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DOMINATION IN PARTITIONED GRAPHS

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Abstract

Let V_1, V_2 be a partition of the vertex set in a graph G, and let γ_i denote the least number of vertices needed in G to dominate V_i. We prove that $\gamma_1 + \gamma_2 \le \frac{4}{5}|V(G)|$ for any graph without isolated vertices or edges, and that equality occurs precisely if G consists of disjoint 5-paths and edges between their centers. We also give upper and lower bounds on $\gamma_1 + \gamma_2$ for graphs with minimum valency δ, and conjecture that $\gamma_1 + \gamma_2 \le \frac{4}{\delta+3}|V(G)|$ for $\delta \le 5$. As δ gets large, however, the largest possible value of $(\gamma_1 + \gamma_2)/|V(G)|$ is shown to grow with the order of $\frac{\log \delta}{\delta}$.

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1. Introduction

Let V_1, V_2 be a partition of the vertices in a graph G, let γ denote the domination number of G and let γ_i denote the least number of vertices needed in G to dominate V_i. Seager ([5]) has proven that $\gamma+\gamma_1+\gamma_2 \leq |V(G)|$ for a graph with minimum valency at least 2. Hartnell and Vestergaard ([1]) have proven that for a tree $\gamma + \gamma_1 + \gamma_2 \leq \frac{5}{4}|V(G)|$. We prove here that $\gamma_1 + \gamma_2 \leq \frac{4}{5}|V(G)|$ for any graph without isolated vertices or edges and that equality occurs precisely if G consists of disjoint 5-paths and edges between their centers. We also prove $\gamma_1 + \gamma_2 \leq \frac{\delta+1}{2\delta}|V(G)|$ for a graph with minimum valency δ, and conjecture that $\gamma_1 + \gamma_2 \leq \frac{4}{\delta+3}|V(G)|$ for δ relatively small. As δ gets large, however, the largest possible value of $(\gamma_1 + \gamma_2)/|V(G)|$ is shown to grow with the order of $\frac{\log \delta}{\delta}$, its supremum (taken over all feasible G for each δ) tending to $\frac{2\log \delta}{\delta}$ as $\delta \to \infty$.

2. Notation and Definitions

By $\overline{G}$ we denote the complementary graph to G, i.e., $V(\overline{G}) = V(G)$ and two vertices are adjacent in $\overline{G}$ precisely if they are nonadjacent in G. A k-path, denoted P_k, is a path on k vertices. If G contains the edge vu_2 and $G-v$ has a component $P_{k-1} = u_2u_3\ldots u_{k-1}u_k$, $k \geq 2$, we say that $vu_2u_3\ldots u_k$ is a k-path pendant from v, or that $vu_2u_3\ldots u_k$ is attached to v. More generally, attaching $P_k = u_1\ldots u_k$ to v by u_i (for some $1 \leq i \leq k$) means that v and u_i get identified, i.e., if $1 \neq i \neq k$ then both $u_1\ldots u_{i-1}$ and $u_{i+1}\ldots u_k$ are components in $G-v$. Furthermore, $N_G(v)$ denotes the set of neighbours to $v \in V(G)$ and we define $N_G[v] = \{v\} \cup N_G(v)$. For $D \subseteq V(G)$ we define $N_G(D) = \bigcup\{N_G(v) \mid v \in D\}$ and $N_G[D] = \bigcup\{N_G[v] \mid v \in D\} = D \cup N_G(D)$. If for $u \in D$ we have $v \in N_G[D]$, but $v \notin N_G[D-u]$, we call v a private neighbour of u with respect to D in G, if e.g., u has no neighbour in D, u by this definition is its own private neighbour. If $X \subseteq V(G)$ satisfies $X \subseteq N_G[D]$, we say that D dominates X in G. If, in particular, $V(G) \subseteq N_G[D]$, we call D a dominating set in G. The cardinality $\gamma(G)$ of a smallest dominating set is called the domination number of G, $\gamma(G) = \min\{|D| \mid N_G[D] = V(G)\}$. Let V_1, V_2 be a partition of $V(G)$ into two disjoint subsets, $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$; $\{V_1, V_2\} = \{\emptyset, V(G)\}$ is permitted. Define $\gamma_G(\emptyset) = 0$ and for $i = 1, 2$ consider a smallest set of vertices D_i in $V(G)$ which in G dominates V_i, $\gamma_i = \gamma_G(V_i) = \min\{|D_i| \mid V_i \subseteq N_G[D_i]\}$. Define f by

$$f(G) = \max\{\gamma_G(V_1) + \gamma_G(V_2) \mid V_1 \cup V_2 = V(G), V_1 \cap V_2 = \emptyset\}\,.$$

When superfluous, we may omit reference to G, e.g., writing γ_i for $\gamma_G(V_i)$. Since $\gamma(G) \le |D_1 \cup D_2| \le \gamma_1 + \gamma_2$, $f(G)$ retains the same value whether or not we in its definition allow one of V_1, V_2 to be empty. By $\delta = \delta(G)$ we denote the minimum valency of G.

For a graph G we wish to determine $f(G)$ or at least to give an upper bound and to indicate some families of graphs for which the number $f(G)$ can be given.

3. Tight Upper Bounds with δ Small

In this section we consider graph classes where no strong assumptions are put on minimum valency.

Theorem 1. *Let G be a graph with at least three vertices in each component. Then*

(1) $f(G) \le \frac{4}{5}|V(G)|$,

(2) *Equality occurs in* (1) *precisely if G can be constructed from a graph H by attaching to each vertex of H a 5-path by its central vertex.*

Proof. We observe that

(i) $f(G_1 \cup G_2) \le f(G_1) + f(G_2)$,

(ii) $f(G) \le f(G - e)$, $\forall e \in E(G)$.

Therefore it suffices by (i) to prove (1) for connected graphs and by (ii) it suffices to prove (1) for a tree. Any tree T with diameter ≥ 5 contains an edge e such that both components of $T - e$ have at least three vertices. Hence it suffices to prove (1) for trees with diameter two, three or four.

A tree with diameter two is a star $K_{1,s}$, $s \ge 2$, and satisfies (1) since $f(K_{1,s}) = 2 < \frac{4}{5}(s+1)$. A tree G with diameter three is a double star, namely two vertex-disjoint stars $K_{1,s}$ and $K_{1,t}$, $s, t \ge 1$, with center u and v, respectively, together with the edge uv. For $|V(G)| > 5$ we see with $D_1 = D_2 = \{u, v\}$ that $f(G) \le |D_1| + |D_2| = 4 < \frac{4}{5}|V(G)|$ as desired. For $|V(G)| = 5$ G is a $K_{1,3}$ with one edge subdivided and $f(G) = 3 < \frac{4}{5} \cdot 5$. If $|V(G)| \le 4$, necessarily $s = t = 1$ and $G = P_4$, then $f(P_4) = 3 < \frac{4}{5} \cdot 4$. So (1) holds with sharp inequality for trees with diameter two or three.

Let G be a tree with diameter four and let $v_1v_2v_3v_4v_5$ be a longest path in G. We may assume that every neighbour v of v_3, in particular also v_2 and v_4, has at most one valency-1 neighbour, otherwise $G - vv_3$ would have three or more vertices in both components and we could apply (i), (ii) and induction. Thus we may assume G is a star with some edges subdivided, i.e., $G-v_3$ consists of k K_2's, $k \geq 2$, and possibly some isolated vertices. For $i = 1, 2$ let D_i consist of v_3 and those vertices in V_i which are non-adjacent to v_3. Then D_i dominates V_i, $i = 1, 2$, and $f(G) \leq 2 + k \leq \frac{4}{5}|V(G)|$, since $k \geq 2$ and $|V(G)| \geq 1 + 2k$. Note that equality in (1) only holds if $k = 2$ and $|V(G)| = 5$, i.e., if $G = P_5$. We have proven (1) for all graphs.

Obviously equality holds in (1) for the graph G constructed in (2) together with a partition V_1, V_2 of $V(G)$, where for $i = 1, 2$, V_i from each attached 5-path contains a neighbour and a non-neighbour to its central vertex, which are at distance 3 apart, while the central vertices may be partitioned arbitrarily. Conversely, let G be a graph with no isolated vertex or edge and with $f(G) = \frac{4}{5}|V(G)|$. This equality implies that the edge deletions described in the proof above for (1) result in components, all of which are 5-paths. We shall now prove that the only way these 5-paths and added edges can form the graph G, with equality in (1) preserved, is by adding edges between central vertices of the 5-paths. Addition of any other edge will cause $f(G) < \frac{4}{5}|V(G)|$. Let namely F consist of $P_5 = u_1u_2u_3u_4u_5$ and $Q_5 = v_1v_2v_3v_4v_5$ together with u_iv_j, $1 \leq i, j \leq 5$, $i \neq 3$. Place v_j in both D_1 and D_2. For any partition V_1, V_2 of $V(F)$ the contribution of $Q_5 - N_{Q_5}[v_j]$ to $\gamma_1 + \gamma_2$ is at most three, because $Q_5 - N_{Q_5}[v_j]$ is a P_3 or has only two vertices. Also, $P_5 - u_i$ is either a 4-path or a 3-path and an isolated vertex. In both cases a vertex with valency two in $P_5 - u_i$ is placed in both D_1 and D_2 while the non-dominated end vertex of the 4-path, respectively the isolated vertex, if belonging to V_i, $i = 1, 2$, is placed in D_i. Thus $f(F) \leq |D_1| + |D_2| = 7 < \frac{4}{5} \cdot 10$ and hence by (i) and (ii) $f(G) < \frac{4}{5}|V(G)|$. This proves the claim above and hence Theorem 1. ■

Seager has proven

Theorem 2 ([5]). *Let G be any graph with minimum valency at least two. Then for any partition V_1, V_2 of $V(G)$,*

$$\gamma(G) + \gamma_G(V_1) + \gamma_G(V_2) \leq |V(G)|\,.$$

Using that we shall prove the following bound for $f(G)$.

Theorem 3. *Let G be any graph with minimum valency at least two. Then $f(G) \leq \frac{2}{3}|V(G)|$.*

Proof. Let V_1, V_2 be a partition of $V(G)$. If $\gamma(G) \le \frac{1}{3}|V(G)|$ we have $\gamma_G(V_i) \le \gamma(G) \le \frac{1}{3}|V(G)|$ for $i = 1, 2$, and $\gamma_G(V_1) + \gamma_G(V_2) \le \frac{2}{3}|V(G)|$ follows. Otherwise $\gamma(G) > \frac{1}{3}|V(G)|$ and from Theorem 2 we obtain $\gamma_G(V_1) + \gamma_G(V_2) \le |V(G)| - \gamma(G) < \frac{2}{3}|V(G)|$. This proves Theorem 3. ■

As one can see, $f(G) = \frac{2}{3}|V(G)|$ implies that $\gamma(G) = \frac{1}{3}|V(G)|$ and that there exists a partition V_1, V_2 of $V(G)$ such that $\gamma_G(V_1) = \gamma_G(V_2) = \frac{1}{3}|V(G)|$.

Circuits on $3k$ vertices satisfy $f(C_{3k}) = 2k$. Let H be any graph and denote by $G = H \circ K_2$ the graph obtained from H by adding for each vertex v in H two new vertices v', v'' and three edges $v'v'', vv', vv''$. With all v' in V_1, all v'' in V_2 and $V(H)$ partitioned arbitrarily we see that $f(G) = \frac{2}{3}|V(G)|$. Based on the same principle, instead of triangles one can attach cycles of any (possibly distinct) lengths divisible by 3 to the vertices of H.

More generally, one can apply the following recursive construction. For $i = 1, 2$ let G_i be a graph with $f(G_i) = \frac{2}{3}|V(G_i)|$ and let D_i be a minimum dominating set of G_i such that each $v \in D_i$ has both a private V_1-neighbour and a private V_2-neighbour belonging to $V(G_i) - D_i$. Then the graph G obtained by joining G_1 and G_2 by any number of D_1D_2-edges has $f(G) = \frac{2}{3}|V(G)|$.

4. General Estimates on $\gamma_1 + \gamma_2$

In this section we investigate the situation when δ gets large. We begin with a constructive general lower bound.

Theorem 4. *For every $\delta > 0$ there exists a graph with minimum valency δ and a partition V_1, V_2 of its vertices such that*

$$f(G) \ge \gamma_G(V_1) + \gamma_G(V_2) = \begin{cases} \dfrac{4}{\delta+3}|V(G)| & \textit{for } \delta \equiv 1 \pmod 4, \\ \dfrac{4}{\delta+4}|V(G)| & \textit{for } \delta \not\equiv 1 \pmod 4. \end{cases}$$

Proof. If $\delta \equiv 1 \pmod 4$, write $\delta = 4t - 3$, $t \ge 1$. Let $\overline{G} = C_{4t} = x_1x_2x_3 \dots x_{4t}$ and let V_1, V_2 consist of alternate vertex pairs on the circuit, i.e.,

$$V_1 = \{x_1, x_2\,;\, x_5, x_6\,;\, x_9, x_{10}\,;\, \dots\,;\, x_{4t-3}, x_{4t-2}\},$$

while

$$V_2 = \{x_3, x_4\,;\, x_7, x_8\,;\, x_{11}, x_{12}\,;\, \dots\,;\, x_{4t-1}, x_{4t}\}.$$

Then $G = \overline{C_{4t}}$ has $\delta(G) = 4t - 3$. In G no single vertex dominates V_1 but two neighbours do, so $\gamma_G(V_1) = \gamma_G(V_2) = 2$ and $\gamma_G(V_1) + \gamma_G(V_2) = 4 = \frac{4}{4t}4t = \frac{4}{\delta+3}|V(G)|$ as desired.

If $\delta \equiv 0, 2 \pmod 4$, choose k even if $\delta \equiv 0 \pmod 4$ and choose k odd if $\delta \equiv 2 \pmod 4$, and let $\overline{G}$ consist of two vertex-disjoint circuits $x_1x_2x_3 \dots x_k$ and $y_1y_2y_3 \dots y_k$ together with the edges x_iy_i, $1 \le i \le k$, and let $V_1 = \{x_1, x_2, \dots, x_k\}$, $V_2 = \{y_1, y_2, \dots, y_k\}$. Then $\delta(G) = 2k - 4$, $\gamma_G(V_1) = \gamma_G(V_2) = 2$ and $\gamma_G(V_1) + \gamma_G(V_2) = 4 = \frac{4}{2k}2k = \frac{4}{\delta+4}|V(G)|$ as desired.

If $\delta \equiv 3 \pmod 4$, let $\overline{G}$ consist of two circuits $x_1x_2x_3 \dots x_{2k}x_{2k+1}$ and $y_1y_2 \dots y_{2k}y_{2k+1}$ together with edges $x_iy_i, 1 \le i \le 2k$, and a vertex x_{2k+2} joined to x_{2k+1} and to y_{2k+1}. Let $V_1 = \{x_1, x_2, \dots, x_{2k}, x_{2k+1}, x_{2k+2}\}, V_2 = \{y_1, y_2, \dots, y_{2k+1}\}$. Then G has $|V(G)| = 4k + 3$, $k \ge 1$, $\delta(G) = 4k - 1$, $\gamma_G(V_1) = \gamma_G(V_2) = 2$. We obtain $\gamma_G(V_1) + \gamma_G(V_2) = 4 = \frac{4}{4k+3}(4k + 3) = \frac{4}{\delta+4}|V(G)|$ as desired and Theorem 4 is proven. ∎

We conjecture the converse of Theorem 4 to be true for not too large δ.

Conjecture 1. For any graph G with minimum valency $\delta(G) = \delta \le 5$ we have

$$f(G) \le \begin{cases} \dfrac{4}{\delta+3}|V(G)| & \text{for } \delta \equiv 1 (\text{mod } 4), \\ \dfrac{4}{\delta+4}|V(G)| & \text{for } \delta \not\equiv 1 (\text{mod } 4). \end{cases}$$

More generally, it would be interesting to determine the largest value of δ for which the construction above is best possible and the formula in Conjecture 1 is valid.

We cannot prove the conjecture, but we can prove a weaker statement.

Theorem 5. *Let G be a graph with minimum valency δ and let $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \emptyset$ be a partition of $V(G)$. Then $\gamma_G(V_1) + \gamma_G(V_2) \le \frac{\delta+1}{2\delta}|V(G)|$.*

Remark. If $0 < d \le \delta$, it follows that $\gamma_G(V_1) + \gamma_G(V_2) \le \frac{d+1}{2d}|V(G)|$, since $\frac{\delta+1}{2\delta} \le \frac{d+1}{2d}$.

First we need a lemma and some definitions.

A set of vertices $S \subseteq V(G)$ is called *distance*-2 *independent* in G if $d_G(u, v) > 2$ for every pair of distinct vertices u, v from S. Define the distance-2 independence number relative to G of a vertex set $X \subseteq V(G)$ to be

$$\alpha_2(X) = \max\{|S| \mid S \subseteq X, d_G(u,v) > 2 \ \forall u,v \in S\}\,.$$

If X is the entire vertex set, we simply write $\alpha_2(G)$ instead of $\alpha_2(V(G))$.

Lemma. *For any graph G and for any subset of vertices $X \subseteq V(G)$ we have $\gamma_G(X) \le \frac{1}{2}(|X| + \alpha_2(X))$.*

Proof. Consider the auxiliary graph $G^2[X]$ with vertex set X and with two vertices $x, x' \in X$ adjacent in $G^2[X]$ if and only if $0 < d_G(x,x') \le 2$. Let $e_i = x_i x_i'$, $1 \le i \le m$, be a maximal matching in $G^2[X]$. By definition there exist vertices $v_1, v_2, \ldots, v_m$ (not necessarily distinct) in $V(G)$ such that each v_i dominates both x_i and x_i'. Moreover, the vertices in X not incident with the e_i are distance-2 independent by maximality of the matching. Therefore, $\alpha_2(X) \ge |X| - 2m$ or $-m \le \frac{1}{2}(\alpha_2(X) - |X|)$ and hence $\gamma(X) \le m + (|X| - 2m) = |X| - m \le \frac{1}{2}(|X| + \alpha_2(X))$. This proves the lemma. ■

***Proof of Theorem* 5.** Applying the lemma in turn to V_1, V_2 and using $\alpha_2(V_i) \le \alpha_2(G)$ (since, also inside V_i, α_2 is defined in terms of distances in the entire G), we obtain

$$\gamma_G(V_1) + \gamma_G(V_2) \le \frac{1}{2}|V(G)| + \alpha_2(G)\,.$$

For another inequality, let S be a largest distance-2 independent set of vertices in G, $|S| = \alpha_2(G)$. We observe that $N[S]$ contains at least $(\delta+1)\,\alpha_2(G)$ vertices, and thus $V(G) - N[S]$ has at most $|V(G)| - (\delta+1)\,\alpha_2(G)$ vertices. For $i = 1, 2$ choose $D_i = S \cup \{(V(G) - N_G[S]) \cap V_i\}$; then $V_i \subseteq N_G[D_i]$ and

$$\begin{aligned} \gamma_G(V_1) + \gamma_G(V_2) &\le |D_1| + |D_2| \le 2\alpha_2(G) + |V(G)| - (\delta+1)\,\alpha_2(G) \\ &= |V(G)| - (\delta-1)\,\alpha_2(G)\,. \end{aligned}$$

Combining the two inequalities yields the desired result that $\gamma_G(V_1) + \gamma_G(V_2) \le \max\limits_{\alpha_2(G)\ge 0} \min\{|V(G)| - (\delta-1)\,\alpha_2(G),\ \frac{1}{2}|V(G)| + \alpha_2(G)\} \le \frac{\delta+1}{2\delta}|V(G)|$. ■

We conclude this section with asymptotically tight estimates on $f(G)$ in terms of minimum valency. Interestingly enough, both the lower and upper bounds are proved by probabilistic methods. Throughout, 'log' means logarithm of base e.

Theorem 6. *There exists a sequence of positive reals ϵ_d, tending to 0 as $d \to \infty$, such that*

$$(1-\epsilon_d)\,\frac{2\log d}{d} \le \sup_{G\colon\ \delta(G)\ge d-1} \frac{f(G)}{|V(G)|} < \frac{1+2\log d}{d}$$

holds for every natural number d.

Proof. *Upper bound.* Let $V(G)=\{v_1,\dots,v_n\}$. We begin with choosing a set $D_0 \subseteq V(G)$ at random, by the rule

$$\mathsf{Prob}(v_i \in D_0) = \frac{\log d}{d}$$

for each $i=1,\dots,n$ independently. Then we set

$$D_j = D_0 \cup \{v_i \in V_j \mid D_0 \cap N_G[v_i] = \emptyset\}$$

for $j=1,2$. Clearly, D_1 dominates V_1 and D_2 dominates V_2, moreover the expected cardinality of D_0 is $\frac{\log d}{d}\,n$. Hence, by the additivity of expectation, the upper bound will follow if we prove

$$\mathsf{Prob}(v_i \mid D_0 \cap N_G[v_i] = \emptyset) < \frac{1}{d}\,.$$

Indeed, the closed neighbourhood $N[v_i]$ of v_i contains at least d vertices by the minimum-valency condition, and each of them is chosen into D_0 independently with probability $\frac{\log d}{d}$. Thus,

$$\mathsf{Prob}(v_i \mid D_0 \cap N_G[v_i] = \emptyset) \le (1-\tfrac{\log d}{d})^d = \left((1-\tfrac{\log d}{d})^{\frac{d}{\log d}}\right)^{\log d} < e^{-\log d} = \frac{1}{d}\,.$$

Lower bound. Assume, without loss of generality, that d is a large even number. We let $n=d^2$, fix two disjoint sets V_1, V_2 of cardinality $d^2/2$ each, and take a random graph G with edge probability $1/d$ on the vertex set $V_1 \cup V_2$. For $j=1,2$ we will prove that, with probability $1-o(1)$ as $d\to\infty$, $\gamma_j := \gamma_G(V_j) \ge (1-o(1))\,d\log d$ holds. This will imply the lower bound of the theorem, because the minimum valency of G is equal to $d-o(d)$ almost surely, by the properties of the binomial distribution.

Consider V_1, and denote $m := \gamma_1$. (For V_2, the argument is analogous.) If $m \ge d\log d$, then the proof is done. Hence, assume $m < d\log d$. Let M be an arbitrary fixed m-element subset of $V(G)$. For any fixed $v \in V_1 \setminus M$,

$$\mathsf{Prob}(M \text{ does not dominate } v) = \left(1-\frac{1}{d}\right)^m,$$

therefore

$$\mathsf{Prob}(M \text{ dominates } v) = 1 - \left(1 - \frac{1}{d}\right)^m < 1 - e^{-\frac{m}{d-1}}\,.$$

Observe that these events are totally independent for the at least $\frac{1}{2}d^2 - m$ vertices of the entire set $V_1 \setminus M$. Consequently,

$$\mathsf{Prob}(M \text{ dominates } V_1) < \left(1 - e^{-\frac{m}{d-1}}\right)^{\frac{1}{2}d^2 - m}.$$

Considering all the possible $\binom{d^2}{m} < \left(\frac{ed^2}{m}\right)^m$ choices of M, we obtain

$$\mathsf{Prob}(\text{some } m\text{-set dominates } V_1) < \left(\frac{ed^2}{m}\right)^m \left(1 - e^{-\frac{m}{d-1}}\right)^{\frac{1}{2}d^2 - m}$$

$$= \exp\left(m\,(1 + \log d^2 - \log m) + (\frac{1}{2}d^2 - m)\log\,(1 - e^{-\frac{m}{d-1}})\right).$$

Since we assumed $m = \gamma_1$, the probability equals 1, to be exceeded by the last formula. Taking logarithm of the inequality and applying the fact $\log(1-x) < -x$,

$$m\,(1 + 2\log d - \log m) > \left(\frac{1}{2}d^2 - m\right) e^{-\frac{m}{d-1}}$$

follows, or, equivalently,

$$e^{\frac{m}{d-1}} > \frac{d^2 - 2m}{2m\,(1 + 2\log d - \log m)}.$$

Taking logarithm again, we conclude

$$m > (d-1)\left(\log(d^2 - 2m) - \log 2 - \log m - \log(1 + 2\log d - \log m)\right).$$

By our assumptions, here $\log(d^2 - 2m) = (2 - o(1))\log d$, $\log m = (1 + o(1))\log d$, while the last term is just $(1 + o(1))\log\log d$. Thus, $m \geq (1 - o(1))d\log d$ for $d \to \infty$, as claimed. ∎

More generally, an analogous argument yields the following result.

Theorem 7. *Let d and k denote natural numbers. There exists a sequence of positive reals ϵ_d (independent of k), tending to 0 as $d \to \infty$, such that*

$$(1-\epsilon_d)\,\frac{k\log d}{d} \leq \sup \frac{\gamma_G(V_1)+\cdots+\gamma_G(V_k)}{|V(G)|} \leq \frac{1+k\log d}{d}$$

where the supremum is taken over all graphs $G=(V,E)$ of minimum valency at least $d-1$ and over all vertex partitions $V_1\cup\cdots\cup V_k = V(G)$.

In the particular case of $k=1$ the upper bound coincides with the one known for the domination number of (non-partitioned) graphs [2, Theorem 2.18], and is asymptotically matched by the lower bound as d gets large.

5. Open Problems

In this concluding section we recall some problems that remain unsolved. Table 1 summarizes bounds conjectured or proved. Define S and T by $S = \limsup_{\delta(G)\to\infty} \frac{\gamma_G(V_1)+\gamma_G(V_2)}{|V(G)|}$ and $T = \limsup_{\delta(G)\to\infty} \frac{\gamma(G)+\gamma_G(V_1)+\gamma_G(V_2)}{|V(G)|}$ where the supremum is taken over all graphs G with minimum valency δ and all partitions V_1, V_2 of $V(G)$. From [2, Theorem 2.18] which states that any graph G with $\delta(G)=\delta$ satisfies $\gamma(G) \leq \frac{1+\ln(\delta+1)}{\delta+1}|V(G)|$, follows that $S=T=0$.

Table 1. Upper bounds conjectured and proved

$\delta(G)$	Conjecture 1	Theorem 5	Comments
1	$f(G) \leq \lvert V(G)\rvert$	$f(G) \leq \lvert V(G)\rvert$	trivially true
2	$f(G) \leq \frac{2}{3}\lvert V(G)\rvert$	$f(G) \leq \frac{3}{4}\lvert V(G)\rvert$	Conj. proven in Th. 3
3	$f(G) \leq \frac{4}{7}\lvert V(G)\rvert$	$f(G) \leq \frac{2}{3}\lvert V(G)\rvert$	
4	$f(G) \leq \frac{1}{2}\lvert V(G)\rvert$	$f(G) \leq \frac{5}{8}\lvert V(G)\rvert$	
5	$f(G) \leq \frac{1}{2}\lvert V(G)\rvert$	$f(G) \leq \frac{3}{5}\lvert V(G)\rvert$	
≥ 6		$f(G) \leq \frac{\delta+1}{2\delta}\lvert V(G)\rvert$	$\frac{f(G)}{\lvert V(G)\rvert}$ is "proportional" to $\frac{\log\delta}{\delta}$ in the sense of Th. 6

Define $s(\delta)$ and $t(\delta)$ by

$$s(\delta) = \limsup_{|V(G)|\to\infty} \frac{\gamma_G(V_1) + \gamma_G(V_2)}{|V(G)|}, \ t(\delta) = \limsup_{|V(G)|\to\infty} \frac{\gamma(G) + \gamma_G(V_1) + \gamma_G(V_2)}{|V(G)|}$$

where the supremum is taken over all graphs G with minimum valency δ and over all partitions V_1, V_2 of $V(G)$. Considering $G = \overline{K_n}$ we have $s(0) = 1$. If we consider graphs with no isolated vertex we get from $\gamma(V_i) \leq \gamma(G) \leq \frac{|V(G)|}{2}$, $i = 1, 2$, and the graphs $G = kK_2$ that $s(1) = 1$. If only graphs with $\delta(G) \geq 2$ are considered, Theorem 3 and $f(C_{3k}) = 2k$ yields $s(2) = \frac{2}{3}$. What can we say if only graphs with $\delta(G) \geq d$ are considered? Theorem 5 and its remark gives $s(\delta) \leq \frac{d+1}{2d}$. Will we have $s(\delta) = \frac{d+1}{2d}$ or $s(\delta) < \frac{d+1}{2d}$ for d small?

Similarly $t(0) = 2$ for $\delta = 0$ and $t(1) = \frac{3}{2}$ for $\delta = 1$. For connected graphs we have $t = \frac{5}{4}$ by [1, Theorem 2] while graphs with $\delta \geq 2$ by Theorem 2 have $t(2) \leq 1$, and in fact $t(2) = 1$, as is seen from the circuits C_{3k}.

Summing up, we ask the questions below.

Question 1. Which graphs G with minimum valency at least two attain the equality $f(G) = \frac{2}{3}|V(G)|$ in Theorem 3 ?

Question 2. For which values of δ is the construction of Theorem 4 optimal for S ?

Question 3. Consider graphs G with $\delta(G) = 3$ or $\delta(G) = 4$. Is $t(\delta)$ equal to 1 or strictly less than 1 ?

Bruce Reed has proven

Theorem 8 ([4])**.** *If a graph G has minimum valency at least* 3 *then* $\gamma(G) \leq \frac{3}{8}|V(G)|$.

We conjecture that $t(3) < 1$. This may be viewed as a weak version of Conjecture 1 since $t(3) < 1$ would follow from the truth of Conjecture 1 giving $\frac{\gamma_G(V_1)+\gamma_G(V_2)}{|V(G)|} \leq \frac{4}{7}$ combined with Theorem 8.

If we conjecture $t(d)$ to be strictly decreasing in d we shall have $t(3) < 1$ since we have earlier found that $t(2) = 1$. We only know that $t(3) \leq \frac{25}{24}$, as $\frac{\gamma_G(V_1)+\gamma_G(V_2)}{|V(G)|} \leq \frac{2}{3}$ by Theorem 4 and $\frac{\gamma(G)}{|V(G)|} \leq \frac{3}{8}$ by Theorem 8. By the same theorems $t(4) \leq 1$, as $\frac{\gamma_G(V_1)+\gamma_G(V_2)}{|V(G)|} \leq \frac{5}{8}$ and $\frac{\gamma(G)}{|V(G)|} \leq \frac{3}{8}$. We have $t(5) < 1$ since $t(5) \leq \frac{\gamma_G(V_1)+\gamma_G(V_2)}{|V(G)|} + \frac{\gamma(G)}{|V(G)|} \leq \frac{6}{10} + \frac{3}{8}$.

Finally, in connection with the case of $\delta = 5$, we raise the following

Conjecture 2. In every 6-uniform 3-regular hypergraph on n vertices there exists a set of at most $n/4$ vertices that meets all edges.

Note that the edge set of such a hypergraph consists of precisely $n/2$ 6-tuples, i.e., the so-called transversal number should be proven not to exceed half of the number of edges.

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