

## SOME TOTALLY MODULAR CORDIAL GRAPHS

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### Abstract

In this paper we define total magic cordial (TMC) and total sequential cordial (TSC) labellings which are weaker versions of magic and simply sequential labellings of graphs. Based on these definitions we have given several results on TMC and TSC graphs.

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## 1. Introduction

After Cahit [3] the meaning of *cordiality* in the graph labelling problems is well understood and studied [4], [5], [11]. Let  $G(m, n)$  (or simply  $G$ ) be a graph with  $m = |E(G)|$  edges and  $n = |V(G)|$  vertices. A binary vertex labelling  $f : V \rightarrow \{0, 1\}$  induces an edge labelling  $\bar{f} : E \rightarrow \{0, 1\}$  defined by  $\bar{f}(\{u, v\}) = |f(u) - f(v)|$ . We call such a labelling *cordial* if the conditions  $|v_f(1) - v_f(0)| \leq 1$  and  $|e_f(1) - e_f(0)| \leq 1$  are satisfied, where  $v_f(i)$  and  $e_f(i)$ ,  $i = 0, 1$ , is the number of vertices and edges of  $G$ , respectively, with label  $i$ . A graph is called cordial if it admits a cordial labelling.

In this work we have investigated another two well-known graph labellings on the ground of cordial labellings. These are the *magic graphs* of Kotzig and Rosa [6] – [9] and *simply sequential graphs* of Bange, Barkauskas and Slater [1], [2], [10]. Note that many problems in these areas are still open [5].

**Definition 1 (A).** A graph  $G(m, n)$  is said to have a *magic labelling* with constant  $C$  if there exists a 1:1 mapping  $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, m+n\}$  such that  $f(a) + f(b) + f(\{a, b\}) = C$  for all  $\{a, b\} \in E(G)$ .

**Definition 2 (B).** A graph  $G$  with  $|V(G) \cup E(G)| = m+n$  is called *simply sequential* if there is a bijection  $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, m+n\}$  such that for each edge  $e = \{ab\} \in E(G)$  one has  $f(e) = |f(a) - f(b)|$ .

We modify these definitions as follows:

**Definition 3 (A').** A graph  $G(m, n)$  is said to have a *totally magic cordial* (TMC) *labelling with constant  $C$*  if there exists a mapping  $f : V(G) \cup E(G) \rightarrow \{0, 1\}$  such that  $f(a) + f(b) + f(\{a, b\}) = C \pmod{2}$  for all  $\{a, b\} \in E(G)$  provided the condition  $|f(0) - f(1)| \leq 1$  is hold, where  $f(0) = v_f(0) + e_f(0)$  and  $f(1) = v_f(1) + e_f(1)$  and  $v_f(i), e_f(i), i \in \{0, 1\}$  are, respectively, the number of vertices and edges labelled with  $i$ .

**Definition 4 (B').** A graph  $G$  with  $|V(G) \cup E(G)| = m+n$  is called *total sequential cordial* (TSC) if there is a total mapping  $f : V(G) \cup E(G) \rightarrow \{0, 1\}$  such that for each edge  $e = \{a, b\}$ ,  $f(e) = |f(a) - f(b)|$  and the condition  $|f(0) - f(1)| \leq 1$  holds.

## 2. Totally Magic Cordial Graphs

**Theorem 5.** *The complete bipartite graph  $K_{m,n}$  is TMC for all  $m, n > 1$ .*

**Proof.** Let us denote vertex bipartition of  $K_{m,n}$  as  $V = U \cup W$ ,  $U = \{u_1, u_2, \dots, u_m\}$  and  $W = \{w_1, w_2, \dots, w_n\}$ .

*Case a.  $m = n$ .* The proof of the theorem is by construction of a special  $(0,1)$ -matrix  $\mathbf{M} = [m_{i,j}]_{n \times n}$ , where  $(i, j)$  entry of  $\mathbf{M}$  corresponds to the edge label  $m_{i,j} = f(u_i, w_j)$ .

(a1) If  $n$  is even then the edge label  $m_{i,j}$  can be rewritten to

$$m_{i,j} = \begin{cases} 0 & \text{if } i \leq \frac{n}{2}, j \leq \frac{n}{2} \text{ and } i > \frac{n}{2}, j > \frac{n}{2} \\ 1 & \text{if } i \leq \frac{n}{2}, j > \frac{n}{2} \text{ and } i > \frac{n}{2}, j \leq \frac{n}{2}. \end{cases}$$

and label the vertices of  $K_{n,n}$  as  $f(u_i) = f(w_i) = 1, i = 1, 2, \dots, \frac{n}{2}, f(u_i) = f(w_i) = 0, i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n$ . The labelling is TMC labelling of  $K_{n,n}$  with  $C = 0$ .

(a2) If  $n$  is odd, along with the vertex and edge labels of Case (a1), (i.e.,  $K_{n-1,n-1}, n-1 = \text{even}$ ) add new vertices  $u_n \in U$  and  $w_n \in W$  with labels  $f(u_n) = 0$  and  $f(w_n) = 1$  and the new edges  $\{u_n, w_i\}, \{u_i, w_n\}, i = 1, 2, \dots, n-1$  with

$$\begin{aligned} f(\{u_n, w_i\}) &= 0, & i &= 1, 2, \dots, \frac{n-1}{2}, \\ f(\{u_n, w_i\}) &= 1, & i &= \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \dots, n-1, \\ f(\{u_i, w_n\}) &= 1, & i &= 1, 2, \dots, \frac{n-1}{2}, \\ f(\{u_i, w_n\}) &= 0, & i &= \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \dots, n-1. \end{aligned}$$

Now it is not difficult to see that the labelling is TMC with  $C = 0$ .

*Case b.*  $n > m$ . Label the sub-complete bipartite graph  $K_{m,m}$  using the labelling techniques of Case (a1) or (a2). Let  $f$  be the partial TMC labelling of  $K_{m,n}$ . Consider the set of unlabeled vertices  $\bar{W} = \{w_{n-m+1}, w_{n-m+2}, \dots, w_n\}$  in  $K_{m,n}$ . Label  $\lceil \frac{n-m}{2} \rceil$  of vertices in  $\bar{W}$  with 1 and the other  $\lfloor \frac{n-m}{2} \rfloor$  vertices in  $\bar{W}$  with 0. Label the edges between  $\bar{W}$  and  $U$  as follows:

For every  $w_j \in \bar{W}, (n-m) \leq j \leq n$ ,

$$f(u_i, w_j) = \begin{cases} 0 & \text{if } f(u_i) = 1, f(w_j) = 1 \text{ or } f(u_i) = 0, f(w_j) = 0, \\ 1 & \text{if } f(u_i) = 1, f(w_j) = 0 \text{ or } f(u_i) = 0, f(w_j) = 1. \end{cases}$$

It can be verified that the labelling  $f$  results in  $C = 0$  and  $|f(0) - f(1)| \leq 1$ . Hence  $f$  is TMC labelling of  $K_{m,n}$  when  $m \neq n$ . ■

**Theorem 6.** *All trees are TMC.*

**Proof.** For any  $n$ -vertex tree  $T_n$ ,  $|V(T_n) \cup E(T_n)| \equiv 1 \pmod{2}$ . Verify TMC labelling of small trees i.e., say, up to 5 vertices and assume that all  $n$ -vertex trees are TMC. Then for an TMC labelling  $f$  of an  $n$ -vertex tree  $T_n$  we have either

- (a)  $f(0) = f(1) + 1$  and  $C = 0$  or
- (b)  $f(0) = f(1) + 1$  and  $C = 1$  or
- (c)  $f(1) = f(0) + 1$  and  $C = 0$  or
- (d)  $f(1) = f(0) + 1$  and  $C = 1$ .

For the general induction step we consider each cases separately: Let  $v \in V(T_n)$  and  $w \notin V(T_n)$ . That is  $w \in V(T_{n+1})$ .

(a) If  $f(v) = 1$  then put either  $f(\{v, w\}) = 0$  and  $f(w) = 1$  or put  $f(\{v, w\}) = 1$  and  $f(w) = 0$  to maintain  $C = 0$  for the  $T_{n+1}$ . For  $T_{n+1}$  we have  $f(0) = f(1) + 1$  again. If  $f(v) = 0$  then put  $f(\{v, w\}) = 1$  and  $f(w) = 1$ . For  $T_{n+1}$  we have  $C = 0$  and  $f(1) = f(0) + 1$  again.

(b) If  $f(v) = 1$  then put  $f(\{v, w\}) = 1$  and  $f(w) = 1$  to maintain  $C = 1$  with  $f(1) = f(0) + 1$  for  $T_{n+1}$ . If  $f(v) = 0$  then put either  $f(\{v, w\}) = 1$  and  $f(w) = 0$  or  $f(\{v, w\}) = 0$  and  $f(w) = 1$ . We have  $C = 1$  and  $f(0) = f(1) + 1$  for  $T_{n+1}$ .

(c) If  $f(v) = 1$  then put either  $f(\{v, w\}) = 1$  and  $f(w) = 0$  or  $f(\{v, w\}) = 0$  and  $f(w) = 1$  to maintain  $C = 0$  and  $f(1) = f(0) + 1$  for  $T_{n+1}$ . If  $f(v) = 0$  then put  $f(\{v, w\}) = 0$  and  $f(w) = 0$ . Then for  $T_{n+1}$  we have  $C = 0$  and  $f(0) = f(1) + 1$ .

(d) If  $f(v) = 1$  then put  $f(\{v, w\}) = 0$  and  $f(w) = 0$  to maintain  $C = 1$  and  $f(0) = f(1) + 1$  for  $T_{n+1}$ . If  $f(v) = 0$  then put either  $f(\{v, w\}) = 1$  and  $f(w) = 0$  or  $f(\{v, w\}) = 0$  and  $f(w) = 1$ . We have  $C = 1$  and  $f(1) = f(0) + 1$  for  $T_{n+1}$ . ■

Definition  $A'$  can be generalized in the following way

**Definition 7.** A graph  $G(m, n)$  is said to have a *totally  $m_3$ -cordial* (or  *$k$ -TMC*) *with constant  $C$*  if there exists a mapping  $f : V(G) \cup E(G) \rightarrow \{0, 1, 2, \dots, k-1\}$  such that  $f(a) + f(b) + f(\{a, b\}) = C \pmod{k}$  for all  $(a, b) \in E(G)$  provided for  $i \neq j$   $|f(i) - f(j)| \leq 1$ ,  $i, j \in \{0, 1, 2, \dots, k-1\}$ , where  $f(x) = v_f(x) + e_f(x)$ ,  $x = 0, 1, 2, \dots, k-1$ .

We give some basic results on 3-TMC labellings of trees. For any  $k$   $k$ -TMC labellings appear to be quite difficult e.g., see for example [4], [11] for the analogy of the problem with  $k$ -equitable labellings.

**Lemma 8.** *The star  $S_n$  is 3-TMC with constant  $C = 0$ .*

**Proof.** Let  $S_n$  denote an star with  $n + 1$  vertices. Label the central vertex of  $S_n$  with 1.

(a) If  $n \equiv 0 \pmod{3}$  label the first  $\lfloor \frac{2n}{3} \rfloor + 1$  edges with 0 and their end-vertices with 2 and label the other unlabeled edges and vertices with 1's. This results in  $f(0) = f(2) = \lfloor \frac{2n}{3} \rfloor + 1$  and  $f(1) = \lfloor \frac{2n}{3} \rfloor$ .

(b) If  $n \equiv 1 \pmod{3}$  add to the star of Case (a) an extra edge labelled with 0 and label its end-vertex with 2. Then we have  $f(0) = f(1) = f(2) = \frac{2n}{3}$ .

(c) If  $n \equiv 2 \pmod{3}$  add to the star of Case (b) an extra edge labelled with 0 and label its end-vertex with 2. Then we have  $f(0) = f(2) = \frac{2n}{3} + 2$  and  $f(1) = \frac{2n}{3} + 1$ . ■

**Lemma 9.** *The path  $P_n$  is 3-TMC for all  $n \geq 2$  with  $C = 0$ .*

**Proof.** Let us denote the path  $P_n$  by the set of alternating vertices and edges as  $\{v_1, e_1, v_2, e_2, v_3, e_3, \dots, v_{n-1}, e_{n-1}, v_n\}$ . If  $n \equiv 2 \pmod{3}$  then label the vertices and edges of  $P_n$  with  $0, 1, 2, 0, 1, 2, 0, 1, 2, \dots$ . The labelling  $f$  satisfies  $f(0) = f(1) = f(2)$  with the constant  $C = 0$ . Hence  $f$  is a 3-TMC labelling of  $P_n$ . We can obtain 3-TMC labelling of  $P_n, n \equiv 1 \pmod{3}, n \geq 4$  by deleting the last edge and vertex labels from the 3-TMC labelling of  $P_n, n \equiv 2 \pmod{3}$ . That is the labelling of  $P_n, n \equiv 1 \pmod{3}$  would be  $0, 1, 2, 0, 1, 2, 0, 1, 2, \dots, 0, 1, 2, 0$  and we have  $f(0) = f(1) - 1 = f(2) - 1$  with  $C = 0$ . Similarly we can obtain 3-TMC of  $P_n, n \equiv 0 \pmod{3}, n \geq 3$  by adding an extra edge to the end-vertex (with label 2) to the 3-TMC labelling of  $P_n, n \equiv 2 \pmod{3}$ . Then label the newly added edge and its end-vertex with 1 and 0. The resulting labelling is again a 3-TMC labelling of  $P_n, n \equiv 0 \pmod{3}$  that satisfies  $f(0) = f(1) = f(2) - 1$  with  $C = 0$ . ■

The statement of the next theorem which is exactly same as the magic labellings of complete graphs but it differs in the length of the proof e.g., see [7], [8].

**Theorem 10.** *The complete graph  $K_n$  is TMC iff  $n \in \{2, 3, 5, 6\}$ .*

**Proof.** Assume that  $f$  is TMC labelling of  $K_n$  and w.l.o.g. assume  $C = 1$ . That is for any edge  $e = (u, v) \in E(K_n)$  we have either  $f(\{u, v\}) = f(u) = f(v) = 1$  or  $f(\{u, v\}) = f(u) = 0$  and  $f(v) = 1$  or  $f(\{u, v\}) = f(v) = 0$  and  $f(u) = 1$  or  $f(u) = f(v) = 0$  and  $f(\{u, v\}) = 1$ . That is under the labelling  $f$  the complete graph  $K_n$  can be decomposed as  $K_n = K_p \cup K_r \cup K_{p,r}$  where  $K_p$  is the sub-complete graph whose vertices and edges are all labelled with 1's,  $K_p$  is the sub-complete graph whose vertices labelled with 0's and but its edges labelled with 1's and  $K_{p,r}$  is the complete bipartite subgraph of  $K_n$  with the bipartitions  $V(K_p) \cup V(K_r)$  which its edges labelled with all 0's. For any total binary labelling  $f$  of  $K_n$  we can write

$$(*) \quad f(0) + f(1) \equiv \begin{cases} 0 \pmod{2} & \text{if } n \equiv 0, 3 \pmod{4} \\ 1 \pmod{2} & \text{if } n \equiv 1, 2 \pmod{4} \end{cases}$$

*Case (i).*  $n \equiv 0, 3 \pmod{4}$ ,  $n > 2$ .

That is for any TMC labelling we must have  $f(0) = f(1)$ . With the constant  $C = 1$  in TMC and using the above decomposition of  $K_n$  we write the following:

$$f(1) = p + \frac{p(p-1)}{2} + \frac{r(r-1)}{2} \quad \text{and} \quad f(0) = r + pr.$$

From  $f(1) = f(0)$  we get the quadratic equation

$$\frac{1}{2}p^2 - \left(r - \frac{1}{2}\right)p + \frac{r}{2}(r-3) = 0.$$

Solving this quadratic for  $p$  we obtain

$$p_{1,2} = \left(r - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{2r^2 - 2r + 1}.$$

The term  $\sqrt{2r^2 - 2r + 1}$  has an integer value only if  $r = 1$  and  $4$ . But for  $r = 1$ , we have  $p = 1$  which leads to  $p + r = 2 \not\equiv 0, 3 \pmod{4}$  and for  $r = 4$  we have  $p = 1$  or  $p = 6$  for which respectively leads to  $p + r = 5, 10 \not\equiv 0, 3 \pmod{4}$ . Thus for  $n \equiv 0, 3 \pmod{4}$  there exists no TMC labelling of  $K_n$ .

*Case (ii).*  $n \equiv 1, 2 \pmod{4}$ .

In this case there are two possibilities i.e.,  $f(1) = f(0) + 1$  or  $f(0) = f(1) + 1$ . Again using exactly the same arguments as used in Case (i) we obtain

$$(a) \quad p_{1,2} = \left(r - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{2r^2 - 2r + 5} \quad \text{for } f(1) = f(0) + 1 \quad \text{and}$$

$$(b) \quad p_{1,2} = \left(r - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{2r^2 - 2r - 3} \quad \text{for } f(0) = f(1) + 1.$$

In case (a)  $\sqrt{2r^2 - 2r + 5}$  has an integer value only for  $r = 2$  which leads to  $n = p + r = 3 + 2 = 5$  and in case (b)  $\sqrt{2r^2 - 2r - 3}$  has an integer value only for  $r = 2, 3$  which leads to  $n = p + r = 1, 2, 6$ . For  $n = 2, 3$  label the

vertices of  $K_2$  and  $K_3$  with 0's and the edges with 1's. For the other feasible values of  $n$ , i.e., 5 and 6 use the labelling described in the decomposition. ■

### 3. Totally Sequential Cordial Graphs

The following simple theorem will show that cordial labellings [3] is stronger than TSC labellings. Cordial labelling  $f$  of  $G$  is a mapping  $f : V(G) \rightarrow \{0, 1\}$  so that when the induced edge labels computed by  $f(\{u, v\}) = |f(u) - f(v)|$ , for all  $\{u, v\} \in E(G)$  the conditions  $|v_f(0) - v_f(1)| \leq 1$  and  $|e_f(0) - e_f(1)| \leq 1$  are satisfied.

**Theorem 11.** *Every cordial graph is TSC.*

**Proof.** Let  $G$  be a cordial graph with  $n$  vertices and  $m$  edges. Let  $f$  be a cordial labelling of  $G$ . If either  $m$  or  $n$  or both are even then the cordial labelling  $f$  is necessarily also TSC since either  $v_f(0) = v_f(1)$  or  $e_f(0) = e_f(1)$  or both are hold. Now assume w.l.o.g. that  $m, n$  are odd and for the cordial labelling  $f$  we have

$$v_f(1) = v_f(0) + 1, e_f(1) = e_f(0) + 1 \implies f(1) - f(0) = 2$$

where  $f(0) = v_f(0) + e_f(0)$  and  $f(1) = v_f(1) + e_f(1)$ . Since under the complementary cordial labelling  $\bar{f}$ , i.e.,  $\bar{f}(v) = 1 - f(v)$  for all  $v \in V(G)$  of  $G$  the edge labels are invariant we may take  $\bar{f}$  so that

$$v_{\bar{f}}(0) - v_{\bar{f}}(1) + e_{\bar{f}}(0) - e_{\bar{f}}(1) = f(0) - f(1) = 0$$

is satisfied. Hence cordial labelling  $\bar{f}$  of  $G$  is also TSC labelling. ■

**Theorem 12.** *The complete graph  $K_n$  is TSC iff*

- (a)  $\sqrt{k}$  has an integer value for  $n = 4k + 1, k \leq 1$ ,
- (b)  $\sqrt{4k + 1}$  has an integer value for  $n = 4k + 2$ ,
- (c)  $\sqrt{4k + 1}$  has an integer value for  $n = 4k$ ,
- (d)  $\sqrt{k + 1}$  has an integer value for  $n = 4k + 3$ .

**Proof.** From expression (\*) in Theorem 10 we have  $f(0) + f(1) = \frac{n(n+1)}{2}$  which is even if  $n \equiv 0, 3 \pmod{4}$  and is odd if  $n \equiv 1, 2 \pmod{4}$ . Again we consider two cases:

Case (i).  $f(0) + f(1) \equiv 0 \pmod{2}$ .

If  $f$  is TSC labelling of  $K_n$  then  $f(0) = f(1)$ . Let  $K_p$  be the labelled sub-complete graph of  $K_n$  whose vertices labelled with 1 and  $K_r$  be the labelled sub-complete graph of  $K_n$  whose vertices labelled with 0, where  $n = p + r$ . Clearly for the labelling  $f$  we write

$$f(1) = rp + p$$

and

$$f(0) = \frac{r(r-1)}{2} + r + \frac{p(p-1)}{2}$$

and using  $f(1) = f(0)$

$$(r-p)^2 - 3p + r = 0.$$

Putting  $p = n - r$  in the above expression we get

$$4r^2 - 4(n-1)r + n^2 - 3n = 0.$$

Solving this for  $r$  we obtain

$$r_{1,2} = \frac{(n-1) \pm \sqrt{n+1}}{2}.$$

Since  $r_{1,2}$  will represent the order of sub-complete graph  $K_r$ , it can easily be seen that for  $n = 4k, k \geq 1$ ,  $K_n$  is TSC only if  $\sqrt{4k+1}$  is an integer and for  $n = 4k+3, k \geq 0$ ,  $K_n$  is TSC only if  $\sqrt{k+1}$  is an integer.

Case (ii).  $f(0) + f(1) \equiv 1 \pmod{2}$ .

That is  $n \equiv 1, 2 \pmod{4}$ . If  $f$  is a TSC labelling of  $K_n$  we may assume w.l.o.g. that  $f(1) > f(0)$  i.e.,  $f(1) = f(0) + 1$  and same decomposition of Case (i). Therefore we write, by using

$$f(1) = pr + p, \quad f(0) = \frac{p(p-1)}{2} + r + \frac{r(r-1)}{2}$$

and

$$p = n - r$$

the following quadratic equation

$$4r^2 - 4(n-1)r + n^2 - 3n + 2 = 0$$



is obtained which has the roots

$$r_{1,2} = \frac{(n-1) \pm \sqrt{n-1}}{2}.$$

Again, in order to have integer values for  $r_{1,2}$  for  $n = 4k + 1$  and  $4k + 2$  respectively  $\sqrt{k}$  and  $\sqrt{4k+1}$  must be integer. ■

**Theorem 13.** *The cycle  $C_n$  is TSC for all  $n > 2$ .*

**Proof.** If  $n \equiv 0 \pmod{4}$  then label the vertices of  $C_n$  with 11001100...00. If  $n \equiv 1 \pmod{4}$  then label the vertices of  $C_n$  with 111001100...00. If  $n \equiv 2 \pmod{4}$  then label the vertices of  $C_n$  with 1111001100...00. Finally if  $n \equiv 3 \pmod{4}$  then label the vertices of  $C_n$  with 10011001100...00. All these vertex binary-labellings are TSC since  $f(0) = f(1) = n$  holds for any  $n > 2$ , where  $f(0) = v_f(0) + e_f(0)$  and  $f(1) = v_f(1) + e_f(1)$ . ■

**Theorem 14.** *Trees are TSC.*

**Proof.** It is well known that trees are cordial cf. [3]. That is there exists binary vertex labelling  $f$  satisfying  $|e_f(0) - e_f(1)| \leq 1$  and  $|v_f(0) - v_f(1)| \leq 1$ , where induce edge labels computed by  $f(e) = |f(u) - f(v)|$ , for all  $e = \{u, v\} \in E$ . Let  $f$  be a cordial labelling of an  $n$ -vertex tree  $T_n$ . We have two cases to consider:

(i)  $n \equiv 0 \pmod{2} \Rightarrow m \equiv 1 \pmod{2} \Rightarrow n + m \equiv 1 \pmod{2}$ . That is  $|e_f(0) - e_f(1)| = 1$  and  $v_f(0) = v_f(1)$ .

Assume w.l.o.g. that  $e_f(1) > e_f(0)$ . Clearly we have  $e_f(1) + v_f(1) - e_f(0) - v_f(0) = 1$ .

Hence  $f(1) - f(0) = 1$  and cordial labelling  $f$  of  $T_n$  is also TSC labelling.

(ii)  $n \equiv 1 \pmod{2} \Rightarrow m \equiv 0 \pmod{2} \Rightarrow n + m \equiv 1 \pmod{2}$ . Then for any  $f$  cordial labelling of  $T_n$  we have  $e_f(0) = e_f(1)$  and if we assume  $v_f(0) = v_f(1) + 1$  then  $f(0) - f(1) = 1$  and if we assume  $v_f(1) = v_f(0) + 1$  then  $f(1) - f(0) = 1$ . That is cordial labelling  $f$  is also TSC labelling. ■

**Theorem 15.** *The wheel  $W_n$  is TSC for all  $n > 3$ .*

**Proof.** Let  $W_n$  denote the wheel with  $n + 1$  vertices.

(a)  $n \equiv 0 \pmod{4}$ . Consider the TSC labellings of  $n$ -vertex cycle given in Theorem 12, e.g., 001100...11. Label the central vertex of  $W_n$  by 0.

Then, together with the induce edge labels of  $W_n$ , it can easily be verified that

$$f(0) = e_f(0) + v_f(0) = n + \frac{n}{2} + 1 = \frac{3n}{2} + 1$$

and

$$f(1) = e_f(1) + v_f(1) = n + \frac{n}{2} = \frac{3n}{2}.$$

(b) If  $n \equiv 1 \pmod{4}$  then insert a new vertex with label 1 between any edge in  $C_n$  of Case (a) with label 1. Then  $f(0) = f(1)$ .

(c) If  $n \equiv 2 \pmod{4}$  then use the TSC labelling of  $C_n$  and label the central vertex with 0. Then  $f(1) = f(0) + 1$ .

(d) Let  $n \equiv 3 \pmod{4}$ . Since  $W_3 = K_4$  (see Theorem 5) is not TSC, assume  $n > 3$ . Consider again TSC labelling  $f$  of  $C_n$  ( $n \equiv 3 \pmod{4}$ ) i.e., 100110011...00. Label this time the central vertex of  $W_n$  with 1. Then it can easily be verified that  $f$  satisfies  $f(0) = f(1) = \frac{3n+1}{2}$ . Hence  $W_n$  is TSC for all  $n > 3$ . ■

Note that a conjecture asserts that the wheel  $W_n$  is simply sequential iff  $4 \nmid n$  [1].

**Theorem 16.** *The complete bipartite graph  $K_{m,n}$  is TSC for all  $m, n \geq 1$ .*

**Proof.** Use cordial labellings of  $K_{m,n}$ , (cf. Theorem 3, [3]). ■

The friendship graph  $F_n$ , ( $n \geq 1$ ) consists of  $n$  triangles with a common vertex.

**Theorem 17.** *The friendship graph  $F_n$  is TSC for all  $n \geq 1$ .*

**Proof.** Label  $\lceil \frac{n}{2} \rceil$  triangles with  $(0, 0, 1)$  and  $\lfloor \frac{n}{2} \rfloor$  triangles with  $(0, 1, 1)$  where the central vertex of  $F_n$  is labelled by 1. Then we have

$$f(0) = \begin{cases} f(1) & \text{if } n \equiv 0, 1 \pmod{3}, \\ f(1) + 1 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

TSC labelling of cubic graphs i.e., regular graphs of degree 3 is appear to be interesting since  $K_4$  (Theorem 12, Case (c)) is not TSC, but we assert the following:

**Conjecture 18.** Cubic graphs other than  $K_4$  are TSC.

Final remark is that, not all Eulerian graphs are TSC as  $K_7$  shows by Theorem 12. On the other hand unlike cf., Theorem 4 [3] that Eulerian graphs with  $e \equiv 2 \pmod{4}$  edges are not cordial but the Eulerian graph with 14 edges shown in Figure 1 is TSC with  $f(0) = f(1) = 11$ .

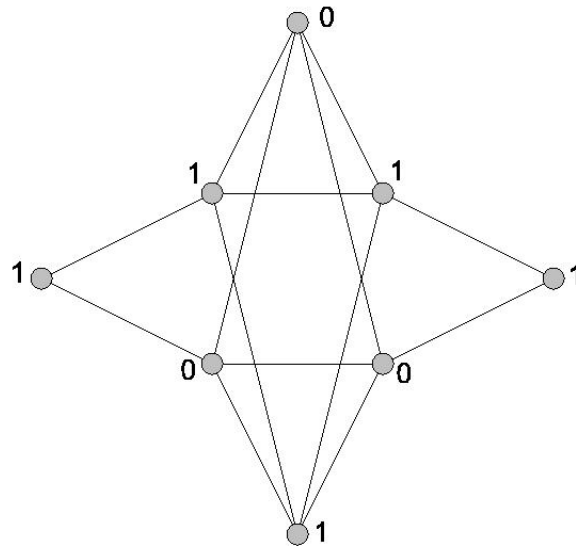


Figure 1

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