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TREES WITH UNIQUE MINIMUM TOTAL DOMINATING SETS

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Abstract

A set S of vertices of a graph G is a *total dominating set* if every vertex of $V(G)$ is adjacent to some vertex in S. We provide three equivalent conditions for a tree to have a unique minimum total dominating set and give a constructive characterization of such trees.

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1. Introduction

For notation and graph theory terminology, we in general follow [1, 7]. Specifically, let $G = (V, E)$ be a graph. For a vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is

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the set $N[S] = N(S) \cup S$. The *private neighborhood* $\text{pn}(v, S)$ of $v \in S$ is defined by $\text{pn}(v, S) = N(v) - N(S - \{v\})$. Equivalently, $\text{pn}(v, S) = \{u \in V \mid N(u) \cap S = \{v\}\}$. Each vertex in $\text{pn}(v, S)$ is called a *private neighbor* of v. The *external private neighborhood* $\text{epn}(v, S)$ of v with respect to S consists of those private neighbors of v in $V - S$, while the *internal private neighborhood* $\text{ipn}(v, S)$ of v with respect to S consists of those private neighbors of v in S. Thus, $\text{epn}(v, S) = \text{pn}(v, S) \cap (V - S)$ and $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$, while $\text{pn}(v, S) = \text{epn}(v, S) \cup \text{ipn}(v, S)$. If G has no isolated vertices, then the set S is a *total dominating set* if every vertex in V is adjacent to a vertex in S, that is, $N(S) = V$. Every graph without isolated vertices has a total dominating set, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of any total dominating set of G. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*. Note that every $\gamma_t(G)$-set is also a dominating set of G, and so $\gamma(G) \le \gamma_t(G)$. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory (see, for example, [4] and [9]).

The literature on domination and its variations in graphs has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8]. Gunther, Hartnell, Markus, and Rall [5] studied graphs with unique minimum dominating sets, and Hopkins and Staton [10] and Gunther, Hartnell, and Rall [6] studied graphs with unique maximum independent sets. We investigate graphs G with unique minimum total dominating sets, that is, unique $\gamma_t(G)$-sets. A graph G will be called a *unique total domination graph*, or just a UTD-*graph*, if G has a unique $\gamma_t(G)$-set.

Observe that the graph mK_2 has its vertex set as its unique minimum total dominating set. For other examples of UTD-graphs, consider the paths P_n with $n \equiv 0 \pmod 4$. Apart from a few minor results on UTD-graphs in general, we study UTD-trees. For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A vertex of degree one is called an *endvertex* or a *leaf* and its neighbor is called a *support* vertex. The set of leaves in T is denoted by $L(T)$ and the set of support vertices by $S(T)$. We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of T is denoted by $B(T)$. A tree T is a *double star* if it contains exactly two vertices that are not leaves.

2. Known Results

We shall need the following properties of minimal total dominating sets established in [2] and [9].

Theorem 1 (Cockayne et al. [2]). *If S is a minimal total dominating set of a connected graph G, then $|\mathrm{epn}(v,S)| \geq 1$ or $|\mathrm{ipn}(v,S)| \geq 1$ for each $v \in S$.*

Theorem 2 (Henning [9]). *If $G \neq K_n$ is a connected graph of order $n \geq 3$, then there exists a $\gamma_t(G)$-set S where for every vertex $v \in S$, $|\mathrm{epn}(v,S)| \geq 1$ or there exists a vertex $u \in \mathrm{ipn}(v,S)$ with $|\mathrm{epn}(u,S)| \geq 1$.*

Cockayne, Henning, and Mynhardt [3] characterized the set of vertices of a tree that are contained in all, or in no, respectively, minimum total dominating sets of the tree. To state this characterization, we introduce the following notation. We define the sets $\mathcal{A}_t(G)$ and $\mathcal{N}_t(G)$ of a graph G by

$$\mathcal{A}_t(G) = \{v \in V(G) \mid v \text{ is in every } \gamma_t(G)\text{-set}\}, \text{ and}$$

$$\mathcal{N}_t(G) = \{v \in V(G) \mid v \text{ is in no } \gamma_t(G)\text{-set}\}.$$

Let T be a tree rooted at a vertex v. The set of leaves in $T = T_v$ distinct from v we denote by $L(v)$, that is, $L(v) = D(v) \cap L(T)$. For $j = 0, 1, 2, 3$, we define

$$L^j(v) = \{u \in L(v) \mid d(u,v) \equiv j \pmod 4\}.$$

We next describe a technique called *tree pruning*, which will allow us to characterize the sets $\mathcal{A}_t(T)$ and $\mathcal{N}_t(T)$ for an arbitrary tree T.

Let T be a tree and let v be a vertex of T that is not a support vertex. The pruning of T is performed with respect to the root. Hence suppose T is rooted at v, that is, $T = T_v$. If $\deg u \leq 2$ for each $u \in V(T_v) - \{v\}$, then let $\overline{T}_v = T$. Otherwise, let u be a branch vertex at maximum distance from v; note that $|C(u)| \geq 2$ and $\deg x \leq 2$ for each $x \in D(u)$. We now apply the following pruning process:

- If $|L^2(u)| \geq 1$, then delete $D(u)$ and attach a path of length 2 to u.
- If $|L^1(u)| \geq 1$, $L^2(u) = \emptyset$ and $|L^3(u)| \geq 1$, then delete $D(u)$ and attach a path of length 2 to u.
- If $|L^1(u)| \geq 1$ and $L^2(u) = L^3(u) = \emptyset$, then delete $D(u)$ and attach a path of length 1 to u.

- If $L^1(u) = L^2(u) = \emptyset$ and $|L^3(u)| \geq 1$, then delete $D(u)$ and attach a path of length 3 to u.
- If $L^1(u) = L^2(u) = L^3(u) = \emptyset$, then delete $D(u)$ and attach a path of length 4 to u.

This step of the pruning process, where all the descendants of u are deleted and a path of length 1, 2, 3, or 4 is attached to u to give a tree in which u has degree 2, is called a *pruning of T_v at u.* Repeat the above process until a tree $\overline{T}_v$ is obtained with $\deg u \leq 2$ for each $u \in V(\overline{T}_v) - \{v\}$. Then, $\overline{T}_v$ is called a *pruning* of T_v. The tree $\overline{T}_v$ is unique. Thus, to simplify notation, we write $\overline{L}^j(v)$ instead of $L^j_{\overline{T}_v}(v)$. The following characterization of the sets $\mathcal{A}_t(T)$ and $\mathcal{N}_t(T)$ for an arbitrary tree T is presented in [3].

Theorem 3 (Cockayne et al. [3]). *Let v be a vertex of a tree T. Then,*

(a) *$v \in \mathcal{A}_t(T)$ if and only if v is a support vertex or $|\overline{L}^1(v) \cup \overline{L}^2(v)| \geq 2$,*

(b) *$v \in \mathcal{N}_t(T)$ if and only if $\overline{L}^1(v) \cup \overline{L}^2(v) = \emptyset$.*

3. Preliminary Results

We first consider induced subgraphs of UTD-graphs. In particular, we show that any graph G without isolated vertices is an induced subgraph of a UTD-graph. The *corona* $\mathrm{cor}(G)$ of a graph G is that graph obtained from G by adding a pendant edge to each vertex of G. Obviously, the graph G is an induced subgraph of $\mathrm{cor}(G)$ and if G has no isolated vertices, then $V(G)$ is the unique $\gamma_t(\mathrm{cor}(G))$-set. Therefore every graph without isolated vertices is an induced subgraph of a UTD-graph, and hence there does not exist a forbidden subgraph characterization of the class of UTD-graphs.

Every endvertex is uniquely dominated by the support vertex adjacent to it, and so any total dominating set contains every support vertex.

Observation 4. *Every support vertex of G is in every $\gamma_t(G)$-set.*

Observation 5. *A path P_n is a* UTD*-graph if and only if $n \in \{2, 5\}$ or $n \equiv 0 \pmod 4$.*

Lemma 6. *If a graph G has a unique $\gamma_t(G)$-set S, then every vertex $v \in S$ is a support vertex or satisfies $|\mathrm{pn}(v, S)| \geq 2$.*

Proof. By Theorem 1, $|\text{pn}(v, S)| \geq 1$ for each $v \in S$. Suppose that $v \in S$ is not a support vertex and $|\text{pn}(v, S)| = 1$. If $|\text{epn}(v, S)| = 1$ (and so $|\text{ipn}(v, S)| = 0$), then let $u \in \text{epn}(v, S)$ and let $w \in N(u) - \{v\}$. Then, $w \in V(G) - S$ and $(S - \{v\}) \cup \{w\}$ is a $\gamma_t(G)$-set, contradicting the uniqueness of S. On the other hand, if $|\text{ipn}(v, S)| = 1$ (and so $|\text{epn}(v, S)| = 0$), then, by Theorem 2, there exists a vertex $u \in \text{ipn}(v, S)$ with $|\text{epn}(u, S)| \geq 1$. Let $w \in \text{epn}(u, S)$. Then, $(S - \{v\}) \cup \{w\}$ is a $\gamma_t(G)$-set, contradicting the uniqueness of S. Hence, $|\text{pn}(v, S)| \geq 2$. ■

As an immediate consequence of Lemma 6 we have the following observation.

Observation 7. *Let G be a connected graph of order $n \geq 3$. If any endvertex u of G is in a $\gamma_t(G)$-set, then G is not a* UTD*-graph.*

The converse of Lemma 6 is not true in general. For example, if G is the 8-cycle $v_1, v_2, \ldots, v_8, v_1$, then $S = \{v_2, v_3, v_6, v_7\}$ is a $\gamma_t(G)$-set; however, S is not a unique $\gamma_t(G)$-set.

Recall that $S(G)$ is the set of support vertices of G.

Lemma 8. *If a graph G is a* UTD*-graph with $\gamma_t(G)$-set S, then $\gamma_t(G-v) \geq \gamma_t(G)$ for every $v \in S - S(G)$.*

Proof. Let G be a UTD-graph with $\gamma_t(G)$-set S, and assume to the contrary that $\gamma_t(G - v) < \gamma_t(G)$ for some $v \in S - S(G)$. Let R be a $\gamma_t(G - v)$-set. Since $|R| < \gamma_t(G)$, R does not dominate v. Furthermore, since G has no isolates, v has a neighbor, say u, in $V - R$. Then $R \cup \{u\}$ is a total dominating set of G that does not contain v. Hence, $R \cup \{u\} \neq S$, contradicting the uniqueness of S. ■

Lemma 9. *If a graph G has a $\gamma_t(G)$-set S for which $\gamma_t(G - v) > \gamma_t(G)$ for every $v \in S - S(G)$, then S is the unique $\gamma_t(G)$-set of G.*

Proof. Suppose there exists a $\gamma_t(G)$-set D that is different from S. Let $v \in S - D$. By Observation 4, v is not a support vertex. In particular, $G - v$ contains no isolated vertex. Since D is a total dominating set of $G - v$, $\gamma_t(G) = |D| \geq \gamma_t(G - v)$, a contradiction. Hence, S is the unique $\gamma_t(G)$-set. ■

The converse of Lemma 9 is not true in general. For example, the set $\{v, x, y\}$ is the unique $\gamma_t(G)$-set for the graph G in Figure 1, and $\gamma_t(G - v) = \gamma_t(G)$.

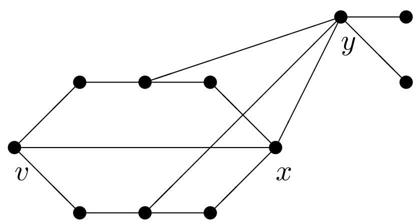

Figure 1: A graph G with the unique $\gamma_t(G)$-set $\{v, x, y\}$

4. Trees

4..1 Equivalent Conditions for UTD-Trees

Our aim in this section is to provide three equivalent conditions for a tree to have a unique minimum total domination set. We begin with the following lemmas.

Lemma 10. *Let T_1 and T_2 be vertex disjoint trees, and let $v \in \mathcal{A}_t(T_1)$. Let T be a tree obtained from $T_1 \cup T_2$ by joining v to a vertex of T_2. Then, $v \in \mathcal{A}_t(T)$.*

Proof. Since $v \in \mathcal{A}_t(T_1)$, Theorem 3 implies that v_1 is a support vertex of T_1 or $|\overline{L}^1(v_1) \cup \overline{L}^2(v_1)| \geq 2$ in T_1. So, certainly, v_1 is a support vertex of T or $|\overline{L}^1(v_1) \cup \overline{L}^2(v_1)| \geq 2$ in T. Thus, by Theorem 3, $v \in \mathcal{A}_t(T)$. ∎

Notice that if a vertex v in a tree T belongs to some but not all $\gamma_t(T)$-sets, then clearly T does not have a unique minimum total domination set. Hence we have the following observation.

Observation 11. *A tree T is a* UTD*-tree if and only if $v \in \mathcal{A}_t(T) \cup \mathcal{N}_t(T)$ for every vertex $v \in V(T)$.*

As an immediate consequence of Theorem 3 and Observation 11, we have the following characterization of UTD-trees.

Theorem 12. *A tree T is a* UTD*-tree if and only if for every vertex $v \in V(T)$, v is a support vertex or $|\overline{L}^1(v) \cup \overline{L}^2(v)| \neq 1$.*

We now establish three equivalent conditions for a tree to be a UTD-tree.

Theorem 13. *Let T be a tree of order $n \geq 2$. Then the following conditions are equivalent:*

(i) *T is a* UTD-*tree.*

(ii) *T has a $\gamma_t(T)$-set S for which every vertex $v \in S$ is a support vertex or satisfies $|\text{pn}(v, S)| \geq 2$.*

(iii) *T has a $\gamma_t(T)$-set S for which $\gamma_t(T - v) > \gamma_t(T)$ for every $v \in S - S(T)$.*

(iv) *For every vertex $v \in V(T)$, v is a support vertex or $|\overline{L}^1(v) \cup \overline{L}^2(v)| \neq 1$.*

Proof. By Theorem 12, (i) $\Leftrightarrow$ (iv). By Lemma 6, (i) $\Rightarrow$ (ii), and by Lemma 9, (iii) $\Rightarrow$ (i). Hence it suffices to prove that (ii) $\Rightarrow$ (iii). Suppose, then, that T has a $\gamma_t(T)$-set S for which every vertex $v \in S$ is a support vertex or satisfies $|\text{pn}(v, S)| \geq 2$. We show that condition (iii) holds. We proceed by induction on the order n of the tree T.

If every vertex of S is a support vertex, then condition (iii) is vacuously true. In particular, the base case when $n = 2$ is true. Assume that for all trees of order less than n, where $n \geq 3$, that (ii) $\Rightarrow$ (iii). Let T be a tree of order n that satisfies condition (ii). We may assume that $S \neq S(T)$, for otherwise condition (iii) is vacuously true. Let $v \in S - S(T)$. We show that $\gamma_t(T - v) > \gamma_t(T)$.

Since v is not a support vertex, $|\text{pn}(v, S)| \geq 2$. In particular, $\deg v \geq 2$. Let $A = \text{epn}(v, S)$, $B = N(v) - (A \cup S)$, $C = \text{ipn}(v, S)$ and $D = N(v) \cap (S - C)$. Since S is a total dominating set of T, $|C \cup D| \geq 1$. Let $N(v) = \{v_1, v_2, \ldots, v_m\}$, where the subscripts are indexed so that if $v_i \in A$, $v_j \in B$, $v_k \in C$ and $v_\ell \in D$, then $i < j < k < \ell$.

Let T be rooted at v. For $i = 1, 2, \ldots, m$, let $T_i = T_{v_i}$ (so T_i is the subtree of T induced by $D[v_i]$), and let $S_i = S \cap V(T_i)$. Since v is not a support vertex, each component T_i of the forest $T - v$ is a nontrivial tree. Let R be a $\gamma_t(T - v)$-set. For $i = 1, 2, \ldots, m$, let $R_i = R \cap V(T_i)$. Then, R_i is a $\gamma_t(T_i)$-set for each i. We proceed further by proving four claims.

Claim 1. *If $v_i \in A$, then $|R_i| = |S_i| + 1$.*

Proof. Suppose $v_i \in A$. Then, S_i does not contain v_i or any child of v_i. Note that since $v_i \notin S_i$, v_i is not a support vertex implying that each component of $T_i - v_i$ is a nontrivial tree. We show first that S_i is the unique $\gamma_t(T_i - v_i)$-set. The set S_i is a total dominating set of $T_i - v_i$. If S_i is not a $\gamma_t(T_i - v_i)$-set, then replacing S_i in S by a $\gamma_t(T_i - v_i)$-set produces a total dominating set of T of cardinality less than $|S| = \gamma_t(T)$, which is impossible. Hence, S_i is a $\gamma_t(T_i - v_i)$-set. Furthermore, every vertex $v \in S_i$ is a support vertex in $T_i - v_i$ or satisfies $|\text{pn}(v, S)| \geq 2$ in $T_i - v_i$. Applying the inductive hypothesis to each component of $T_i - v_i$, each component of $T_i - v_i$ satisfies condition (iii). Since (iii) $\Rightarrow$ (i), it follows that S_i is the unique $\gamma_t(T_i - v_i)$-set.

Since R_i is a $\gamma_t(T_i)$-set, R_i contains a child of v_i, and so $S_i \neq R_i$. If $|R_i| < |S_i|$, then $(S - S_i) \cup R_i$ is a total dominating set of T of cardinality less than $|S| = \gamma_t(T)$, which is impossible. Hence, $|R_i| \geq |S_i|$.

If $v_i \in \mathcal{A}_t(T_i)$, then, by Lemma 10, $v_i \in \mathcal{A}_t(T)$ contradicting the fact that S is a $\gamma_t(T)$-set not containing v_i. Hence, $v_i \notin \mathcal{A}_t(T_i)$. We may assume therefore that R_i is chosen so that $v_i \notin R_i$. But then R_i is also a total dominating set of $T_i - v_i$. Since R_i is a $\gamma_t(T_i)$-set, R_i contains a child of v_i, and so $S_i \neq R_i$. Thus, since S_i is the unique $\gamma_t(T_i - v_i)$-set, R_i is not a $\gamma_t(T_i - v_i)$-set, and so $|R_i| \geq |S_i| + 1$. Since $S_i \cup \{v_i'\}$ is a total dominating set of T_i where $v_i' \in C(v_i)$, $|S_i| + 1 \leq |R_i| = \gamma_t(T_i) \leq |S_i| + 1$. Consequently, $|R_i| = |S_i| + 1$. ∎

Claim 2. *If $v_i \in B$, then $|R_i| = |S_i|$.*

Proof. Suppose $v_i \in B$. Then, S_i is a total dominating set of T_i, and so $\gamma_t(T_i) \leq |S_i|$. If $|R_i| < |S_i|$, then $(S - S_i) \cup R_i$ is a total dominating set of T of cardinality less than $|S| = \gamma_t(T)$, which is impossible. Hence, $|S_i| \leq |R_i| = \gamma_t(T_i) \leq |S_i|$. Consequently, $|R_i| = |S_i|$. ∎

Claim 3. *If $v_i \in C$, then $|R_i| = |S_i| + 1$.*

Proof. Suppose $v_i \in C$. Then, $v_i \in S_i$ but S_i does not contain any child of v_i. Let H be the tree obtained from T_i by joining v to v_i and to a new vertex u. Then, H is a tree of order less than n in which the vertex v is a support vertex and therefore belongs to every $\gamma_t(H)$-set.

The set $S_i \cup \{v\}$ is a total dominating set of H. If $S_i \cup \{v\}$ is not a $\gamma_t(H)$-set, then replacing $S_i \cup \{v\}$ in S by a $\gamma_t(H)$-set (which necessarily contains v) produces a total dominating set of T of cardinality less than $|S| = \gamma_t(T)$, which is impossible. Hence, $S_i \cup \{v\}$ is a $\gamma_t(H)$-set. Furthermore, every vertex $w \in S_i \cup \{v\}$ is a support vertex in H or satisfies $|\text{pn}(w, S)| \geq 2$ in H. Applying the inductive hypothesis to H, the tree H satisfies condition (iii). Since (iii) $\Rightarrow$ (i), it follows that $S_i \cup \{v\}$ is the unique $\gamma_t(H)$-set.

We show now that $v_i \notin \mathcal{N}_t(T_i)$. Since $S_i \cup \{v\}$ is the unique $\gamma_t(H)$-set, we know by Observation 11 that $v_i \in \mathcal{A}_t(H)$. Hence, by Theorem 3, v_i is a support vertex of H or $|\overline{L}^1(v_i) \cup \overline{L}^2(v_i)| \geq 2$ in the tree H. If v_i is a support vertex in H, then, since v is not a leaf, v_i is also a support vertex in T_i, and so in the tree T_i, $|\overline{L}^1(v_i)| \geq 1$ and therefore $|\overline{L}^1(v_i) \cup \overline{L}^2(v_i)| \geq 1$ in T_i. On the other hand, if $|\overline{L}^1(v_i) \cup \overline{L}^2(v_i)| \geq 2$ in the tree H, then $|\overline{L}^1(v_i) \cup \overline{L}^2(v_i)| \geq 1$ in the tree T_i. In any event, $|\overline{L}^1(v_i) \cup \overline{L}^2(v_i)| \geq 1$ in the tree T_i. Hence, by Theorem 3, $v_i \notin \mathcal{N}_t(T_i)$.

Since $v_i \notin \mathcal{N}_t(T_i)$, there exists a $\gamma_t(T_i)$-set that contains v_i. We may assume that R_i is chosen so that $v_i \in R_i$. The desired result now follows as in the proof of Claim 1. ■

Claim 4. *If $v_i \in D$, then $|R_i| = |S_i|$.*

Proof. Suppose $v_i \in D$. Then, $v_i \in S_i$ and S_i contains a child of v_i. Let H be defined as in Claim 3. Then, as shown in Claim 3, $v_i \notin \mathcal{N}_t(T)$ and we may assume that $v_i \in R_i$. The desired result now follows as in the proof of Claim 2. ■

We now return to our proof of Theorem 13. Since $|\text{pn}(v, S)| \geq 2$, $|A| + |C| = |\text{epn}(v, S)| + |\text{ipn}(v, S)| = |\text{pn}(v, S)| \geq 2$. By Claims 1, 2, 3 and 4, $|R| = \sum_{i=1}^{m} |R_i| \geq 2 + \sum_{i=1}^{m} |S_i| = 2 + (|S| - 1) = |S| + 1$. Thus, $\gamma_t(T - v) = |R| > |S| = \gamma_t(T)$. Since v is an arbitrary vertex of $S - S(T)$, the set S satisfies condition (iii). Hence, (ii) $\Rightarrow$ (iii) as desired. ■

4..2 Combining UTD-Trees

Our aim in this section is to provide a constructive characterization of UTD-trees. For this purpose, we introduce the following notation. Let T be a UTD-tree of order at least 4 and let S be the unique $\gamma_t(T)$-set. Let the vertices of T be partitioned into sets S_A, S_B, S_C, S_D, and S_E as follows:

$$S_A = \{v \in S \mid v \in \text{ipn}(w, S) \text{ for some } w \in S - S(T) \text{ with } |\text{pn}(w, S)| = 2\},$$
$$S_B = S - S_A,$$
$$S_C = \{v \in V - S \mid \text{pn}(w, S) = \{v\} \text{ for some } w \in S\},$$
$$S_D = \{v \in V - S \mid v \in \text{pn}(w, S) \text{ for some } w \in S - S(T - v) \text{ with } |\text{pn}(w, S)| = 2\},$$
$$S_E = (V - S) - (S_C \cup S_D).$$

Note that if $v \in S_C$, then $v \in L(T)$. We say that the vertices of S_X have status X where $X \in \{A, B, C, D, E\}$.

The following lemma will prove to be useful.

Lemma 14. *Let T_1 and T_2 be vertex disjoint trees, and let $v \in \mathcal{A}_t(T_1)$. Let T be a tree obtained from $T_1 \cup T_2$ by joining v to a vertex of T_2. Let D be a $\gamma_t(T)$-set. If T_1 is a* UTD*-tree of order at least* 3*, then $|D \cap V(T_1)| \geq \gamma_t(T_1)$.*

Proof. Let S_1 be the unique $\gamma_t(T_1)$-set and let $D_1 = D \cap V(T_1)$. If D_1 is a total dominating set of T_1, then $|D_1| \geq \gamma_t(T_1)$, as desired. Suppose, then, that D_1 is not a total dominating set of T_1. Then, D_1 contains no neighbor of v. Now, $D_1 \cup \{v'\}$ is a total dominating set of T_1 for any neighbor v' of v in T_1. Suppose in the tree T_1, $N(v) \subset S_1$. Then, v cannot be a support vertex (since no leaf belongs to S_1), and so $\deg v \geq 2$. Thus, since $D_1 \cup \{v'\}$ contains only one neighbor of v, the uniqueness of S_1 implies that $D_1 \cup \{v'\}$ is not a $\gamma_t(T_1)$-set. Hence, $|D_1| + 1 \geq \gamma_t(T_1) + 1 = |S_1| + 1$, and so $|D_1| \geq |S_1|$. On the other hand, if in the tree T_1, $N(v) \not\subset S_1$, then we choose $v' \in N(v) - S_1$. Since $D_1 \cup \{v'\} \neq S_1$, the uniqueness of S_1 once again implies that $|D_1| \geq |S_1|$. The result follows. ∎

In what follows, we shall adopt the following notation. Let T_1 and T_2 be two vertex disjoint UTD-trees each of order at least 4. For $i \in \{1, 2\}$, let S_i denote the unique $\gamma_t(T_i)$-set. Then, S_i consists of the vertices of status A and B. We now present three operations which allow us to link up T_1 and T_2 to produce a new UTD-tree T.

Operation $\mathcal{T}_1$. Join a vertex u_1 of status D or E in T_1 to a vertex u_2 of status D or E in T_2.

Operation $\mathcal{T}_2$. Join a vertex u_1 of S_1 to a vertex u_2 of status E in T_2.

Operation $\mathcal{T}_3$. Join a vertex u_1 of status B in T_1 to a vertex u_2 of status B in T_2.

In the next lemma, for the tree T obtained from $T_1 \cup T_2$ using one of these three operations, let D be a $\gamma_t(T)$-set, and let $D_i = D \cap V(T_i)$ for $i \in \{1, 2\}$.

Lemma 15. *$S_1 \cup S_2$ is the unique $\gamma_t(T)$-set of T produced by Operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.*

Proof. (i) Suppose T is produced by Operation $\mathcal{T}_1$. We show first that $S_1 \cup S_2$ is a $\gamma_t(T)$-set. The set $S_1 \cup S_2$ is a total dominating set of T, and so

$\gamma_t(T) \le |S_1| + |S_2|$. If $|D_1| + |D_2| \ge |S_1| + |S_2|$, then $\gamma_t(T) \ge |S_1| + |S_2|$ and consequently $\gamma_t(T) = |S_1| + |S_2|$. Hence it suffices to show that $|D_1| + |D_2| \ge |S_1| + |S_2|$.

Suppose $u_1, u_2 \notin D$. Then D_i is a total dominating set of T_i, and so $|D_i| \ge \gamma_t(T_i) = |S_i|$. Thus, $|D_1| + |D_2| \ge |S_1| + |S_2|$.

Suppose $u_1 \in D$ and $u_2 \notin D$. Then, D_1 is a total dominating set of T_1. Since S_1 is the unique $\gamma_t(T_1)$-set and $u_1 \notin S_1$, $|D_1| \ge |S_1| + 1$. Also, $D_2 \cup \{u_2'\}$ is a total dominating set of T_2 where u_2' is any neighbor of u_2 in T_2. Thus, $|D_2| + 1 \ge \gamma_t(T_2) = |S_2|$. Hence, $|D_1| + |D_2| \ge |S_1| + |S_2|$. Similarly, if $u_1 \notin D$ and $u_2 \in D$, then $|D_1| + |D_2| \ge |S_1| + |S_2|$.

Suppose $u_1, u_2 \in D$. Then, for $i \in \{1, 2\}$, $D_i \cup \{u_i'\}$ is a total dominating set of T_i where u_i' is any neighbor of u_i in T_i. Since S_i is the unique $\gamma_t(T_i)$-set, $D_i \cup \{u_i'\}$ is not a $\gamma_t(T_i)$-set, and so $|D_i| + 1 \ge \gamma_t(T_i) + 1 = |S_i| + 1$. Hence, $|D_i| \ge |S_i|$ for each i. Thus, $|D_1| + |D_2| \ge |S_1| + |S_2|$. Hence, $S_1 \cup S_2$ is a $\gamma_t(T)$-set.

Since u_1 and u_2 are vertices of status D or E in T_1 and T_2, respectively, in the tree T every vertex $v \in S_1 \cup S_2$ is a support vertex or satisfies $|\text{pn}(v, S_1 \cup S_2)| \ge 2$. Thus, since $S_1 \cup S_2$ is a $\gamma_t(T)$-set, it follows from Theorem 13 that T is a UTD-tree and $S_1 \cup S_2$ is the unique $\gamma_t(T)$-set.

(ii) Suppose T is produced by Operation $\mathcal{T}_2$. We show that $|D_i| \ge |S_i|$ for each i. It follows from Lemma 14 that $|D_1| \ge |S_1|$. If D_2 is a total dominating set of T_2, then $|D_2| \ge |S_2|$ as desired. Suppose D_2 is not a total dominating set of T_2. Then D_2 contains no neighbor of u_2 and $D_2 \cup \{u_2'\}$ is a total dominating set of T_2 for any neighbor u_2' of u_2 in T_2. If $D_2 \cup \{u_2'\} = S_2$, then since u_2 has status E in T_2, $|\text{pn}(u_2, S_2)| \ge 2$, contradicting the fact that $D_2 = S_2 - \{u_2'\}$ is a total dominating set of $T_2 - u_2$. Hence, $D_2 \cup \{u_2'\} \ne S_2$. Since S_2 is the unique $\gamma_t(T_2)$-set, $|D_2| + 1 \ge \gamma_t(T_2) + 1 = |S_2| + 1$, and so $|D_2| \ge |S_2|$. Thus, $|D_i| \ge |S_i|$ for each i.

The set $S_1 \cup S_2$ is a total dominating set of T, and so $\gamma_t(T) \le |S_1| + |S_2|$. However, $\gamma_t(T) = |D| = |D_1| + |D_2| \ge |S_1| + |S_2|$. Consequently, $\gamma_t(T) = |S_1| + |S_2|$.

Since $u_1 \in S_1$ and u_2 is a vertex of status E in T_2, in the tree T every vertex $v \in S_1 \cup S_2$ is a support vertex or satisfies $|\text{pn}(v, S_1 \cup S_2)| \ge 2$. Thus, since $S_1 \cup S_2$ is a $\gamma_t(T)$-set, it follows from Theorem 13 that T is a UTD-tree and $S_1 \cup S_2$ is the unique $\gamma_t(T)$-set.

(iii) Suppose T is produced by Operation $\mathcal{T}_3$. It follows from Lemma 14 that $|D_i| \ge |S_i|$ for each i and therefore that $S_1 \cup S_2$ is a $\gamma_t(T)$-set. Since $u_i \in S_i$ has status B in T_i for $i \in \{1, 2\}$, in the tree T every vertex $v \in$

$S_1 \cup S_2$ is a support vertex or satisfies $|\text{pn}(v, S_1 \cup S_2)| \geq 2$. It follows from Theorem 13 that T is a UTD-tree and $S_1 \cup S_2$ is the unique $\gamma_t(T)$-set. ■

Let $\mathcal{T}$ be the family of trees T with $V(T) = L(T) \cup S(T)$, $|S(T)| \geq 2$. Let $\mathcal{F}$ be the family of trees that can be obtained from a star T with at least two leaves by adding at least one leaf adjacent to each leaf of T (so each leaf of T is a support vertex in the resulting tree). We are now in a position to present a constructive characterization of UTD-trees.

Theorem 16. *Let T be a tree of order at least 4. Then T is a* UTD*-tree if and only if T can be constructed from disjoint trees in $\mathcal{T} \cup \mathcal{F}$ by a sequence of Operations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$.*

Proof. Each tree in $\mathcal{T} \cup \mathcal{F}$ is a UTD-tree, and so the sufficiency follows from Lemmas 15. To prove the neccessity, we proceed by induction on $\gamma_t(T)$. If $\gamma_t(T) = 2$, then T is a double star, and so $T \in \mathcal{T}$. Hence the base case holds. Assume the result is true for all UTD-trees T' with $\gamma_t(T') < m$, where $m \geq 3$. Let $T = (V, E)$ be a UTD-tree with $\gamma_t(T) = m$. Let S be the unique $\gamma_t(T)$-set.

Let u_1u_2 be an edge of T, and let T_1 and T_2 be the components of $T - u_1u_2$ containing u_1 and u_2, respectively. For $i \in \{1, 2\}$, let $S_i = S \cap V(T_i)$ and let D_i be a $\gamma_t(T_i)$-set. We proceed further by proving three claims.

Claim 5. *For $i \in \{1, 2\}$, if S_i is a total dominating set of T_i, then S_i is the unique $\gamma_t(T_i)$-set.*

Proof. For $i \in \{1, 2\}$, $|D_i| \leq |S_i|$. Now, $D_1 \cup D_2$ is a total dominating set of T, and so $|S_1|+|S_2| \geq |D_1|+|D_2| \geq \gamma_t(T) = |S_1|+|S_2|$. Hence $|D_i| = |S_i|$ for each i. Thus, $D_1 \cup D_2$ is a $\gamma_t(T)$-set. The uniqueness of S implies that $D_1 \cup D_2 = S$, and therefore $D_i = S_i$ for each i. Hence, T_i is a UTD-tree and S_i is the unique $\gamma_t(T_i)$-set. ■

Claim 6. *If $u_1, u_2 \in V - S$, then T can be constructed as claimed.*

Proof. Since S_i is a total dominating set of T_i for $i \in \{1, 2\}$, T_i is a UTD-tree and S_i is the unique $\gamma_t(T_i)$-set by Claim 5. Applying the inductive hypothesis to T_i, each T_i can be constructed from disjoint trees in $\mathcal{T} \cup \mathcal{F}$ by a sequence of Operations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$. Since $u_i \notin S_i$, u_i has status C, D or E in T_i. If u_i has status C in T_i, then in the tree T_i, u_i is a leaf and $\text{pn}(w, S_i) = \{u_i\}$ for some $w \in S_i$. Thus in T, $|\text{pn}(w, S)| = 1$ and

$w \in S - S(T)$, contradicting Theorem 13. Hence, u_i has status D or E in T_i, and T can be obtained from $T_1 \cup T_2$ by Operation $\mathcal{T}_1$. The result follows. ■

Claim 7. *If $u_1 \in S$ and if $u_2 \in V - S$ is not an external private neighbor of any vertex in S, then T can be constructed as claimed.*

Proof. Since S_i is a total dominating set of T_i for $i \in \{1, 2\}$, T_i is a UTD-tree and S_i is the unique $\gamma_t(T_i)$-set by Claim 5. Applying the inductive hypothesis to T_i, each T_i can be constructed from disjoint trees in $\mathcal{T} \cup \mathcal{F}$ by a sequence of Operations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$.

We show next that u_2 has status E in T_2. Since $u_2 \notin S_2$, u_2 has status C, D or E in T_2. If u_2 has status C in T_2, then $\text{pn}(w, S_2) = \{u_2\}$ for some $w \in S_2$. Thus in T, $|\text{pn}(w, S)| = 0$, contradicting Theorem 1. If u_2 has status D in T_2, then in the tree T_2, $u_2 \in \text{pn}(w, S_2)$ where w is adjacent to no leaf except possibly u_2 and $|\text{pn}(w, S_2)| = 2$. Thus in T, $|\text{pn}(w, S)| = 1$ and $w \in S - S(T)$, contradicting Theorem 13. Hence, u_2 has status E in T_2. Thus, T can be obtained from $T_1 \cup T_2$ by Operation $\mathcal{T}_2$. The result follows. ■

We now return to the proof of Theorem 16. By Claim 6, we may assume that no edge joins two vertices of $V - S$ and by Claim 7, we may assume that each vertex in $V - S$ is the external private neighbor of some vertex in S. Hence, each vertex in $V - S$ is a leaf in T. If $S = S(T)$, then $T \in \mathcal{T}$. Hence, we may assume that $S \neq S(T)$.

Let $u_1 \in S - S(T)$. By assumption, $N(u_1) \subseteq S$. Hence, by Theorem 13, $|\text{ipn}(u_1, S)| \geq 2$. For each $w \in \text{ipn}(u_1, S)$, $N(w) \cap S = \{u_1\}$ and $w \in S(T)$. If $S = N[u_1]$, then $T \in \mathcal{F}$. Hence we may assume that some neighbor u_2 of u_1 is not an internal private neighbor of u_1.

For $i \in \{1, 2\}$, let T_i, S_i, and D_i be as defined earlier. Since S_i is a total dominating set of T_i for $i \in \{1, 2\}$, T_i is a UTD-tree and S_i is the unique $\gamma_t(T_i)$-set by Claim 5. Applying the inductive hypothesis to T_i, each T_i can be constructed from disjoint trees in $\mathcal{T} \cup \mathcal{F}$ by a sequence of Operations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$.

In the tree T_1, $|\text{ipn}(u_1, S_1)| \geq 2$, and so $u_1 \notin \text{ipn}(w, S_1)$ for any $w \in S_1$. Thus, u_1 has status B in T_1. In the tree T_2, if u_2 has status A, then $u_2 \in \text{ipn}(w, S_2)$ for some $w \in S_2 - S(T)$ where $|\text{pn}(w, S_2)| = 2$. But then in the tree T, $w \in S - S(T)$ and $|\text{pn}(w, S)| = 1$, contradicting Theorem 13. Hence, u_2 has status B in T_2. Thus, T can be obtained from $T_1 \cup T_2$ by Operation $\mathcal{T}_3$. ■

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