{S}_n^k$. Let $G = (n+1)H + K_1$. Since every subgraph of $(n+1)H$ has a vertex of degree at most $k$, every subgraph of $G$ has a vertex of degree at most $k + 1$. Thus $G \in \mathcal{D}_{k+1}$.

Also, $G \notin \mathcal{S}_n^{k+1}$: Suppose, to the contrary, that $\{V_1, V_2, \ldots, V_{k+1}\}$ is an $\mathcal{S}_n^{k+1}$-partition of $V(G)$. Let $v$ be the universal vertex of $G$ and suppose, without loss of generality, that $v \in V_1$. Since $G[V_1] \in \mathcal{S}_n$ it follows that $|V_1| \leq n + 1$. Since there are $n + 1$ copies of $H$ in $G$ we have that for some copy $F$ of $H$, $F \cap V_1 = \emptyset$. This contradicts the fact that $H \notin \mathcal{S}_n^k$. ■

The lattice of (additive) hereditary properties is discussed in [1] — we use the supremum and infimum of properties in our next result without further discussion.

**Theorem 1.5.** *Let $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{Q}$ be additive hereditary properties such that $\chi_{\mathcal{Q}}(\mathcal{P}_1)$ and $\chi_{\mathcal{Q}}(\mathcal{P}_2)$ are finite. The following hold:*

(i) $\chi_{\mathcal{Q}}(\mathcal{P}_1 \cup \mathcal{P}_2) = \chi_{\mathcal{Q}}(\mathcal{P}_1 \vee \mathcal{P}_2) = \max\{\chi_{\mathcal{Q}}(\mathcal{P}_1), \chi_{\mathcal{Q}}(\mathcal{P}_2)\}$.

(ii) $\chi_{\mathcal{Q}}(\mathcal{P}_1 \cap \mathcal{P}_2) \leq \min\{\chi_{\mathcal{Q}}(\mathcal{P}_1), \chi_{\mathcal{Q}}(\mathcal{P}_2)\}$.

(iii) $\max\{\chi_{\mathcal{Q}}(\mathcal{P}_1), \chi_{\mathcal{Q}}(\mathcal{P}_2)\} \leq \chi_{\mathcal{Q}}(\mathcal{P}_1 \circ \mathcal{P}_2) \leq \chi_{\mathcal{Q}}(\mathcal{P}_1) + \chi_{\mathcal{Q}}(\mathcal{P}_2)$. ■

We remark that the inequality in Theorem 1.5(ii) may be strict. For example $\chi_{\mathcal{O}}(\mathcal{T}_3) = 4$ and $\chi_{\mathcal{O}}(\mathcal{I}_1)$ is infinite but $\chi_{\mathcal{O}}(\mathcal{T}_3 \cap \mathcal{I}_1) = 3$. (See [2].)

## 2. Results on $\mathcal{W}_k$

In this section we investigate the value of $\chi_{\mathcal{W}_n}(\mathcal{W}_k)$. The problem of determining it has been discussed in (or is related to problems in) several papers (see for example [3], [4], [6], [7], [8] and [11]) and the following conjecture has been made in at least three of them:

**Conjecture 2.1** [3], [6], [7]**.** *$\mathcal{W}_{a+b+1} \subseteq \mathcal{W}_a \circ \mathcal{W}_b$ for all positive integers $a$ and $b$.*

This conjecture implies the following for $\chi_{\mathcal{W}_n}(\mathcal{W}_k)$:

**Conjecture 2.2.** *For every $n, k \geq 1$, the following holds:*

$$\chi_{\mathcal{W}_n}(\mathcal{W}_k) = \left\lceil \frac{k+1}{n+1} \right\rceil .$$

In [6] the bound $\chi_{\mathcal{W}_n}(\mathcal{W}_k) \leq \lfloor \frac{k-n+1}{2} \rfloor + 2$ is proved. The following theorem will enable us to improve on this bound.

**Theorem 2.3.** $\mathcal{W}_{\lceil \frac{2a}{3} \rceil + b + 1} \subseteq \mathcal{W}_a \circ \mathcal{W}_b$ *for all $a \geq 15$ and $b \geq 1$.*

***Proof.*** Consider any graph $G$ in $\mathcal{W}_{\lceil \frac{2a}{3} \rceil + b + 1}$. Take $V_1$ to be a maximal subset of $V(G)$ such that $G[V_1]$ is in $\mathcal{W}_a$. Let $V_2 = V(G) - V_1$. Suppose that there is a path $P$ in $G[V_2]$ of length $b+1$ and let $v_1$ and $v_2$ denote the end-vertices of $P$. Since $V_1$ is maximal in $\mathcal{W}_a$ it follows that there is a path $P_1$ of length $a+1$ in $G[V_1 \cup \{v_1\}]$ and a path $P_2$ of length $a+1$ in $G[V_1 \cup \{v_2\}]$. Note that if either $v_1$ or $v_2$ is an end-vertex of $P_1$ or $P_2$ respectively, then in both cases we get a path of length at least $a+b+3$

in $G$, a contradiction. Therefore the vertices $v_1$ and $v_2$ are not end-vertices of their respective paths. Let $P_{11}$ and $P_{12}$ denote the paths on either side of $v_1$ such that $P_{11} \cup \{v_1\} \cup P_{12} = P_1$. Similarly, let $P_{21} \cup \{v_2\} \cup P_{22} = P_2$. Now suppose, without loss of generality, that $x = |E(P_{11})|+1 \leq y = |E(P_{12})|+1$, so that $x + y = a + 1$.

It is easily seen that if $y \geq \lfloor \frac{2a+2}{3} \rfloor + 1$, then by simply taking the path $P_{12} \cup P$, we get a path of length at least $\lfloor \frac{2a+2}{3} \rfloor + 1 + b + 1 \geq \frac{2a+2-2}{3} + b + 2 > \lceil \frac{2a}{3} \rceil + b + 1$ in $G$, a contradiction. Therefore $\lceil \frac{a+1}{2} \rceil \leq y \leq \lfloor \frac{2a+2}{3} \rfloor$. Moreover, each $P_{ij}$, $i, j \in \{1, 2\}$ has length at least $\lfloor \frac{a-5}{3} \rfloor$, since $x = a + 1 - y \geq a - \lfloor \frac{2a+2}{3} \rfloor + 1 \geq a - \frac{2a+5}{3} = \frac{a-5}{3} \geq \lfloor \frac{a-5}{3} \rfloor$.

Note that $P_{11}$ and $P_{12}$ are neccessarily disjoint as are $P_{21}$ and $P_{22}$, and that $v_1$ and $v_2$ are not on any of these paths.

$P_{12}$ must intersect both $P_{21}$ and $P_{22}$: Firstly, $P_{12}$ must intersect the longer of $P_{21}$ and $P_{22}$ since otherwise we get a too long path in $G$; containing the two longer paths and $P$. Furthermore, if $P_{12}$ does not intersect the shorter of $P_{21}$ and $P_{22}$, then we get a path of length at least $\lceil \frac{a+1}{2} \rceil + b + 1 + \lfloor \frac{a-5}{3} \rfloor \geq \frac{a+1}{2} + \frac{a-7}{3} + b + 1 = \frac{5}{6}(a-1) > \lceil \frac{2a}{3} \rceil + 1 + b$ (since $a \geq 15$) in $G$; containing $P_{12}$, $P$ and the shorter of $P_{21}$ and $P_{22}$, a contradiction. Similarly, the longer of $P_{21}$ and $P_{22}$ must intersect both $P_{11}$ and $P_{12}$.

Note that since $P_{11}$ and $P_{12}$ are disjoint and $P_{21}$ and $P_{22}$ are disjoint, $P_{2i}$, $i \in \{1, 2\}$ can only intersect one of $P_{11}$ and $P_{12}$ first and vice-versa.

Suppose that both $P_{21}$ and $P_{22}$ intersect $P_{12}$ first. Then we obtain a path of length at least $x + b + 1 + 1 + \lfloor \frac{y}{2} \rfloor \geq a + 1 - y + \frac{y-1}{2} + b + 2 \geq a - \frac{1}{2} \lfloor \frac{2a+2}{3} \rfloor + \frac{5}{2} + b \geq a - \frac{1}{2}(\frac{2a+2}{3}) + \frac{5}{2} + b = \frac{2a}{3} + \frac{13}{6} + b > \lceil \frac{2a}{3} \rceil + 1 + b$ in $G$; containing $P_{11}$, $P$, at least one edge of either $P_{21}$ or $P_{22}$ and at least a half of $P_{12}$, a contradiction.

Now, suppose that $P_{21}$ or $P_{22}$ intersects $P_{11}$ first, say $P_{21}$. Then we obtain a path of length at least $y + \lfloor \frac{x}{2} \rfloor + b + 1 + 1 = y + \lfloor \frac{1}{2}(a + 1 - y) \rfloor + b + 2 \geq y + \frac{a+1-y-1}{2} + b + 2 = \frac{y}{2} + \frac{a}{2} + b + 2 \geq \frac{1}{2} \lceil \frac{a+1}{2} \rceil + \frac{a}{2} + b + 2 \geq \frac{3a+9}{4} + b > \lceil \frac{2a}{3} \rceil + 1 + b$ in $G$; containing $P_{12}$, $P$, at least one edge of $P_{21}$ and at least a half of $P_{11}$, a contradiction. ∎

**Theorem 2.4.** *$\chi_{\mathcal{W}_n}(\mathcal{W}_k) \leq \left\lceil \frac{3k}{2n+3} \right\rceil$ for all $n \geq 15$ and $k \geq 1$.*

***Proof.*** $\mathcal{W}_{c\lceil \frac{2n+3}{3} \rceil} \subseteq \mathcal{W}_n^c$ for all positive integers $c$ and $n$: the proof is by induction on $c$. The result holds for $c = 1$. Suppose now that the result holds for $c$. Note that $\mathcal{W}_{(c+1)\lceil \frac{2n+3}{3} \rceil} = \mathcal{W}_{\lceil \frac{2n}{3} \rceil + 1 + c\lceil \frac{2n+3}{3} \rceil}$ which by Theorem 2.3 is contained in $\mathcal{W}_n \circ \mathcal{W}_{c\lceil\frac{2n+3}{3}\rceil}$ which by the induction hypothesis is contained in $\mathcal{W}_n \circ \mathcal{W}_n^c = \mathcal{W}_n^{c+1}$.

Now, with $c = \lceil\frac{3k}{2n+3}\rceil$, since $k \leq \lceil\frac{3k}{2n+3}\rceil\lceil\frac{2n+3}{3}\rceil$ we have that $\mathcal{W}_k \subseteq \mathcal{W}_{c\lceil\frac{2n+3}{3}\rceil} \subseteq \mathcal{W}_n^c$. ■

This result is close to the bound $\chi_{\mathcal{W}_n}(\mathcal{W}_k) \leq \left\lceil\frac{3(k-n)}{2n+2}\right\rceil + 1$ presented in [7] but our method of proof is completely different.

## 3. Results Relating $\mathcal{S}_k$ and $\mathcal{W}_n$

**Theorem 3.1.** *For positive integers $n$ and $k$ we have that*

$$\left\lceil\frac{k+1}{n+1}\right\rceil \leq \chi_{\mathcal{W}_n}(\mathcal{S}_k) \leq \left\lceil\frac{k+1}{2}\right\rceil.$$

***Proof.*** The left inequality holds since $K_{k+1} \in \mathcal{S}_k$. The right inequality follows as a corollary to Theorem 1.3. ■

The first inequality in Theorem 3.1 may be strict, for example $\chi_{\mathcal{W}_2}(\mathcal{S}_2) = 2 > \lceil\frac{2+1}{2+1}\rceil$ (since $\mathcal{S}_2 \not\subseteq \mathcal{W}_2$). Equality in both the inequalities may be achieved, for example, by Theorem 1.3 we have that $\chi_{\mathcal{S}_n}(\mathcal{S}_k) = \left\lceil\frac{k+1}{n+1}\right\rceil$ and therefore $\chi_{\mathcal{W}_1}(\mathcal{S}_k) = \left\lceil\frac{k+1}{2}\right\rceil$.

Note that whether or not the second inequality proved in Theorem 3.1 may be strict still remains an open problem.

We now start working towards bounds on $\chi_{\mathcal{S}_n}(\mathcal{W}_k)$.

**Theorem 3.2.** $\mathcal{W}_4 \subseteq \mathcal{S}_2 \circ \mathcal{S}_1$.

***Proof.*** Consider any graph $G$ in $\mathcal{W}_4$. Take $V_1$ to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $\mathcal{S}_2$,

(ii) $G[V_1]$ contains a maximum number of 4-components,

(iii) $G[V_1]$ contains a maximum number of components isomorphic to $K_3$,

(iv) $G[V_1]$ contains a maximum number of 2-components and

(v) $G[V_1]$ contains a maximum number of isolated vertices.

(In other words, we consider all subsets $V$ of $V(G)$ such that $G[V] \in \mathcal{S}_2$. Amongst these we consider all subsets $V$ for which $G[V]$ has a maximum

number of 4-components. Amongst these we consider all subsets inducing a maximum number of components isomorphic to $K_3$ etc.)
Let $V_2 = V(G) - V_1$. We will show that $G[V_2] \in \mathcal{S}_1$. Suppose, to the contrary, that $G[V_2] \notin \mathcal{S}_1$ and let $v$ be a vertex in $G[V_2]$ of degree at least two with $u$ and $w$ two of its neighbours in $G[V_2]$. Note that by choice of $V_1$ every component in $G[V_1]$ is a 4-component, $K_3$, $K_2$ or $K_1$.

Moreover, by (v) it follows that $u$, $v$ and $w$ each have at least one neighbour in $V_1$. Furthermore, $v$ is adjacent to a nontrivial component in $G[V_1]$: If this is not the case, then we can replace the vertices in $V_1$ that are adjacent to $v$ with a 2-component; still satisfying (i) through (iii) but contradicting (iv). Similarly, $u$ and $w$ are adjacent to nontrivial components in $G[V_1]$.

Suppose that $v$ is adjacent to a triangle in $G[V_1]$. Note that neighbours of both $u$ and $w$ in $V_1$ can only lie on this triangle, otherwise we obtain at least a $P_6$ in $G$. However then we obtain a $P_6$ in $G$; containing all three vertices of the triangle in $G[V_1]$ as well as the $P_3$ formed by $u, v$ and $w$. Thus $v$ cannot be adjacent to a triangle in $G[V_1]$.

Furthermore, $v$ cannot be adjacent to a 4-component in $G[V_1]$. This case is analogous to the above case since a 4-component will also contribute three vertices to give a $P_6$ in $G$. Moreover, neither $u$ nor $w$ are adjacent to 4-components or triangles in $G[V_1]$, since otherwise we obtain at least a $P_6$ in $G$.

Therefore $v$ must be adjacent to a $K_2$ in $G[V_1]$. Note that $u$ and $w$ must each have at least one neighbour on the $K_2$ adjacent to $v$ in $G[V_1]$, otherwise we obtain a $P_6$ in $G$. If $v$ is adjacent to both vertices on the $K_2$ in $G[V_1]$, then we can replace the components in $G[V_1]$ that are adjacent to $u, v$ and $w$ with a triangle; still satisfying (i) and (ii), but contradicting (iii).

Thus $v$ has only one neighbour on any $K_2$ in $G[V_1]$. If $u$ or $w$ is adjacent to the same vertex as $v$ on the $K_2$ adjacent to $v$ in $G[V_1]$, then once again we can replace the components in $G[V_1]$ that are adjacent to $u, v$ and $w$ with a triangle; still satisfying (i) and (ii), but contradicting (iii). Therefore, both $u$ and $w$ are adjacent to the vertex on the $K_2$ in $G[V_1]$ that is not adjacent to $v$. However, then we can replace the components in $G[V_1]$ that are adjacent to $u, v$ and $w$ with a 4-component; containing the $K_2$ in $G[V_1]$ and the vertices $v$ and either $u$ or $w$; still satisfying (i), but contradicting (ii). Therefore $G[V_2] \in \mathcal{S}_1$. ■

**Corollary 3.3.** *For all $n \geq 2$ and $k$, we have that $\chi_{\mathcal{S}_n}(\mathcal{W}_k) \leq 2\left\lceil \frac{k+1}{5} \right\rceil$.*

***Proof.*** It is known that $\mathcal{W}_{4+k+1} \subseteq \mathcal{W}_4 \circ \mathcal{W}_k$ (see [3]). Similar to the proof of Theorem 1.3 it follows that $\mathcal{W}_k \subseteq \mathcal{W}_4^{\left\lceil \frac{k+1}{5} \right\rceil}$. The result now follows from Theorem 3.2. ■

The inequality in Corollary 3.3 may be strict, for example $\chi_{\mathcal{S}_2}(\mathcal{W}_1) = 1 < 2 = 2\lceil \frac{2}{5} \rceil$. Equality may also be obtained, for example $\chi_{\mathcal{S}_2}(\mathcal{W}_2) = 2 = 2\lceil \frac{3}{5} \rceil$.

Having proved Corollary 3.3 we naturally ask: Can this bound be improved and if so under what conditions? Corollary 3.5 gives us an answer for $n \geq 5$ and Theorem 3.7 for $n \geq 9$.

**Theorem 3.4.** *For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 5$, the following holds: $\mathcal{W}_k \subseteq \mathcal{Q}^{\lceil \frac{k}{3} \rceil} \circ \mathcal{O}$.*

***Proof.*** Let $c = \left\lceil \frac{k}{3} \right\rceil$. Consider any graph $G$ in $\mathcal{W}_k$. Take $V_1$ to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $\mathcal{Q}$,

(ii) $G[V_1]$ contains a maximum number of 6-components,

(iii) $G[V_1]$ contains a maximum number of 4-components,

(iv) $G[V_1]$ contains a maximum number of 2-components and

(v) $G[V_1]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take $V_i$ to be a subset of $V(G) - \bigcup_{j=1}^{i-1} V_j$ such that for each $i$, $G[V_i]$ satisfies the above list. Let $S = V(G) - \bigcup_{j=1}^{c} V_j$. We will show that $G[S] \in \mathcal{O}$. Suppose, to the contrary, that $G[S] \notin \mathcal{O}$ and let $v$ be a vertex in $G[S]$ of degree at least one and $u$ be a neighbour of $v$ in $G[S]$. Suppose that $v$ is not an end-vertex of a $P_4$ in $G[V_c \cup \{v\}]$ and that $u$ is not an end-vertex of a $P_5$ in $G[V_c \cup \{u\}]$. Note that for every $i$, the choice of $V_i$ gives that every component in $G[V_i]$ is a 6-component, a 4-component, $K_2$ or $K_1$. Moreover, by (v) it follows that $u$ and $v$ have at least one neighbour in $V_c$ each and by (iv) both $u$ and $v$ are adjacent to nontrivial components in $G[V_c]$. Since $v$ is not an end-vertex of a $P_4$ in $G[V_c \cup \{v\}]$ it follows that $v$ is not adjacent to a 6-component or a 4-component in $G[V_c]$. Similarly, $u$ is not adjacent to a 6-component in $G[V_c]$.

Suppose that $u$ is adjacent to a 4-component in $G[V_c]$. Then, since $v$ is not adjacent to a 4-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u$ and $v$ with a 6-component; still satisfying (i) since $K_6 \in \mathcal{Q}$, but contradicting (ii), since neither $u$ nor $v$ is adjacent to a 6-component in $G[V_c]$.

Therefore $u$ is adjacent to a 2-component in $G[V_c]$. However, then we can replace the components in $G[V_c]$ that are adjacent to $u$ and $v$ with a 4-component; satisfying (i) and (ii) but contradicting (iii).

Therefore, $u$ is an end-vertex of a $P_5$ in $G[V_c \cup \{u\}]$ or $v$ is an end-vertex of a $P_4$ in $G[V_c \cup \{v\}]$. In both cases it follows that there is a path $P$ of length four in $G[S \cup V_c]$. Let $x$ be the end-vertex of $P$ in $V_c$ and $y$ the neighbour of $x$ on $P$. By repeating this argument it follows that $x$ is an end-vertex of a $P_4$ in $G[V_{c-1} \cup \{x\}]$ or $y$ is an end-vertex of a $P_5$ in $G[V_{c-1} \cup \{y\}]$. Continuing in this way we obtain a path of length at least $3c + 1 \geq k + 1$ in $G$, a contradiction. Therefore, $G[S] \in \mathcal{O}$. ■

**Corollary 3.5.** *For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 5$, the following holds:* $\chi_{\mathcal{Q}}(\mathcal{W}_k) \leq \left\lceil \frac{k}{3} \right\rceil + 1$. ■

The inequality in Corollary 3.5 may be strict, for example we have that $\chi_{\mathcal{I}_5}(\mathcal{W}_k) = \lceil \frac{k+1}{6} \rceil < \frac{k+3}{3} \leq \lceil \frac{k}{3} \rceil + 1$ with $K_6 \in \mathcal{I}_5$ and $K_7 \notin \mathcal{I}_5$. Equality can also be obtained: In Theorem 3.8 (still to follow) we prove that $\chi_{\mathcal{S}_n}(\mathcal{W}_k) \geq \lfloor \log_2(k+2) \rfloor$ thus for all $n$ we have that $\chi_{\mathcal{S}_n}(\mathcal{W}_6) \geq 3$ and by Corollary 3.5 we have $\chi_{\mathcal{S}_n}(\mathcal{W}_6) \leq \lceil \frac{6}{3} \rceil + 1 = 3$.

**Theorem 3.6.** *For an additive hereditary property $\mathcal{Q}$ with $c(\mathcal{Q}) \geq 9$, the following holds: $\mathcal{W}_k \subseteq \mathcal{Q}^{\lceil \frac{k-1}{4} \rceil} \circ \mathcal{S}_1$.*

***Proof.*** Let $c = \left\lceil \frac{k-1}{4} \right\rceil$. Consider any graph $G$ in $\mathcal{W}_k$. Take $V_1$ to be a subset of $V(G)$ such that, in order of priority:

(i) $G[V_1]$ is in $\mathcal{Q}$,

(ii) $G[V_1]$ contains a maximum number of 10-components,

(iii) $G[V_1]$ contains a maximum number of 8-components,

(iv) $G[V_1]$ contains a maximum number of 6-components,

(v) $G[V_1]$ contains a maximum number of 4-components,

(vi) $G[V_1]$ contains a maximum number of 2-components and

(vii) $G[V_1]$ contains a maximum number of isolated vertices.

Now, for $2 \leq i \leq c$ take $V_i$ to be a subset of $V(G) - \bigcup_{j=1}^{i-1} V_j$ such that for each $i$, $G[V_i]$ satisfies the above list. Let $S = V(G) - \bigcup_{j=1}^{c} V_j$. We will show that $G[S] \in \mathcal{S}_1$. Suppose, to the contrary, that $G[S] \notin \mathcal{S}_1$ and let $v$ be a vertex in $G[S]$ of degree at least two with $u$ and $w$ neighbours of $v$ in $G[S]$.

Suppose that $u$ is not an end-vertex of a $P_7$ in $G[V_c \cup \{u\}]$ and that $v$ is not an end-vertex of a $P_6$ in $G[V_c \cup \{v\}]$ and that $w$ is not an end-vertex of a $P_5$ in $G[V_c \cup \{w\}]$. Note that for every $i$, the choice of $V_i$ gives that every component in $G[V_i]$ is a 10-component, an 8-component, a 6-component, a 4-component, $K_2$ or $K_1$. Moreover, by (vii) it follows that $u, v$ and $w$ have at least one neighbour in $V_c$ each and by (vi) each of $u$, $v$ and $w$ is adjacent to a nontrivial component in $G[V_c]$. Since $u$ is not an end-vertex of a $P_7$ in $G[V_c \cup \{u\}]$ it follows that $u$ is not adjacent to a 10-component in $G[V_c]$. Similarly, $v$ is not adjacent to a 10-component or an 8- component in $G[V_c]$ and $w$ is not adjacent to a 10-component, an 8- component or a 6-component in $G[V_c]$.

Suppose that $u$ is adjacent to an 8-component in $G[V_c]$. Then, since neither $v$ nor $w$ are adjacent to an 8-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u, v$ and $w$ with a 10-component; still satisfying (i) but contradicting (ii).

Suppose that $v$ is adjacent to a 6-component in $G[V_c]$. Then, since $w$ is not adjacent to a 6-component in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u, v$ and $w$ with an 8-component; still satisfying (i) and (ii) but contradicting (iii), since none of $u, v$ and $w$ is adjacent to a 10-component or an 8-component in $G[V_c]$, Similarly, $u$ is not adjacent to a 6-component in $G[V_c]$.

Suppose that $v$ is adjacent to a 4-component in $G[V_c]$. Note that since $u$, $v$ and $w$ are not adjacent to 6-components in $G[V_c]$ it follows that neither $u$ nor $w$ is adjacent to a 4-component in $G[V_c]$ — otherwise we can replace the components in $G[V_c]$ that are adjacent to $u, v$ and $w$ with a 6-component; satisfying (i) through (iii) but contradicting (iv). Therefore, since $u$ and $w$ are not adjacent to 4-components in $G[V_c]$, we can replace the components in $G[V_c]$ that are adjacent to $u, v$ and $w$ with a 6-component; containing three vertices of the 4-component, $u$ and its neighbour in $V_c$.

Therefore $v$ is adjacent to a 2-component in $G[V_c]$. However, then we can replace the components in $G[V_c]$ that are adjacent to $u, v$ and $w$ with a 4-component; satisfying (i) through (iv) but contradicting (v).

Therefore, $u$ is an end-vertex of a $P_7$ in $G[V_c \cup \{u\}]$ or $v$ is an end-vertex of a $P_6$ in $G[V_c \cup \{v\}]$ or $w$ is an end-vertex of a $P_5$ in $G[V_c \cup \{w\}]$. In each case it follows that there is a path $P$ of length 6 in $G[S \cup V_c]$. Let $z$ be the end-vertex of $P$ in $V_c$, $y$ the neighbour of $z$ on $P$ and $x$ the other neighbour of $y$ on $P$. By repeating the above argument it follows that $z$ is an end-vertex of a $P_5$ in $G[V_{c-1} \cup \{z\}]$ or $y$ is an end-vertex of a $P_6$ in $G[V_{c-1} \cup \{y\}]$ or $x$

is an end-vertex of a $P_7$ in $G[V_{c-1} \cup \{x\}]$. Continuing in this way we obtain a path of length at least $4c + 2 \geq k + 1$ in $G$, a contradiction. Therefore, $G[S] \in \mathcal{S}_1$. ■

**Theorem 3.7.** *For $n \geq 9$, the following holds:*

$$\left\lceil \frac{k+1}{n+1} \right\rceil \leq \chi_{\mathcal{S}_n}(\mathcal{W}_k) \leq \left\lceil \frac{k-1}{4} \right\rceil + 1.$$

***Proof.*** The left inequality holds since $K_{k+1} \in \mathcal{W}_k$. The right inequality follows as a corollary of Theorem 3.6 since $K_{10} \in \mathcal{S}_n$ for each $n \geq 9$. ■

Our next result improves on the lower bound in Theorem 3.7 for large values of $n$.

**Theorem 3.8** *For all positive integers $k$ and $n$, $\chi_{\mathcal{S}_n}(\mathcal{W}_k) \geq \lfloor \log_2(k+2) \rfloor$.*

***Proof.*** We first prove, by induction on $m$, that for all positive integers $m$ and $n$, $\mathcal{W}_{2^{m+1}-2} \not\subseteq \mathcal{S}_n^m$: For the case where $m = 1$ the result holds since $\mathcal{W}_2 \not\subseteq \mathcal{S}_n$. Assume therefore that the result holds for $m - 1$, thus there exists a graph $H$ such that $H \in \mathcal{W}_{2^m-2}$ and $H \notin \mathcal{S}_n^{m-1}$. Now let $G = (n+1)H + K_1$. Clearly $G \in \mathcal{W}_{2(2^m-2)+2} = \mathcal{W}_{2^{m+1}-2}$. As in the proof of Theorem 1.4 $G \notin \mathcal{S}_n^m$.

Now, let $k$ and $n$ be any positive integers. We have that $\mathcal{W}_k \supseteq \mathcal{W}_{2^{\lfloor \log_2(k+2) \rfloor}-2} \not\subseteq \mathcal{S}_n^{\lfloor \log_2(k+2) \rfloor - 1}$ and the result follows. ■

Corollary 3.5 and Theorem 3.7 seem to suggest that for every $k$ and $m$ we can get $\mathcal{W}_k \subseteq \mathcal{S}_n^{\lceil \frac{k}{m} \rceil + 1}$ for all $n$ sufficiently large. However, Theorem 3.8 implies that $\mathcal{W}_6 \not\subseteq \mathcal{S}_n^{\lceil \frac{6}{6} \rceil + 1}$ for all $n$ since $\chi_{\mathcal{S}_n}(\mathcal{W}_6) \geq \lfloor \log_2(8) \rfloor = 3$. The method of proof in Theorem 3.6 does not extend. If we try to maximize with respect to 12-components, 10-components etc. the argument fails, and assuming that $k$ is large makes no difference.

## Acknowledgement

The authors would like to thank Michael Dorfling and an anonymous referee for their helpful suggestions.

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Received 10 March 2001
Revised 3 December 2001