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ISOMORPHISMS AND TRAVERSABILITY OF DIRECTED PATH GRAPHS

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Abstract

The concept of a line digraph is generalized to that of a directed path graph. The directed path graph $\overrightarrow{P}_k(D)$ of a digraph D is obtained by representing the directed paths on k vertices of D by vertices. Two vertices are joined by an arc whenever the corresponding directed paths in D form a directed path on $k+1$ vertices or form a directed cycle on k vertices in D. In this introductory paper several properties of $\overrightarrow{P}_3(D)$ are studied, in particular with respect to isomorphism and traversability. In our main results, we characterize all digraphs D with $\overrightarrow{P}_3(D) \cong D$, we show that $\overrightarrow{P}_3(D_1) \cong \overrightarrow{P}_3(D_2)$ "almost always" implies $D_1 \cong D_2$, and we characterize all digraphs with Eulerian or Hamiltonian $\overrightarrow{P}_3$-graphs.

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1. Introduction

We refer to [2] for any undefined terminology.

In [3] path graphs were introduced as a generalization of line graphs of (undirected) graphs. In the next section we shall introduce an analogous concept for directed graphs. But first we recall some basic definitions and notation concerning directed graphs.

We define a directed graph or *digraph* D to be a pair $(V(D), A(D))$, where $V(D)$ is a finite non-empty set of elements called *vertices*, and $A(D)$ is a (finite) set of distinct ordered pairs of distinct elements of $V(D)$ called *arcs*. For convenience we shall denote the arc (v, w) (where $v, w \in V(D)$) by vw. If $a = vw$ is an arc of D, then we say that v and w are *adjacent*, and that a is an *out-arc* of v and an *in-arc* of w; we call w an *out-neighbour* of v and v an *in-neighbour* of w. The *in-degree* $d^-(v)$ of v is the number of in-arcs of v; the *out-degree* $d^+(v)$ of v is the number of out-arcs of v; v is a *source* or *sink* if $d^-(v) = 0$ or $d^+(v) = 0$, respectively. The *underlying graph* $U(D)$ of a digraph D is the graph (or multigraph) obtained from D by replacing each arc by an (undirected) edge joining the same pair of vertices. A digraph D is called *strongly connected* if, for each pair of vertices v and w, there is a directed path in D from v to w, and *connected* if there is a path from v to w in $U(D)$. A directed subgraph of D corresponding to a path of $U(D)$ is called a *semipath* of D. We denote by $\overrightarrow{P}_k$ a directed path on k vertices ($k \geq 1$), i.e., a semipath on k vertices with one source and one sink, in which all arcs are oriented from source to sink. A directed cycle $\overrightarrow{C}_k$ ($k \geq 2$) consists of a $\overrightarrow{P}_k$ with source v and sink w together with the arc wv. Two arcs $a, b \in A(D)$ are said to be *adjacent* if $\{a, b\} = \{vw, wz\}$ for some vertices $v, w, z \in V(D)$; to stress the head-to-tail adjacency, we say that *a hits b* if $a = vw$ and $b = wz$. We call two adjacent arcs $a, b \in A(D)$ a $\overrightarrow{P}_3$*-pair* or a $\overrightarrow{C}_2$*-pair* if they form a $\overrightarrow{P}_3$ or a $\overrightarrow{C}_2$ in D, respectively. If $\{a, b\} \subseteq A(D)$ is a $\overrightarrow{P}_3$-pair and a hits b, then we denote the $\overrightarrow{P}_3$ formed by a and b simply by ab.

2. Directed Path Graphs

Let k be a positive integer, and let D be a digraph containing at least one $\overrightarrow{P}_k$. Denote by $\overrightarrow{\Pi}_k(D)$ the set of all $\overrightarrow{P}_k$'s of D. Then the $\overrightarrow{P}_k$*-graph* of D, denoted by $\overrightarrow{P}_k(D)$, is the digraph with vertex set $\overrightarrow{\Pi}_k(D)$; pq is an arc of $\overrightarrow{P}_k(D)$ if and only if there is a $\overrightarrow{P}_{k+1}$ or $\overrightarrow{C}_k$ $v_1v_2 \ldots v_{k+1}$ in D (with $v_1 = v_{k+1}$

in the case of a $\overrightarrow{C}_k$) such that $p = v_1v_2\dots v_k$ and $q = v_2\dots v_kv_{k+1}$. Note that $\overrightarrow{P}_1(D) = D$ and $\overrightarrow{P}_2(D) = \overrightarrow{L}(D)$, the line digraph of D, as it was introduced in [4].

For a nice survey of results on line graphs and line digraphs we refer to [5]. In the sequel we shall restrict ourselves to $\overrightarrow{P}_3$-graphs. In Section 3 we give some elementary results on $\overrightarrow{P}_3$-graphs, in Section 4 we discuss isomorphisms of $\overrightarrow{P}_3$-graphs, and in Section 5 we consider the traversability of $\overrightarrow{P}_3$-graphs.

3. Elementary Results

3..1 Vertices, Arcs and Degrees

Let D be a digraph containing at least one $\overrightarrow{P}_3$ and let $G = \overrightarrow{P}_3(D)$. To express the number of vertices, the number of arcs, and the degrees of the vertices of G in terms of D, we first introduce some additional terminology.

For a vertex $v \in V(D)$, we set

$$\overline{A}_v = \{u \in V(D) \mid \{uv, vu\} \subseteq A(D)\},$$

and we define

$$\overline{A}(D) = \{uv \in A(D) \mid vu \in A(D)\}.$$

Now the number of $\overrightarrow{P}_3$'s in D with middle vertex v is equal to

$$(d^-(v) - |\overline{A}_v|)d^+(v) + |\overline{A}_v|(d^+(v) - 1) = d^-(v)d^+(v) - |\overline{A}_v|.$$

Hence

$$|V(G)| = \sum_{v\in V(D)} (d^-(v)d^+(v) - |\overline{A}_v|) = \sum_{v\in V(D)} d^-(v)d^+(v) - |\overline{A}(D)|.$$

The number of arcs of G can be counted by summing up, for each arc a of D, the number of $\overrightarrow{P}_3$'s of D "joined" together by having the arc a "in common", as follows: each arc $uv \in A(D)\backslash\overline{A}(D)$ joins $d^-(u)d^+(v)$ $\overrightarrow{P}_3$'s, while each arc $uv \in \overline{A}(D)$ joins $(d^-(u) - 1)(d^+(v) - 1)$ $\overrightarrow{P}_3$'s. Hence

$$|A(G)| = \sum_{uv\in A(D)} (d^-(u)d^+(v)) + |\overline{A}(D)| - \sum_{uv\in \overline{A}(D)} (d^-(u) + d^+(v)).$$

The in-degree and out-degree of a vertex in G corresponding to a $\overrightarrow{P}_3$ uvw in D are

$$d^-(u) - |\{u\} \cap \overline{A}_v| \quad \text{and}$$
$$d^+(w) - |\{w\} \cap \overline{A}_v|,$$

respectively.

3..2 Cycle Structure

By considering the possible adjacency structures of $\overrightarrow{P}_3$'s in a digraph D, one easily obtains the following result on (short) cycles in $\overrightarrow{P}_3(D)$. We only give a proof of (iv) and remark that similar results can be deduced for longer cycles. Recall that $U(D)$ denotes the underlying (undirected) graph of D.

Theorem 1. *Let D be a digraph containig at least one $\overrightarrow{P}_3$. Then*

(i) *$\overrightarrow{P}_3(D)$ contains no $\overrightarrow{C}_2$;*

(ii) *Each C_3 in $U(\overrightarrow{P}_3(D))$ is a $\overrightarrow{C}_3$ in $\overrightarrow{P}_3(D)$;*

(iii) *Each C_4 in $U(\overrightarrow{P}_3(D))$ is induced (has no chords) and is a $\overrightarrow{C}_4$ or is oriented with alternating arc directions in $\overrightarrow{P}_3(D)$;*

(iv) *No C_k ($k \geq 5$) of $U(\overrightarrow{P}_3(D))$ is both induced and oriented with alternating arc directions in $\overrightarrow{P}_3(D)$.*

Proof. (iv) Suppose $C_k = v_1v_2v_3v_4\dots v_1$ ($k \geq 5$) is both induced and oriented, and assume v_2v_1, v_2v_3, and v_4v_3 are arcs. Considering the arrangement of the $\overrightarrow{P}_3$'s in D corresponding to the vertices of C_k according to the given arcs, it is easy to observe that the chord v_4v_1 of C_k must be present in $\overrightarrow{P}_3(D)$. ■

3..3 Splitting Vertices

Let D be a digraph and $v \in V(D)$ a source with out-arcs $vu_1, \dots, vu_k$. Suppose D' is obtained from D by replacing v by two (or more) new vertices v_1, v_2 and splitting the out-arcs $vu_1, \dots, vu_k$ into two (or more) disjoint (non-empty) sets $v_1u_1, \dots, v_1u_{k_1}$, $v_2u_{k_1+1}, \dots, v_2u_k$. Then it is clear that $\overrightarrow{P}_3(D') \cong \overrightarrow{P}_3(D)$. A similar splitting preserving the $\overrightarrow{P}_3$-structure can be applied to sinks. Of course the reverse operation of combining sources or sinks is also preserving the $\overrightarrow{P}_3$-structure, as long as sources or sinks do not have common out-neighbours or in-neighbours, respectively.

Splitting an arbitrary vertex v of D into two new vertices v_1, v_2 and dividing the in-arcs and out-arcs at v among v_1 and v_2, we obtain a digraph D' with the property that $\overrightarrow{P}_3(D')$ is an induced subgraph of $\overrightarrow{P}_3(D)$. We leave the details to the reader.

4. Isomorphisms of $\overrightarrow{P}_3$-Graphs

In this section we consider two questions:

(1) For which digraphs D is $\overrightarrow{P}_3(D) \cong D$?

(2) For which digraphs D_1 and D_2 does $\overrightarrow{P}_3(D_1) \cong \overrightarrow{P}_3(D_2)$ imply $D_1 \cong D_2$?

We refer to [5] for results related to similar questions concerning line (di)graphs, and to [3] for analogous results on P_3-graphs of (undirected) graphs.

In this section we shall characterize all digraphs for which $\overrightarrow{P}_3(D) \cong D$, and we shall see that $\overrightarrow{P}_3(D_1) \cong \overrightarrow{P}_3(D_2)$ "almost always" implies $D_1 \cong D_2$. Before we present the results we introduce some additional terminology.

Let D be a digraph. A directed tree T of D is an *out-tree* of D if $V(T) = V(D)$ and precisely one vertex of T has in-degree zero (the *root* of T), while all other vertices of T have in-degree one. An *in-tree* of D is defined analogously with respect to out-degrees. Note that any strongly-connected digraph contains an in-tree and an out-tree, and no sources or sinks. We first give a short proof of the following result.

Theorem 2. *Let D be a connected digraph without sources or sinks. If D has an in-tree or an out-tree, then $\overrightarrow{P}_3(D) \cong D$ if and only if $D \cong \overrightarrow{C}_n$ for some $n \geq 3$. Hence, if D is strongly connected, then $\overrightarrow{P}_3(D) \cong D$ if and only if $D \cong \overrightarrow{C}_n$ for some $n \geq 3$.*

Proof. If $D \cong \overrightarrow{C}_n$ for some $n \geq 3$, then clearly $\overrightarrow{P}_3(D) \cong \overrightarrow{C}_n \cong D$.

For the converse, assume without loss of generality that D has an out-tree T with root v. Let t denote the number of vertices with out-degree zero in T. Denote $V(D) = \{v, v_1, \ldots, v_{n-1}\}$, where $v_1, \ldots, v_t$ are the vertices with out-degree zero in T. Note that $\overrightarrow{P}_3(D)$ does not contain $\overrightarrow{C}_2$. Since v has at least one in-arc in D, and each of $v_1, \ldots, v_t$ has at least one out-arc in D, we know that $\overrightarrow{P}_3(D)$ has at least

$$d_T^+(v) + t + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1})$$

vertices. From $\overrightarrow{P}_3(D) \cong D$ we obtain

$$n = |V(D)| = |V(\overrightarrow{P}_3(D))| \geq d_T^+(v) + t + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1}),$$

hence

$$d_T^+(v) + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1}) \leq n - t. \tag{1}$$

On the other hand, since T is an out-tree, we obtain

$$d_T^+(v) + (d_T^+(v_{t+1}) + 1) + \ldots + (d_T^+(v_{n-1}) + 1) + t = 2(n-1),$$

hence

$$d_T^+(v) + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1}) = n - 1. \tag{2}$$

Combining (1) and (2), we get that $t \leq 1$, implying that $t = 1$, and that $T = \overrightarrow{P}_n$. Similar arguments show that any in-tree of D is a $\overrightarrow{P}_n$. This is only possible if $D \cong \overrightarrow{C}_n$. ■

Let D be a digraph obtained from a $\overrightarrow{P}_{m_1}$ ($m_1 \geq 2$) and a vertex disjoint $\overrightarrow{C}_{m_2}$ ($m_2 \geq 3$) by identifying either the first or the last vertex of $\overrightarrow{P}_{m_1}$ with one vertex of $\overrightarrow{C}_{m_2}$. Then one easily checks that $\overrightarrow{P}_3(D) \cong D$, that D contains either precisely one source or precisely one sink, and that D contains an in-tree or an out-tree. This class of graphs can be extended by taking a directed cycle and a collection of vertex disjoint out-trees (or in-trees) and identifying the roots of the trees with different vertices of the cycle. Note that we require all (nontrivial) trees to be out-trees or all (nontrivial) trees to be in-trees.

In the sequel we characterize all connected digraphs satisfying $\overrightarrow{P}_3(D) \cong D$.

First we present a useful relationship between $\overrightarrow{P}_3$-graphs and iterated line digraphs. Given a digraph D, we denote by $Asym(D)$ the graph obtained from D by deleting all $\overrightarrow{C}_2$'s, i.e., by deleting all $\overrightarrow{C}_2$-pairs $\{uv, vu\} \subseteq A(D)$.

Theorem 3. *For any digraph D containing at least one $\overrightarrow{P}_3$, $\overrightarrow{P}_3(D) \cong \overrightarrow{L}(Asym(\overrightarrow{L}(D)))$.*

Proof. Let D be a digraph containing at least one $\overrightarrow{P}_3$. Then $\overrightarrow{P}_3(D)$ and $\overrightarrow{L}(Asym(\overrightarrow{L}(D)))$ exist, and $v \in V(\overrightarrow{L}(Asym(\overrightarrow{L}(D))))$ if and only if $v \in A(Asym(\overrightarrow{L}(D)))$. This is equivalent to saying that $v = xy$ for some $x, y \in V(Asym(\overrightarrow{L}(D)))$, or, equivalently, for some $x, y \in V(\overrightarrow{L}(D))$ such that $\{x, y\}$ is not a $\overrightarrow{C}_2$-pair of D. It is clear that this is equivalent to saying that v is a $\overrightarrow{P}_3$ of D, hence $v \in V(\overrightarrow{P}_3(D))$.

Moreover, uv is an arc of $\overrightarrow{L}(Asym(\overrightarrow{L}(D)))$ if and only if uv corresponds to a $\overrightarrow{P}_3$ in $Asym(\overrightarrow{L}(D))$, and hence in $\overrightarrow{L}(D)$. It is again clear that this is equivalent to saying that u and v correspond to two $\overrightarrow{P}_3$'s in D forming a $\overrightarrow{P}_4$ or $\overrightarrow{C}_3$ in D, or, equivalently, that uv is an arc of $\overrightarrow{P}_3(D)$. ■

Corollary 4. *For any digraph D containing at least one $\overrightarrow{P}_3$ and no $\overrightarrow{C}_2$, $\overrightarrow{P}_3(D) \cong \overrightarrow{L}(\overrightarrow{L}(D)) = \overrightarrow{L}^2(D)$.*

Proof. This follows immediately from Theorem 3 and the observation that D contains a $\overrightarrow{C}_2$ if and only if $\overrightarrow{L}(D)$ contains a $\overrightarrow{C}_2$. ■

Corollary 5. *For any digraph D containing at least one $\overrightarrow{P}_3$,*
$\limsup_{n\to\infty} |V(\overrightarrow{P}_3^n(D))| < \infty$ *if and only if*
$\limsup_{n\to\infty} |V(\overrightarrow{L}^{2n-1}(Asym(\overrightarrow{L}(D))))| < \infty$.

Proof. This follows from the fact that
$\overrightarrow{P}_3^2(D) = \overrightarrow{L}(Asym(\overrightarrow{L}(\overrightarrow{L}(Asym(\overrightarrow{L}(D)))))) = \overrightarrow{L}^3(Asym(\overrightarrow{L}(D)))$, hence $\overrightarrow{P}_3^n(D) = \overrightarrow{L}^{2n-1}(Asym(\overrightarrow{L}(D)))$. ■

Corollary 6. *For any digraph D containing at least one $\overrightarrow{P}_3$, $\overrightarrow{P}_3(D)$ is strongly connected if and only if $Asym(\overrightarrow{L}(D))$ is strongly connected.*

Proof. This is an immediate consequence of Theorem 3 and the following result which holds for our digraph D ([5, Theorem 7.4 (i)]). ■

Theorem 7. *Let D be a digraph on at least three vertices* (*none of which is isolated*). *Then $\overrightarrow{L}(D)$ is strongly connected if and only if D is strongly connected.*

In particular, if $Asym(D)$ is strongly connected, then $Asym(\overrightarrow{L}(D))$ is strongly connected and so is $\overrightarrow{P}_3(D)$, but for the following digraph D, $Asym(D)$ is disconnected, while $Asym(\overrightarrow{L}(D))$ is strongly connected.

The digraph D consists of two vertex-disjoint $\overrightarrow{C}_3$'s and two additional arcs uv and vu between two vertices u and v of different $\overrightarrow{C}_3$'s. It is easy to find infinitely many examples of the same type. Another infinite class of examples can be obtained from the class of (undirected) trees (with at least two edges) by replacing all edges of the trees by distinct $\overrightarrow{C}_2$'s; for such a resulting graph D, $A(Asym(D)) = \emptyset$, while $Asym(\overrightarrow{L}(D))$ contains a Hamilton cycle, e.g., a directed cycle through all vertices. (See also Section 5.)

4..1 Characterizing all Connected Digraphs D with $\overrightarrow{P}_3(D) \cong D$

In this subsection we assume D is a connected digraph satisfying $\overrightarrow{P}_3(D) \cong D$. Then by Theorem 1 (i), D contains no $\overrightarrow{C}_2$, hence by Corollary 4, $D \cong \overrightarrow{P}_3(D) \cong \overrightarrow{L}^2(D)$, hence $D \cong \overrightarrow{P}_3^n(D) \cong \overrightarrow{L}^{2n}(D)$ for any integer $n \geq 1$. By Theorem 7, if D is strongly connected, then $\overrightarrow{L}(D)$ is strongly connected, hence $\overrightarrow{P}_3^n(D)$ is strongly connected for all $n \geq 1$; then the counting arguments from Section 3.1 imply $|V(\overrightarrow{P}_3^n(D))| \geq |V(\overrightarrow{P}_3^{n-1}(D))| \geq \ldots \geq |V(D)|$, with equality only if D is a directed cycle. Since $|V(D)|$ is finite this shows that all strongly connected components of D are directed cycles. The next three statements are [5, Theorem 9.1 (i), (ii), and Theorem 7.2 (iii)], respectively.

Theorem 8. *Let D be a digraph. Then*

(i) $A(\overrightarrow{L}^n(D)) = \emptyset$ *for some n if and only if D has no directed cycles.*

(ii) $|V(\overrightarrow{L}^n(D))|$ *gets arbitrarily large if and only if D has two directed cycles joined by a directed path (possibly of length 0).*

(iii) *If no two directed cycles of D are joined by a directed path, then for all sufficiently large values of n, each connected component of $\overrightarrow{L}^n(D)$ has at most one directed cycle.*

These results together imply that D has precisely one directed cycle C. Following [5, pp. 298–299] we define the *basic configuration* of D to be the subdigraph formed by the union of all directed paths of D directed to or from vertices of C. Clearly, D includes the cycle C. As remarked in [5], the line digraph of D and each of its iterates have a single directed cycle (all of the same length), and for sufficiently large n, the basic configuration of $\overrightarrow{L}^n(D)$ consists of a directed cycle and an out-tree and in-tree (disjoint except for roots) at each vertex of the directed cycle. Hence D is of this form. It is not difficult to check one cannot have an out-tree and in-tree attached at the same vertex of C, since this creates an isolated vertex in $\overrightarrow{P}_3(D)$ or a subdigraph attached to a vertex of the directed cycle that is neither an out-tree nor an in-tree.

We can derive more if we look more carefully at the way the arrangement of the attached trees changes if we turn from D to $\overrightarrow{P}_3(D) \cong D$. Suppose that D consists of a directed cycle $C = v_0v_1 \ldots v_kv_0$ and at least one nontrivial in-tree T_1 and at least one nontrivial out-tree T_2 attached to C at vertices v_{i_1} and v_{i_2}, respectively. Then $v_{i_1} \neq v_{i_2}$ by the above arguments. Let T_1, v_{i_1} and T_2, v_{i_2} be chosen in such a way that the directed path P from v_{i_1} to v_{i_2} along C is as short as possible. We may assume $i_1 = 0$ and let $j = i_2$. Then P has length j. Denote by H the subdigraph of D consisting of C, T_1, and T_2. Then one easily checks that $\overrightarrow{P}_3(H)$ consists of a cycle $C' \cong C$, an in-tree $T_1' \cong T_1$, an out-tree $T_2' \cong T_2$, and at least one additional arc from a vertex of T_1' to a vertex of T_2' if $j = 1$. The latter is obviously impossible, since we assume $\overrightarrow{P}_3(D) \ncong D$, so $j \geq 2$. It is equally easy to observe that the directed path from $V(T_1') \cap V(C')$ to $V(T_2') \cap V(C')$ along C' in $\overrightarrow{P}_3(H)$ has length $j - 2$, again a contradiction. Hence we conclude that the only connected digraphs D with $\overrightarrow{P}_3(D) \cong D$ consist of a directed cycle with in-trees or out-trees attached to its vertices, with at most one nontrivial tree per vertex, and only one type of nontrivial trees.

4..2 Periodic $\overrightarrow{P}_3$-graphs

We present a result on $\overrightarrow{P}_3$-graphs analogous to Theorem 8 (i) and (ii), and the next statements ([5, Theorem 9.1 (iii) and Corollary 7.3]).

Theorem 9. *Let D be a digraph. Then:*

(i) *D is $\overrightarrow{L}$-periodic if and only if D has directed cycles, no two of which are joined by a directed path (possibly of length 0).*

(ii) *If D is strongly connected and $\overrightarrow{L}^n(D) \cong D$ for some $n \geq 1$, then $\overrightarrow{L}(D) \cong D$ and D is a directed cycle.*

We omit the proof of Theorem 11 below, since the statements are easy consequences of the fact that $\overrightarrow{P}_3^n(D) = \overrightarrow{L}^{2n-1}(Asym(\overrightarrow{L}(D)))$, in combination with the above results, as well as the fact that directed cycles in $\overrightarrow{L}(D)$ correspond to directed closed walks in D, and directed paths in $\overrightarrow{L}(D)$ to directed walks in D.

To be able to understand the results from [5], we have to introduce some more definitions from [5].
Given the digraph D consisting of a directed cycle $C = v_0v_1 \dots v_{k-1}v_0$ together with an out-tree A_i or an in-tree B_i at each vertex v_i $(0 \leq i \leq k-1)$, the cyclic (modulo k) sequences $\{A_i\}$ and $\{B_i\}$ are called the *out-tree* and *in-tree sequences*. As mentioned in [5], the line digraph of D has the same sequences with the in-trees advanced one vertex of the cycle relative to the out-trees. The *out-tree index* of D is then defined as the minimum positive integer r for which $A_{i+r} \cong A_i$ for all i; the *in-tree index* is defined similarly. If, for some positive integers n and k, we have $\overrightarrow{L}^{n+k}(D) \cong \overrightarrow{L}^n(D)$, then we call D $\overrightarrow{L}$*-periodic*, and the smallest value of k for which this holds its $\overrightarrow{L}$*-period*. The next result (Theorem 9.2 of [5]) relates the $\overrightarrow{L}$-period of D to its out-tree and in-tree indices.

Theorem 10. *Let D be an $\overrightarrow{L}$-periodic digraph with a single directed cycle. Then the $\overrightarrow{L}$-period of D is the greatest common divisor of its out-tree and in-tree indices.*

Analogous to the above definition, here a digraph D is called $\overrightarrow{P}_3$*-periodic* if $\overrightarrow{P}_3^{n+k}(D) \cong \overrightarrow{P}_3^n(D)$ for some positive integers k and n. We obtain the following result.

Theorem 11. *Let D be a digraph. Then*

(i) *$A(\overrightarrow{P}_3^n(D)) = \emptyset$ for some n if and only if D has no directed cycles except for $\overrightarrow{C}_2$'s.*

(ii) *$|V(\overrightarrow{P}_3^n(D))|$ gets arbitrarily large for sufficiently large n if and only if D has two directed cycles of length at least 3 joined by a directed path (possibly of length 0).*

(iii) *D is $\overrightarrow{P}_3$-periodic if and only if D has directed cycles of length at least 3, no two of which are joined by a directed path (possibly of length 0).*

(iv) *If D is strongly connected and $\overrightarrow{P}_3^n(D) \cong D$ for some $n \geq 1$, then $\overrightarrow{P}_3(D) \cong D$ and D is a directed cycle.*

4..3 Which Digraphs Have Isomorphic $\overrightarrow{P}_3$-graphs?

Before we turn to Question 2 we introduce some additional terminology concerning isomorphisms.

Let D and D' be two digraphs. An *isomorphism* of D onto D' is a bijection $f : V(D) \to V(D')$ such that $uv \in A(D)$ if and only if $f(u)f(v) \in$

$A(D')$. An *arc-isomorphism* of D onto D' is a bijection $f : A(D) \to A(D')$ such that $a \in A(D)$ hits $b \in A(D)$ if and only if $f(a) \in A(D')$ hits $f(b) \in A(D')$. Hence an arc-isomorphism of D onto D' is an isomorphism of $\overrightarrow{L}(D)$ onto $\overrightarrow{L}(D')$. A $\overrightarrow{P}_3$-*isomorphism* of D onto D' is an isomorphism of $\overrightarrow{P}_3(D)$ onto $\overrightarrow{P}_3(D')$. We say that a $\overrightarrow{P}_3$-isomorphism f of D onto D' is *induced by an arc-isomorphism* of D onto D' if there exists an isomorphism f^* of D onto D' such that $f(uvw) = f^*(uv)f^*(vw)$ for each $\overrightarrow{P}_3 = uvw$ of D.

Question 2 can be rephrased as follows.

(2′) Which $\overrightarrow{P}_3$-isomorphisms of D onto D' are induced by isomorphisms of D onto D'?

The related question for arc-isomorphisms was answered in [4].

Theorem 12. *Let D and D' be two digraphs without sources or sinks. Then every arc-isomorphism of D onto D' is induced by an isomorphism of D onto D', hence $\overrightarrow{L}(D) \cong \overrightarrow{L}(D')$ if and only if $D \cong D'$.*

We can prove a similar result on $\overrightarrow{P}_3$-isomorphisms if we make a "weak" additional assumption concerning the digraphs D and D'. This additional assumption is reasonable and is the natural counterpart of the assumption in Theorem 12 that D and D' contain no sources or sinks.

Theorem 13. *Let D and D' be two connected digraphs. If for each arc $a = uv \in A(D) \cup A(D')$ there exist arcs $b = xu$ and $c = vy$ in the same digraph with $x \neq v$ and $y \neq u$, then every $\overrightarrow{P}_3$-isomorphism of D onto D' is induced by an arc-isomorphism of D onto D'.*

Proof. Let f denote a $\overrightarrow{P}_3$-isomorphism of D onto D', where D and D' satisfy the conditions of the theorem. For any arc $x \in A(D)$, there exist two arcs $y, z \in A(D)$ such that yx and xz correspond to two $\overrightarrow{P}_3$'s of D. Since f is a $\overrightarrow{P}_3$-isomorphism, for some $\overrightarrow{P}_3$-pairs $\{a, b\}, \{c, d\} \subseteq A(D')$, $f(yx) = ab$ and $f(xz) = cd$. But this implies $b = c$, since adjacencies of $\overrightarrow{P}_3$'s are preserved by f. Fixing z we see that b is independent of y, and fixing y we see that c is independent of z, so $b = c$ depends only on x. Denoting $b = c$ by $f^*(x)$ for each x, we get a function $f^* : A(D) \to A(D')$ such that $f(pq) = f^*(p)f^*(q)$ for each pq corresponding to a $\overrightarrow{P}_3$ of D. By similar reasoning there is a function f_* so that $f^{-1}(ab) = f_*(a)f_*(b)$ for each ab corresponding to a $\overrightarrow{P}_3$ of D'. Clearly $f_* \cdot f^*$ and $f^* \cdot f_*$ are identity functions, so f^* and f_* are inverse bijections. The function f^* induces f. We claim that

f^* is an arc-isomorphism from D to D'. From above, both f^* and f_* preserve arcs pq of the line digraph corresponding to a $\overrightarrow{P}_3$. Now we must show that f^* and f_* preserve arcs xy of the line digraph corresponding to a $\overrightarrow{C}_2$. By symmetry, it suffices to prove this for f^*. Assume $\{x, y\}$ is a $\overrightarrow{C}_2$-pair in D. If D is a digraph obtained from a cycle C_n by replacing each edge uv by two arcs uv and vu, then one easily checks that $D \cong D'$, unless D' consists of two disjoint $\overrightarrow{C}_n$'s, contradicting the connectivity of D'. In the other case, there exist arcs $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$ $(k > 2)$ in D such that $\{x_1, y_1\}, \{y_k, x_k\}$, and $\{x_i, x_{i+1}\}, \{y_{i+1}, y_i\}$ $(i = 1, \ldots, k-1)$ are $\overrightarrow{P}_3$-pairs in D, while $\{x_i, y_i\}$ $(i = 2, \ldots, k-1)$ are $\overrightarrow{C}_2$-pairs in D, and $\{x, y\} = \{x_i, y_i\}$ for some $i \in \{2, \ldots, k-1\}$. We complete the proof by showing that, for each $i \in \{2, \ldots, k-1\}$, $\{f^*(x_i), f^*(y_i)\}$ is a $\overrightarrow{C}_2$-pair in D', in particular $\{f^*(x), f^*(y)\}$. Assume, to the contrary, that i is the smallest index in $\{2, \ldots, k-1\}$ such that $\{f^*(x_i), f^*(y_i)\}$ is not a $\overrightarrow{C}_2$-pair. Suppose first that $f^*(x_i)f^*(y_i)$ (or $f^*(y_i)f^*(x_i)$) is a $\overrightarrow{P}_3$ in D'. Then, since f_* preserves $\overrightarrow{P}_3$-pairs, x_iy_i is a $\overrightarrow{P}_3$ in D, a contradiction. Hence $f^*(x_i)$ and $f^*(y_i)$ are nonadjacent arcs in D'. Considering the $\overrightarrow{P}_3$'s $f^*(x_{i-1})f^*(x_i)$ and $f^*(y_i)f^*(y_{i-1})$ in D', it is clear that $\{f^*(x_{i-1}), f^*(y_{i-1})\}$ is not a $\overrightarrow{C}_2$-pair in D'. The choice of i implies that $i = 2$. Hence $\{f^*(x_2), f^*(y_2)\}$ is not a $\overrightarrow{C}_2$-pair (nor a $\overrightarrow{P}_3$-pair) in D'. Now, considering the $\overrightarrow{P}_3$'s $f^*(x_1)f^*(x_2)$, $f^*(x_1)f^*(y_1)$, and $f^*(y_2)f^*(y_1)$ in D', we easily obtain a contradiction. ■

From the above proof we already note that we cannot omit the connectivity condition in Theorem 13.

To illustrate the necessity of the condition on the sources and sinks we can use the splitting technique from Section 3.3. As a small example consider the next pair of nonisomorphic digraphs with isomorphic $\overrightarrow{P}_3$-graphs. The first digraph consists of a $\overrightarrow{C}_3$, one additional vertex v, and arcs from v to two vertices of the $\overrightarrow{C}_3$; the second one of a $\overrightarrow{C}_3$, two additional vertices v_1, v_2, and arcs v_1u and v_2w to two vertices u and w of the $\overrightarrow{C}_3$.

With respect to the necessity of the condition on the arcs, consider the following pair of nonisomorphic digraphs with isomorphic $\overrightarrow{P}_3$-graphs. The first digraph is a $\overrightarrow{C}_4$; the second one is obtained from a $\overrightarrow{C}_4$ by replacing two vertex-disjoint arcs by $\overrightarrow{C}_2$'s. In general, if we add to D any arc not contained in a $\overrightarrow{P}_3$ to get D', then $\overrightarrow{P}_3(D) \cong \overrightarrow{P}_3(D')$. In particular we can add an arc from a source to a sink, or, if uv is an arc where $d^+(u) = d^-(v) = 1$, we can add the arc vu.

Combining Theorems 12 and 13 it is clear we have the following consequences for digraphs D and D' satisfying the conditions in the hypothesis of Theorem 13.

Corollary 14. *Let D and D' be two connected digraphs. If for each arc $a = uv \in A(D) \cup A(D')$ there exist arcs $b = xu$ and $c = vy$ in the same digraph with $x \neq v$ and $y \neq u$, then $\overrightarrow{P}_3(D) \cong \overrightarrow{P}_3(D')$ if and only if $D \cong D'$.*

Corollary 15. *Let D be a connected digraph. If for each arc $a = uv \in A(D)$ there exist arcs $b = xu$ and $c = vy$ in D with $x \neq v$ and $y \neq u$, then $Aut(D) \cong Aut(\overrightarrow{L}(D)) \cong Aut(\overrightarrow{P}_3(D))$.*

Remark. Recently, in [1] the equation $P_3(G) \cong P_3(G')$ for undirected graphs has been solved completely, building on earlier work in [3] and [6].

5. Traversability of $\overrightarrow{P}_3$-Graphs

In this section we consider (*directed*) *Euler tours* and (*directed*) *Hamilton cycles* in $\overrightarrow{P}_3$-graphs.

For line digraphs of strongly connected digraphs, the following result ([5, Theorem 10.1]) characterizes the traversability.

Theorem 16. *Let D be a strongly connected digraph. Then*

(i) *$\overrightarrow{L}(D)$ is Eulerian if and only if $d^-(v) = d^+(w)$ for each arc vw in D;*

(ii) *$\overrightarrow{L}(D)$ is Hamiltonian if and only if D is Eulerian.*

Combining Theorems 3 and 16 we immediately obtain the following characterization of Eulerian and Hamiltonian $\overrightarrow{P}_3$-graphs of digraphs D in terms of properties of $Asym(\overrightarrow{L}(D))$.

Corollary 17. *Let D be a digraph such that $Asym(\overrightarrow{L}(D))$ is strongly connected. Then*

(i) *$\overrightarrow{P}_3(D)$ is Eulerian if and only if $d^-(v) = d^+(w)$ for each arc vw in $Asym(\overrightarrow{L}(D))$;*

(ii) *$\overrightarrow{P}_3(D)$ is Hamiltonian if and only if $Asym(\overrightarrow{L}(D))$ is Eulerian.*

The properties of $Asym(\overrightarrow{L}(D))$ in Corollary 17 can be translated into properties of D as follows.

For an arc $xy \in A(D)$, let

$$d^-(xy) = d^-(x) - |\{yx\} \cap \overline{A}(D)| \quad \text{and}$$
$$d^+(xy) = d^+(y) - |\{yx\} \cap \overline{A}(D)|.$$

Then $d^-(v) = d^+(w)$ for each arc vw in $Asym(\overrightarrow{L}(D))$ if and only if $d^-(ab) = d^+(bc)$ for each $\overrightarrow{P}_3$ abc in D.

We say that a Euler tour T of D is a $\overrightarrow{C}_2$*-tour* if the arcs of each $\overrightarrow{C}_2$ of D are successive arcs in T. Then $Asym(\overrightarrow{L}(D))$ is Eulerian if and only if $\overrightarrow{L}(D)$ has a $\overrightarrow{C}_2$-tour and $\overrightarrow{L}(D) \ncong \overrightarrow{C}_2$. Furthermore, $Asym(\overrightarrow{L}(D))$ is a vertex-disjoint union of Eulerian digraphs if $\overrightarrow{L}(D)$ is Eulerian (and $\ncong \overrightarrow{C}_2$). Hence we obtain the following result.

Theorem 18. *Let D be a digraph such that $Asym(\overrightarrow{L}(D))$ is strongly connected. Then*

(i) *$\overrightarrow{P}_3(D)$ is Eulerian if and only if $d^-(ab) = d^+(bc)$ for each $\overrightarrow{P}_3$ abc in D;*

(ii) *$\overrightarrow{P}_3(D)$ is Hamiltonian if and only if $\overrightarrow{L}(D)$ has a $\overrightarrow{C}_2$-tour;*

(iii) *$\overrightarrow{P}_3(D)$ contains a 2-factor if and only if $\overrightarrow{L}(D)$ is Eulerian, or, equivalently if $d^-(v) = d^+(w)$ for each arc vw in D;*

(iv) *$\overrightarrow{P}_3(D)$ is Hamiltonian if $d^-(v) = d^+(w)$ for each arc vw in D, and D contains no $\overrightarrow{C}_2$.*

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