A NOTE ON DOMINATION IN BIPARTITE GRAPHS

TOBIAS GERLACH AND JOCHEN HARANT

Department of Mathematics Technical University of Ilmenau D-98684 Ilmenau, Germany

Abstract

DOMINATING SET remains *NP*-complete even when instances are restricted to bipartite graphs, however, in this case VERTEX COVER is solvable in polynomial time. Consequences to VECTOR DOMINATING SET as a generalization of both are discussed.

Keywords: bipartite graph, domination.

2000 Mathematics Subject Classification: 05C35.

For terminology and notation not defined here we refer to [2, 3]. Given a finite, simple, and undirected graph G without isolated vertices, $V(G) = \{1, \ldots, n\}, E(G), G[V], \text{ and } d_i(G)$ denote its vertex set, its edge set, the subgraph of G induced by $V \subseteq V(G)$, and the degree of $i \in V(G)$ in G, respectively. Furthermore, let $\vec{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and $\vec{d}(G) = (d_1(G), \ldots, d_n(G))$. Given an integral vector $\vec{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i(G)$ for $i = 1, \ldots, n$, a set $D \subseteq V(G)$ is called \vec{k} -dominating if $d_i(G[V(G) \setminus D]) \leq d_i(G) - k_i$ for $i \in V(G) \setminus D$. Consider the corresponding decision problem, which was investigated in [4]:

VECTOR DOMINATING SET

Instance: A graph G on $V(G) = \{1, ..., n\}$, an integral vector $\vec{k} = (k_1, ..., k_n)$ with $1 \le k_i \le d_i(G)$ for i = 1, ..., n, and a positive integer l. Question: Does G contain a \vec{k} -dominating set D with $|D| \le l$?

The restriction of VECTOR DOMINATING SET to $\vec{k} = \vec{1}$ is the decision problem DOMINATING SET, remaining *NP*-complete even when instances are restricted to bipartite graphs [1]. It is easy to see that $D \subseteq V(G)$ is $\vec{d}(G)$ dominating if and only if each edge of *G* has at least one endvertex in *D*. The corresponding decision problem VERTEX COVER is NP-complete in general, however, is solvable in polynomial time for bipartite graphs [3]. The question arises in how many components \vec{k} may differ from $\vec{1}$ or from $\vec{d}(G)$ such that the restriction of VECTOR DOMINATING SET to bipartite graphs remains NP-complete or is solvable in polynomial time, respectively. Theorem 1 and Theorem 2 give partial answers to this question.

Theorem 1. Given $0 < c < \frac{1}{2}$, the following restriction of VECTOR DOMINATING SET remains NP-complete.

Instance: A bipartite graph G on $V(G) = \{1, \ldots, n\}$, an integral vector $\vec{k} = (k_1, \ldots, k_n)$ with $1 \le k_i \le d_i(G)$ for $i = 1, \ldots, n$ and $|\{i \in V(G) \mid k_i > 1\}| = \lceil cn \rceil$, and a positive integer l.

Question: Does G contain a \vec{k} -dominating set D with $|D| \leq l$?

Theorem 2. Given c > 0, the following restriction of VECTOR DOMI-NATING SET is solvable in polynomial time.

Instance: A bipartite graph G on $V(G) = \{1, ..., n\}$, an integral vector $\vec{k} = (k_1, ..., k_n)$ with $1 \le k_i \le d_i(G)$ for i = 1, ..., n and $|E(G[\{i \in V(G) \mid k_i < d_i(G)\}])| \le c \log_2 n$, and a positive integer l.

Question: Does G contain a \vec{k} -dominating set D with $|D| \leq l$?

Proof of Theorem 1. Given $0 < c < \frac{1}{2}$, we shall transform DOMI-NATING SET to the restriction of VERTEX DOMINATING SET of Theorem 1. Let a graph H on $V(H) = \{1, \ldots, m\}$ and a positive integer l be an instance of DOMINATING SET, the positive integer r be choosen such that $\frac{2cm}{1-2c} \leq r < \frac{2cm}{1-2c} + 1$, hence, $c(2m+2r) \leq r < c(2m+2r) + (1-2c)$ implying $r = \lceil c(2m+2r) \rceil \geq 1$, and G be constructed with $V(G) = \{1, \dots, 2m + 2r\}$ and $E(G) = \{(i, m + j), (j, m + i) \mid (i, j) \in E(H)\}$ $\cup \{(i, m+i), (m+i, 2m+j), (2m+j, 2m+r+j) \mid i = 1, \dots, m, j = 1, \dots, r\}.$ Following the ideas in [1], it is easy to see that G is bipartite and that for $D(H) \subseteq V(H), D(G) = \{i + m \mid i \in D(H)\} \cup \{2m + 1, \dots, 2m + r\}$ is a $\vec{1}$ -dominating set of G if and only if D(H) is a $\vec{1}$ -dominating set of H. Let $\vec{k} = (k_1, \ldots, k_{2m+2r})$ be an arbitrary integral vector with $1 \le k_p \le d_p(G)$ for $p = 1, \ldots, 2m + 2r$, and $k_p = 1$ if and only if $p \in V(G) \setminus \{2m+1, \ldots, 2m+r\}$. Then $|\{i \in V(G) \mid k_i > 1\}| = r = [c|V(G)|]$, and D(G) is even a \vec{k} -dominating set of G if D(H) is a 1-dominating set of H. Hence, H contains a 1-dominating set of cardinality l if and only if G contains a k-dominating set of cardinality l+r. With |V(G)| = 2m+2r < 2m+2c(2m+2r)+2(1-2c) = 2m+2r2|V(H)|+2c|V(G)|+2(1-2c), hence, $|V(G)| < \frac{2}{1-2c}|V(H)|+2$ we are done.

NOTE ON DOMINATION IN BIPARTITE GRAPHS

Proof of Theorem 2. For $D \subseteq V(G)$, let E(D) be the set of edges having no endvertex in D, and H(D) be the graph arising from G by deleting the edges of E(D). It is easy to see that D is a \vec{k} -dominating set of G if and only if $E(D) \subseteq E(G[\{i \in V(G) \mid k_i < d_i(G)\}])$, each endvertex i of an edge in E(D) is endvertex of at most $d_i(G) - k_i$ edges in E(D), and D is a $\vec{d}(H(D))$ dominating set of H(D). Since VERTEX COVER is solvable in polynomial time and the number of sets E(D) is at most $2^{|E(G[\{i \in V(G) \mid k_i < d_i(G)\}])|} \le n^c$ we are done.

Remark. It remains open whether the bounds cn $(0 < c < \frac{1}{2})$ and $c \log_2 n$ (c > 0) of Theorem 1 and Theorem 2 can be made significantly greater such that VECTOR DOMINATING SET is still NP-complete or solvable in polynomial time, respectively.

References

- G.J. Chang and G.L. Nemhauser, The k-domination and k-stability problems in sun-free chordal graphs, SIAM J. Algebraic Discrete Methods 5 (1984) 332–345.
- [2] R. Diestel, Graph Theory (Springer-Verlag, New York, 2000).
- [3] M.R. Garey and D.S. Johnson, Computers and Intractability (W.H. Freeman and Company, San Francisco, 1979).
- [4] J. Harant, A. Pruchnewski and M. Voigt, On dominating sets and independent sets of graphs, Combinatorics, Probability and Computing 8 (1999) 547–553.

Received 24 August 2000