Discussiones Mathematicae
Graph Theory 22 (2002) 247–258

SOME TOTALLY MODULAR CORDIAL GRAPHS

Ibrahim Cahit

Department of Computer Science and Engineering
European University of Lefke
Lefke, Mersin 10, Turkey

e-mail: ica@lefke.edu.tr

Abstract

In this paper we define total magic cordial (TMC) and total sequential cordial (TSC) labellings which are weaker versions of magic and simply sequential labellings of graphs. Based on these definitions we have given several results on TMC and TSC graphs.

Keywords: graph labeling, cordial labeling, magic and sequential graphs.

2000 Mathematics Subject Classification: 05C78.

1. Introduction

After Cahit [3] the meaning of *cordiality* in the graph labelling problems is well understood and studied [4], [5], [11]. Let $G(m, n)$ (or simply G) be a graph with $m = |E(G)|$ edges and $n = |V(G)|$ vertices. A binary vertex labelling $f : V \to \{0, 1\}$ induces an edge labelling $\overline{f} : E \to \{0, 1\}$ defined by $\overline{f}(\{u, v\}) = |f(u) - f(v)|$. We call such a labelling *cordial* if the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$ are satisfied, where $v_f(i)$ and $e_f(i)$, $i = 0, 1$, is the number of vertices and edges of G, respectively, with label i. A graph is called cordial if it admits a cordial labelling.

In this work we have investigated another two well-known graph labellings on the ground of cordial labellings. These are the *magic graphs* of Kotzig and Rosa [6] – [9] and *simply sequential graphs* of Bange, Barkauskas and Slater [1], [2], [10]. Note that many problems in these areas are still open [5].

Definition 1 (A). A graph $G(m,n)$ is said to have a *magic labelling* with constant C if there exists a 1:1 mapping $f : V(G) \cup E(G) \to \{1,2,\ldots,m+n\}$ such that $f(a)+f(b)+f(\{a,b\}) = C$ for all $\{a,b\} \in E(G)$.

Definition 2 (B). A graph G with $|V(G) \cup E(G)| = m+n$ is called *simply sequential* if there is a bijection $f : V(G) \cup E(G) \to \{1,2,\ldots,m+n\}$ such that for each edge $e = \{ab\} \in E(G)$ one has $f(e) = |f(a) - f(b)|$.

We modify these definitions as follows:

Definition 3 (A′). A graph $G(m,n)$ is said to have a *totally magic cordial* (TMC) *labelling with constant* C if there exists a mapping $f : V(G) \cup E(G) \to \{0,1\}$ such that $f(a)+f(b)+f(\{a,b\}) = C \pmod 2$ for all $\{a,b\} \in E(G)$ provided the condition $|f(0) - f(1)| \le 1$ is hold, where $f(0) = v_f(0) + e_f(0)$ and $f(1) = v_f(1) + e_f(1)$ and $v_f(i)$, $e_f(i), i \in \{0,1\}$ are, respectively, the number of vertices and edges labelled with i.

Definition 4 (B′). A graph G with $|V(G) \cup E(G)| = m+n$ is called *total sequential cordial* (TSC) if there is a total mapping $f : V(G) \cup E(G) \to \{0,1\}$ such that for each edge $e = \{a,b\}$, $f(e) = |f(a) - f(b)|$ and the condition $|f(0) - f(1)| \le 1$ holds.

2. Totally Magic Cordial Graphs

Theorem 5. *The complete bipartite graph $K_{m,n}$ is TMC for all $m,n > 1$.*

Proof. Let us denote vertex bipartition of $K_{m,n}$ as $V = U \cup W$, $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$.

Case a. $m = n$. The proof of the theorem is by construction of a special (0,1)-matrix $\mathbf{M} = [m_{i,j}]_{n\times n}$, where (i,j) entry of $\mathbf{M}$ corresponds to the edge label $m_{i,j} = f(u_i, w_j)$.

(a1) If n is even then the edge label $m_{i,j}$ can be rewritten to

$$m_{i,j} = \begin{cases} 0 & \text{if} \quad i \le \frac{n}{2}, j \le \frac{n}{2} \quad \text{and} \quad i > \frac{n}{2}, j > \frac{n}{2} \\ 1 & \text{if} \quad i \le \frac{n}{2}, j > \frac{n}{2} \quad \text{and} \quad i > \frac{n}{2}, j \le \frac{n}{2}. \end{cases}$$

and label the vertices of $K_{n,n}$ as $f(u_i) = f(w_i) = 1$, $i = 1, 2, \dots, \frac{n}{2}$, $f(u_i) = f(w_i) = 0$, $i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n$. The labelling is TMC labelling of $K_{n,n}$ with $C = 0$.

(a2) If n is odd, along with the vertex and edge labels of Case (a1), (i.e., $K_{n-1,n-1}$, $n - 1 = \mathit{even}$) add new vertices $u_n \in U$ and $w_n \in W$ with labels $f(v_n) = 0$ and $f(w_n) = 1$ and the new edges $\{u_n, w_i\}, \{u_i, w_n\}$, $i = 1, 2, \dots, n-1$ with

$$\begin{aligned}
&f(\{u_n, w_i\}) = 0, && i = 1, 2, \dots, \tfrac{n-1}{2},\\
&f(\{u_n, w_i\}) = 1, && i = \tfrac{n-1}{2} + 1, \tfrac{n-1}{2} + 2, \dots, n-1,\\
&f(\{u_i, w_n\}) = 1, && i = 1, 2, \dots, \tfrac{n-1}{2},\\
&f(\{u_i, w_n\}) = 0, && i = \tfrac{n-1}{2} + 1, \tfrac{n-1}{2} + 2, \dots, n-1.
\end{aligned}$$

Now it is not difficult to see that the labelling is TMC with $C = 0$.

Case b. $n > m$. Label the sub-complete bipartite graph $K_{m,m}$ using the labelling techniques of Case (a1) or (a2). Let f be the partial TMC labelling of $K_{m,n}$. Consider the set of unlabeled vertices $\bar{W} = \{w_{n-m+1}, w_{n-m+2}, \dots, w_n\}$ in $K_{m,n}$. Label $\lceil \frac{n-m}{2} \rceil$ of vertices in $\bar{W}$ with 1 and the other $\lfloor \frac{n-m}{2} \rfloor$ vertices in $\bar{W}$ with 0. Label the edges between $\bar{W}$ and U as follows:

For every $w_j \in \bar{W}, (n - m) \le j \le n$,

$$f(u_i, w_j) = \begin{cases} 0 & \text{if} \quad f(u_i) = 1, f(w_j) = 1 \quad \text{or} \quad f(u_i) = 0, f(w_j) = 0,\\ 1 & \text{if} \quad f(u_i) = 1, f(w_j) = 0 \quad \text{or} \quad f(u_i) = 0, f(w_j) = 1. \end{cases}$$

It can be verified that the labelling f results in $C = 0$ and $|f(0) - f(1)| \le 1$. Hence f is TMC labelling of $K_{m,n}$ when $m \neq n$. ■

Theorem 6. *All trees are TMC.*

Proof. For any n-vertex tree T_n, $|V(T_n) \cup E(T_n)| \equiv 1 \pmod 2$. Verify TMC labelling of small trees i.e., say, up to 5 vertices and assume that all n-vertex trees are TMC. Then for an TMC labelling f of an n-vertex tree T_n we have either

(a) $f(0) = f(1) + 1$ and $C = 0$ or

(b) $f(0) = f(1) + 1$ and $C = 1$ or

(c) $f(1) = f(0) + 1$ and $C = 0$ or

(d) $f(1) = f(0) + 1$ and $C = 1$.

For the general induction step we consider each cases separately: Let $v \in V(T_n)$ and $w \notin V(T_n)$. That is $w \in V(T_{n+1})$.

(a) If $f(v) = 1$ then put either $f(\{v,w\}) = 0$ and $f(w) = 1$ or put $f(\{v,w\}) = 1$ and $f(w) = 0$ to maintain $C = 0$ for the T_{n+1}. For T_{n+1} we have $f(0) = f(1) + 1$ again. If $f(v) = 0$ then put $f(\{v,w\}) = 1$ and $f(w) = 1$. For T_{n+1} we have $C = 0$ and $f(1) = f(0) + 1$ again.

(b) If $f(v) = 1$ then put $f(\{v,w\}) = 1$ and $f(w) = 1$ to maintain $C = 1$ with $f(1) = f(0)+1$ for T_{n+1}. If $f(v) = 0$ then put either $f(\{v,w\}) = 1$ and $f(w) = 0$ or $f(\{v,w\}) = 0$ and $f(w) = 1$. We have $C = 1$ and $f(0) = f(1)+1$ for T_{n+1}.

(c) If $f(v) = 1$ then put either $f(\{v,w\}) = 1$ and $f(w) = 0$ or $f(\{v,w\}) = 0$ and $f(w) = 1$ to maintain $C = 0$ and $f(1) = f(0) + 1$ for T_{n+1}. If $f(v) = 0$ then put $f(\{v,w\}) = 0$ and $f(w) = 0$. Then for T_{n+1} we have $C = 0$ and $f(0) = f(1) + 1$.

(d) If $f(v) = 1$ then put $f(\{v,w\}) = 0$ and $f(w) = 0$ to maintain $C = 1$ and $f(0) = f(1) + 1$ for T_{n+1}. If $f(v) = 0$ then put either $f(\{v,w\}) = 1$ and $f(w) = 0$ or $f(\{v,w\}) = 0$ and $f(w) = 1$. We have $C = 1$ and $f(1) = f(0)+1$ for T_{n+1}. ■

Definition A' can be generalized in the following way

Definition 7. A graph $G(m,n)$ is said to have a *totally* m_3*-cordial* (or k-TMC) *with constant* C if there exists a mapping $f : V(G) \cup E(G) \to \{0,1,2,\ldots,k-1\}$ such that $f(a) + f(b) + f(\{a,b\}) = C \pmod k$ for all $(a,b) \in E(G)$ provided for $i \neq j$ $|f(i) - f(j)| \leq 1$, $i,j \in \{0,1,2,\ldots,k-1\}$, where $f(x) = v_f(x) + e_f(x)$, $x = 0,1,2,\ldots,k-1$.

We give some basic results on 3-TMC labellings of trees. For any k k-TMC labellings appear to be quite difficult e.g., see for example [4], [11] for the analogy of the problem with k-equitable labellings.

Lemma 8. *The star S_n is 3-TMC with constant $C = 0$.*

Proof. Let S_n denote an star with $n+1$ vertices. Label the central vertex of S_n with 1.

(a) If $n \equiv 0 \pmod 3$ label the first $\lfloor \frac{2n}{3} \rfloor + 1$ edges with 0 and their end-vertices with 2 and label the other unlabeled edges and vertices with 1's. This results in $f(0) = f(2) = \lfloor \frac{2n}{3} \rfloor + 1$ and $f(1) = \lfloor \frac{2n}{3} \rfloor$.

(b) If $n \equiv 1 \pmod 3$ add to the star of Case (a) an extra edge labelled with 0 and label its end-vertex with 2. Then we have $f(0) = f(1) = f(2) = \frac{2n}{3}$.

(c) If $n \equiv 2 \pmod 3$ add to the star of Case (b) an extra edge labelled with 0 and label its end-vertex with 2. Then we have $f(0) = f(2) = \frac{2n}{3} + 2$ and $f(1) = \frac{2n}{3} + 1$. ∎

Lemma 9. *The path P_n is 3-TMC for all $n \geq 2$ with $C = 0$.*

Proof. Let us denote the path P_n by the set of alternating vertices and edges as $\{v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_{n-1}, e_{n-1}, v_n\}$. If $n \equiv 2 \pmod 3$ then label the vertices and edges of P_n with $0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots$. The labelling f satisfies $f(0) = f(1) = f(2)$ with the constant $C = 0$. Hence f is a 3-TMC labelling of P_n. We can obtain 3-TMC labelling of $P_n, n \equiv 1 \pmod 3), n \geq 4$ by deleting the last edge and vertex labels from the 3-TMC labelling of $P_n, n \equiv 2 \pmod 3$. That is the labelling of $P_n, n \equiv 1 \pmod 3$ would be $0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots, 0, 1, 2, 0$ and we have $f(0) = f(1) - 1 = f(2) - 1$ with $C = 0$. Similarly we can obtain 3-TMC of $P_n, n \equiv 0 \pmod 3), n \geq 3$ by adding an extra edge to the end-vertex (with label 2) to the 3-TMC labelling of $P_n, n \equiv 2 \pmod 3$. Then label the newly added edge and its end-vertex with 1 and 0. The resulting labelling is again a 3-TMC labelling of $P_n, n \equiv 0 \pmod 3$ that satisfies $f(0) = f(1) = f(2) - 1$ with $C = 0$. ∎

The statement of the next theorem which is exactly same as the magic labellings of complete graphs but it differs in the length of the proof e.g., see [7], [8].

Theorem 10. *The complete graph K_n is TMC iff $n \in \{2, 3, 5, 6\}$.*

Proof. Assume that f is TMC labelling of K_n and w.l.o.g. assume $C = 1$. That is for any edge $e = (u, v) \in E(K_n)$ we have either $f(\{u, v\}) = f(u) = f(v) = 1$ or $f(\{u, v\}) = f(u) = 0$ and $f(v) = 1$ or $f(\{u, v\}) = f(v) = 0$ and $f(u) = 1$ or $f(u) = f(v) = 0$ and $f(\{u, v\}) = 1$. That is under the labelling f the complete graph K_n can be decomposed as $K_n = K_p \cup K_r \cup K_{p,r}$ where K_p is the sub-complete graph those its vertices and edges are all labelled with 1's, K_p is the sub-complete graph whose its vertices labelled with 0's and but its edges labelled with 1's and $K_{p,r}$ is the complete bipartite subgraph of K_n with the bipartitions $V(K_p) \cup V(K_r)$ which its edges labelled with all 0's. For any total binary labelling f of K_n we can write

$$(*)\qquad f(0)+f(1)\equiv\begin{cases}0 \pmod 2 & \text{if} \quad n\equiv 0,3 \pmod 4\\ 1 \pmod 2 & \text{if} \quad n\equiv 1,2 \pmod 4\end{cases}$$

Case (i). $n\equiv 0,3 \pmod 4$, $n>2$.
That is for any TMC labelling we must have $f(0)=f(1)$. With the constant $C=1$ in TMC and using the above decomposition of K_n we write the following:

$$f(1)=p+\frac{p(p-1)}{2}+\frac{r(r-1)}{2} \quad \text{and} \quad f(0)=r+pr.$$

From $f(1)=f(0)$ we get the quadratic equation

$$\frac{1}{2}p^2-\left(r-\frac{1}{2}\right)p+\frac{r}{2}(r-3)=0.$$

Solving this quadratic for p we obtain

$$p_{1,2}=\left(r-\frac{1}{2}\right)\pm\frac{1}{2}\sqrt{2r^2-2r+1}.$$

The term $\sqrt{2r^2-2r+1}$ has an integer value only if $r=1$ and 4. But for $r=1$, we have $p=1$ which leads to $p+r=2\not\equiv 0,3 \pmod 4$ and for $r=4$ we have $p=1$ or $p=6$ for which respectively leads to $p+r=5,10\not\equiv 0,3 \pmod 4$. Thus for $n\equiv 0,3 \pmod 4$ there exists no TMC labelling of K_n.

Case (ii). $n\equiv 1,2 \pmod 4$.
In this case there are two possibilities i.e., $f(1)=f(0)+1$ or $f(0)=f(1)+1$. Again using exactly the same arguments as used in Case (i) we obtain

$$\text{(a)}\qquad p_{1,2}=\left(r-\frac{1}{2}\right)\pm\frac{1}{2}\sqrt{2r^2-2r+5} \quad \text{for} \ f(1)=f(0)+1 \ \text{and}$$

$$\text{(b)}\qquad p_{1,2}=\left(r-\frac{1}{2}\right)\pm\frac{1}{2}\sqrt{2r^2-2r-3} \quad \text{for} \ f(0)=f(1)+1.$$

In case (a) $\sqrt{2r^2-2r+5}$ has an integer value only for $r=2$ which leads to $n=p+r=3+2=5$ and in case (b) $\sqrt{2r^2-2r-3}$ has an integer value only for $r=2,3$ which leads to $n=p+r=1,2,6$. For $n=2,3$ label the vertices of K_2 and K_3 with 0's and the edges with 1's. For the other feasible values of n, i.e., 5 and 6 use the labelling described in the decomposition. ■

3. Totally Sequential Cordial Graphs

The following simple theorem will show that cordial labellings [3] is stronger than TSC labellings. Cordial labelling f of G is a mapping $f : V(G) \to \{0,1\}$ so that when the induced edge labels computed by $f(\{u,v\}) = |f(u) - f(v)|$, for all $\{u,v\} \in E(G)$ the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ are satisfied.

Theorem 11. *Every cordial graph is TSC.*

Proof. Let G be a cordial graph with n vertices and m edges. Let f be a cordial labelling of G. If either m or n or both are even then the cordial labelling f is necessarily also TSC since either $v_f(0) = v_f(1)$ or $e_f(0) = e_f(1)$ or both are hold. Now assume w.l.o.g. that m, n are odd and for the cordial labelling f we have

$$v_f(1) = v_f(0) + 1, e_f(1) = e_f(0) + 1 \Longrightarrow f(1) - f(0) = 2$$

where $f(0) = v_f(0) + e_f(0)$ and $f(1) = v_f(1) + e_f(1)$. Since under the complementary cordial labelling $\bar{f}$, i.e., $\bar{f}(v) = 1 - f(v)$ for all $v \in V(G)$ of G the edge labels are invariant we may take $\bar{f}$ so that

$$v_{\bar{f}}(0) - v_{\bar{f}}(1) + e_{\bar{f}}(0) - e_{\bar{f}}(1) = f(0) - f(1) = 0$$

is satisfied. Hence cordial labelling $\bar{f}$ of G is also TSC labelling. ■

Theorem 12. *The complete graph K_n is TSC iff*
(a) $\sqrt{k}$ *has an integer value for* $n = 4k+1, k \leq 1$,
(b) $\sqrt{4k+1}$ *has an integer value for* $n = 4k+2$,
(c) $\sqrt{4k+1}$ *has an integer value for* $n = 4k$,
(d) $\sqrt{k+1}$ *has an integer value for* $n = 4k+3$.

Proof. From expression $(*)$ in Theorem 10 we have $f(0) + f(1) = \frac{n(n+1)}{2}$ which is even if $n \equiv 0, 3 \pmod 4$ and is odd if $n \equiv 1, 2 \pmod 4$. Again we consider two cases:

Case (i). $f(0) + f(1) \equiv 0 \pmod 2$.
If f is TSC labelling of K_n then $f(0) = f(1)$. Let K_p be the labelled sub-complete graph of K_n whose vertices labelled with 1 and K_r be the labelled sub-complete graph of K_n whose vertices labelled with 0, where $n = p + r$. Clearly for the labelling f we write

$$f(1) = rp + p$$

and

$$f(0) = \frac{r(r-1)}{2} + r + \frac{p(p-1)}{2}$$

and using $f(1) = f(0)$

$$(r-p)^2 - 3p + r = 0.$$

Putting $p = n - r$ in the above expression we get

$$4r^2 - 4(n-1)r + n^2 - 3n = 0.$$

Solving this for r we obtain

$$r_{1,2} = \frac{(n-1) \pm \sqrt{n+1}}{2}.$$

Since $r_{1,2}$ will represent the order of sub-complete graph K_r, it can easily be seen that for $n = 4k, k \geq 1$, K_n is TSC only if $\sqrt{4k+1}$ is an integer and for $n = 4k+3, k \geq 0$, K_n is TSC only if $\sqrt{k+1}$ is an integer.

Case (ii). $f(0) + f(1) \equiv 1 \pmod 2$.
That is $n \equiv 1, 2 \pmod 4$. If f is a TSC labelling of K_n we may assume w.l.o.g. that $f(1) > f(0)$ i.e., $f(1) = f(0) + 1$ and same decomposition of Case (i). Therefore we write, by using

$$f(1) = pr + p, \;\; f(0) = \frac{p(p-1)}{2} + r + \frac{r(r-1)}{2}$$

and

$$p = n - r$$

the following quadratic equation

$$4r^2 - 4(n-1)r + n^2 - 3n + 2 = 0$$

is obtained which has the roots

$$r_{1,2} = \frac{(n-1) \pm \sqrt{n-1}}{2}.$$

Again, in order to have integer values for $r_{1,2}$ for $n = 4k+1$ and $4k+2$ respectively $\sqrt{k}$ and $\sqrt{4k+1}$ must be integer. ■

Theorem 13. *The cycle C_n is TSC for all $n > 2$.*

Proof. If $n \equiv 0 \pmod 4$ then label the vertices of C_n with $11001100\ldots00$. If $n \equiv 1 \pmod 4$ then label the vertices of C_n with $111001100\ldots00$. If $n \equiv 2 \pmod 4$ then label the vertices of C_n with $1111001100\ldots00$. Finally if $n \equiv 3 \pmod 4$ then label the vertices of C_n with $10011001100\ldots00$. All these vertex binary-labellings are TSC since $f(0) = f(1) = n$ holds for any $n > 2$, where $f(0) = v_f(0) + e_f(0)$ and $f(1) = v_f(1) + e_f(1)$. ■

Theorem 14. *Trees are TSC.*

Proof. It is well known that trees are cordial cf. [3]. That is there exists binary vertex labelling f satisfying $|e_f(0)-e_f(1)| \leq 1$ and $|v_f(0)-v_f(1)| \leq 1$, where induce edge labels computed by $f(e) = |f(u) - f(v)|$, for all $e = \{u, v\} \in E$. Let f be a cordial labelling of an n-vertex tree T_n. We have two cases to consider:

(i) $n \equiv 0 \pmod 2 \Rightarrow m \equiv 1 \pmod 2 \Rightarrow n+m \equiv 1 \pmod 2$. That is $|e_f(0) - e_f(1)| = 1$ and $v_f(0) = v_f(1)$.

Assume w.l.o.g. that $e_f(1) > e_f(0)$. Clearly we have $e_f(1) + v_f(1) - e_f(0) - v_f(0) = 1$.

Hence $f(1) - f(0) = 1$ and cordial labelling f of T_n is also TSC labelling.

(ii) $n \equiv 1 \pmod 2 \Rightarrow m \equiv 0 \pmod 2 \Rightarrow n+m \equiv 1 \pmod 2$. Then for any f cordial labelling of T_n we have $e_f(0) = e_f(1)$ and if we assume $v_f(0) = v_f(1) + 1$ then $f(0) - f(1) = 1$ and if we assume $v_f(1) = v_f(0) + 1$ then $f(1) - f(0) = 1$. That is cordial labelling f is also TSC labelling. ■

Theorem 15. *The wheel W_n is TSC for all $n > 3$.*

Proof. Let W_n denote the wheel with $n+1$ vertices.

(a) $n \equiv 0 \pmod 4$. Consider the TSC labellings of n-vertex cycle given in Theorem 12, e.g., $001100\ldots11$. Label the central vertex of W_n by 0.

Then, together with the induce edge labels of W_n, it can easily be verified that

$$f(0) = e_f(0) + v_f(0) = n + \frac{n}{2} + 1 = \frac{3n}{2} + 1$$

and

$$f(1) = e_f(1) + v_f(1) = n + \frac{n}{2} = \frac{3n}{2}.$$

(b) If $n \equiv 1 \pmod 4$ then insert a new vertex with label 1 between any edge in C_n of Case (a) with label 1. Then $f(0) = f(1)$.

(c) If $n \equiv 2 \pmod 4$ then use the TSC labelling of C_n and label the central vertex with 0. Then $f(1) = f(0) + 1$.

(d) Let $n \equiv 3 \pmod 4$. Since $W_3 = K_4$ (see Theorem 5) is not TSC, assume $n > 3$. Consider again TSC labelling f of C_n ($n \equiv 3 \pmod 4$) i.e., $100110011\ldots00$. Label this time the central vertex of W_n with 1. Then it can easily be verified that f satisfies $f(0) = f(1) = \frac{3n+1}{2}$. Hence W_n is TSC for all $n > 3$. ∎

Note that a conjecture asserts that the wheel W_n is simply sequential iff $4 \not| n$ [1].

Theorem 16. *The complete bipartite graph $K_{m,n}$ is TSC for all $m, n \geq 1$.*

Proof. Use cordial labellings of $K_{m,n}$, (cf. Theorem 3, [3]). ∎

The friendship graph $F_n, (n \geq 1)$ consists of n triangles with a common vertex.

Theorem 17. *The friendship graph F_n is TSC for all $n \geq 1$.*

Proof. Label $\lceil \frac{n}{2} \rceil$ triangles with $(0, 0, 1)$ and $\lfloor \frac{n}{2} \rfloor$ triangles with $(0, 1, 1)$ where the central vertex of F_n is labelled by 1. Then we have

$$f(0) = \begin{cases} f(1) & \text{if} \quad n \equiv 0, 1 \pmod 3, \\ f(1) + 1 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

TSC labelling of cubic graphs i.e., regular graphs of degree 3 is appear to be interesting since K_4 (Theorem 12, Case (c)) is not TSC, but we assert the following:

Conjecture 18. Cubic graphs other than K_4 are TSC.

Final remark is that, not all Eulerian graphs are TSC as K_7 shows by Theorem 12. On the other hand unlike cf., Theorem 4 [3] that Eulerian graphs with $e \equiv 2 \pmod 4$ edges are not cordial but the Eulerian graph with 14 edges shown in Figure 1 is TSC with $f(0) = f(1) = 11$.

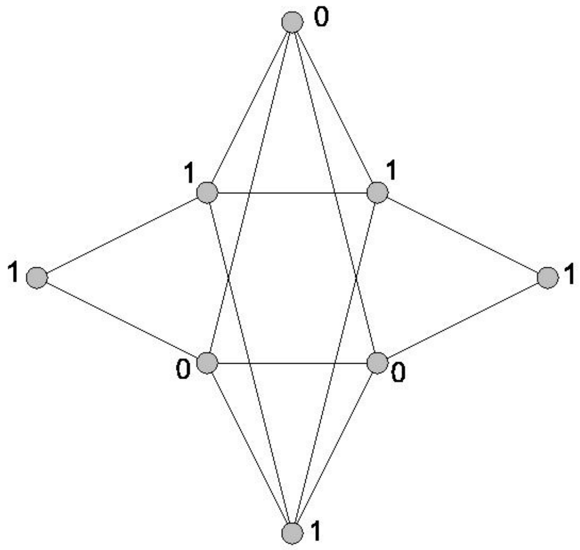

Figure 1

Acknowledgement

The author would like to thank the referee for his comments and suggestions on the paper.

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Received 9 March 2001
Revised 5 March 2002