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ON k-TRESTLES IN POLYHEDRAL GRAPHS

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Abstract

A k-trestle of a graph G is a 2-connected spanning subgraph of G of maximum degree at most k. We show that a polyhedral graph G has a 3-trestle, if the separator-hypergraph of G contains no two different cycles joined by a path of 3-separators of length ≥ 0. There are graphs not satisfying this condition that have no 3-trestles. Further, for each integer k every graph with toughness smaller than $\frac{2}{k}$ has no k-trestle.

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1. Introduction

By Steinitz's theorem a polyhedral graph is a planar and 3-connected graph. Let G be a connected graph. A subset S of the vertex set of G *separates* G if the graph $G - S$ obtained from G by deleting the vertices of S is disconnected. If $|S| = k$, S is said to be *a k-separator of G*. If no

$S_p \subset S$ (a proper subset of the set S) separates G then the S is said to be *a proper k-separator of G*. A subgraph of G is *a spanning subgraph of G* if it contains all vertices of G. 2-connected spanning subgraphs in which all vertices have degree at most k are called *k-trestles*. We will say that a graph G is *k-trestled* if G has a k-trestle [6]. Note that a graph G has a 2-trestle if and only if G is Hamiltonian.

A graph G is said to be *t-tough* if for every separating set $S \subseteq V(G)$ the number $\omega(G-S)$ of components of $G-S$ is at most $\frac{|S|}{t}$. The toughness $\tau(G)$ of a non-complete graph G is defined to be the largest integer $t > 0$ such that G is t-tough. For a complete graph G let $\tau(G) = \infty$. The concept of toughness was introduced by Chvátal [4]. It is easy to see that every graph with toughness less than one has no 2-trestles. The following Lemma shows that every graph has a similar property with respect to k-trestles, $k \geq 3$.

Lemma 1. *Every graph G with toughness $\tau(G) < \frac{2}{k}$ (where the integer k is greater than one) has no k-trestle.*

In [4] Chvátal conjectured:

Conjecture 1 (Chvátal)**.** There is a real number $t_0 > 0$ such that every t_0-tough graph has a Hamiltonian cycle, i.e., a 2-trestle.

It seems to be interesting to consider relations between t-tough and k-trestled graphs in general. We pose the following conjecture.

Conjecture 2. For every integer k greater than one there is a real number $t_k > 0$ such that every t_k-tough graph has a k-trestle.

There are several papers which deal with k-trestled polyhedral graphs. In [1] Barnette showed that there is a polyhedral graph with no 5-trestles. In [5] Gao proved that every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle.

The well known theorem of Tutte [8] states that every 4-connected planar graph contains a Hamiltonian cycle, which means that every polyhedral graph with no 3-separators has a 2-trestle. Moreover, Tutte [8] proved

Theorem 1. *Let G be a 4-connected planar graph and let e and f be two edges of a facial cycle of G. Then G has a Hamiltonian cycle through e and f.*

Let H_1 and H_2 be two disjoint subsets of the vertex set $V(G)$ of a graph G. The length of a minimal path in G with one end in H_1 and the second in H_2 is said to be *the distance of H_1 and H_2 in G*.

Böhme, Harant and Tkáč in [3] showed that every maximal planar graph G in which no 3-separator has any common vertex with a proper 4-separator and every two distinct 3-separators have distance at least three, has a 2-trestle. In [2] Böhme and Harant presented examples of maximal planar graphs with no 2-trestles in which the minimal distances between two 3-separators are arbitrarily large.

Our next theorems partially supplement these results but in a more general case.

For each polyhedral graph G we will construct a separator-hypergraph $\mathcal{H}(G)$ with the same set of vertices, such that the edges of $\mathcal{H}(G)$ are the 3-separators of G. A cycle (and a path) of a hypergraph is a sequence $P_1e_1P_2e_2\cdots P_ke_kP_{k+1}$, where $P_1, P_2, \cdots, P_k$ are pairwise distinct vertices, $e_1, e_2, \cdots, e_k$ are pairwise distinct edges, the edge e_i is incident with both P_i and P_{i+1}, $1 \leq i \leq k$, and $P_{k+1} = P_1$ (and $P_{k+1} \notin \{P_1, P_2, \cdots, P_k\}$, respectively).

Theorem 2. *Let G be a polyhedral graph. Let each component of the separator-hypergraph $\mathcal{H}(G)$ have at most one cycle. Then G has a 3-trestle.*

Theorem 3. *There are polyhedral graphs with more than one cycle in their separator-hypergraph which have no 3-trestles.*

The polyhedral graphs constructed for Theorem 3 have separator-hypergraphs with many cycles; even 2-cycles are present.

2. Proofs of Theorems

***The Proof of Lemma* 1.** Let G be a graph with toughness $\tau(G) < \frac{2}{k}$ (where the integer k is greater than one). Suppose that G has a k-trestle H. Since $\tau(G) < \frac{2}{k}$ there exists a subset S_0 of the vertex set of G ($S_0 \subset V(G)$) with

$$\frac{|S_0|}{\omega(G - S_0)} = \tau(G) < \frac{2}{k}.$$

So G contains a vertex set S_0 such that

$$2\omega(G - S_0) > k|S_0|.$$

If G has a k-trestle H then $S_0 \subset V(G) = V(H)$ and every vertex from S_0 has in H a degree at most k. Since H is 2-connected, every component of $G - S_0$ is adjacent with at least two vertices from S_0. This means that the following inequality holds

$$2\omega(G - S_0) \leq k|S_o|.$$

But this contradicts the before stated inequality.

Instead of Theorem 2 we shall prove the slightly stronger but more technical Theorem 4.

Theorem 4. *Let G be a polyhedral graph. Let each component of the separator-hypergraph $\mathcal{H}(G)$ have at most one cycle. Label a vertex in each cycle-free component of $\mathcal{H}(G)$. Then G has a 3-trestle H such that every 3-valent vertex of H is an unlabelled vertex of a 3-separator in G.*

***The Proof of Theorem* 4.** The proof is by induction on the number of 3-separators of the considered graphs. If G has no 3-separator then G is 4-connected and by Tutte's Theorem 1 the graph G has a Hamiltonian cycle. Thus G has a special 3-trestle with the required properties.

Assume that Theorem 4 is true for all polyhedral graphs with at most m 3-separators, $m \geq 0$. Let G be a polyhedral graph with $m + 1$ 3-separators such that each component of the "separator"-hypergraph $\mathcal{H}(G)$ has at most one cycle.

A 3-separator $S = \{x, y, z\}$ is called *elementary* if one component $I(S)$ of $G - S$ has no 3-separators. W.l.o.g. we may suppose that G is mapped into the plane so that $I(S)$ is the interior of the cycle (x, y, z). Now we prove the following

Claim 1. If $S = \{x, y, z\}$ is an elementary 3-separator of G then $\langle I(S) \cup S \rangle_G$, the subgraph induced by $I(S) \cup S$ in G, contains an x, y-path through all vertices of $I(S) \cup S \setminus \{z\}$ avoiding z.

***Proof of Claim* 1.** Since $S = \{x, y, z\}$ is elementary the subgraph $H := \langle I(S) \cup S \rangle_G \cup (x, y, z)$ has no 3-separators and H is 4-connected or K_4 (a complete graph on four vertices). By Tutte's Theorem 1 the subgraph H has a Hamiltonian cycle h through the edges (x, z) and (z, y). The path $p = h \setminus \{z\}$ has the required properties, and the proof of Claim 1 is complete.

The graph G obviously contains an elementary 3-separator $S = \{x, y, z\}$. This 3-separator S is a hyperedge of a component K of $\mathcal{H}(G)$.

Case 1. Let K have no cycle in $\mathcal{H}(G)$.
The subhypergraph $K \setminus \{S\}$ of $\mathcal{H}(G)$ has at most three cycle-free components K_x, K_y and K_z containing x, y, and z, respectively. Note that some of these components can be trivial. W.l.o.g. let K_x have the vertex with the label of K (it may be that x has this label). In K_y and K_z we label the vertices y and z, respectively.

Case 2. Let K have a cycle C in $\mathcal{H}(G)$.
Note that K has no label.

Case 2.1. Let $S \notin C$.
The subhypergraph $K \setminus \{S\}$ of $\mathcal{H}(G)$ has at most three components K_x, K_y and K_z containing x, y and z, respectively. W.l.o.g. let $C \subseteq K_x$, and K_y, K_z are cycle-free in $\mathcal{H}(G)$. In K_y and K_z we label the vertices y and z, respectively.

Case 2.2. Let $S \in C$.
Two vertices of S belong to C, say, x and y. The subhypergraph $K \setminus S$ of $\mathcal{H}(G)$ has at most two components $K_{x,y}$ and K_z containing $\{x, y\}$ or $\{z\}$, respectively. The path $C \setminus \{S\} \subseteq K_{x,y}$ and both components $K_{x,y}$ and K_z are cycle-free in $\mathcal{H}(G)$. We label y and z.
In all cases we proceed in the same way.

The graphs G_1 and G_2 are obtained from G by deleting the interior or the exterior of (x, y, z), respectively, and adding the cycle (x, y, z). Thus G has a separation: $G = G_1 \cup G_2$, $G_1 \cap G_2 = (x, y, z)$, $K \setminus \{S\} \subseteq G_1$.

By the induction hypothesis G_1 contains a 3-trestle T_1 with the required properties. The degrees $\deg_{T_1}(y) = \deg_{T_1}(z) = 2$.

By Claim 1 the subgraph G_2 contains a y, z-path T_2 through all vertices of $G_2 \setminus \{x\}$ avoiding x. Then $T_1 \cup T_2$ is a 3-trestle of G with the required properties. ■

***The Proof of Theorem* 3.** Theorem 3 will be proved by constructing an appropriate graph. A double-cube is obtained from two disjoint copies C_1 and C_2 of the cube by identifying a face of C_1 with a face of C_2. This polyhedral graph has $n = 12$ vertices and $f = 10$ quadrangles. In each quadrangle with bounding 4-cycle (v_0, v_1, v_2, v_3) we introduce a 4-cycle

(w_0, w_1, w_2, w_3) so that for every $i \pmod 4$ a vertex v_i is connected with w_i and w_{i+1} by an edge, introduce a new vertex α_i in each triangle face with bounding cycle (v_i, v_{i+1}, w_{i+1}) and join α_i to each vertex of the bounding 3-cycle (v_i, v_{i+1}, w_{i+1}) by an edge.

The resulting graph H is polyhedral and its connected separator-hypergraph has more than one cycle.

We claim that H has no 3-trestle.

Suppose H has a 3-trestle T. By construction each vertex α_i is joined to the vertex v_i or v_{i+1} of the double-cube by at least one edge of T. Thus the subgraph T has at least $4f$ such edges. Consequently, the double-cube has at least one vertex v of degree

$$\deg_T(v) \geq \frac{4f}{n} = \frac{40}{12} > 3.$$

Hence v has a degree $\deg_T(v) \geq 4$ and T is no 3-trestle. This contradiction shows that H has no 3-trestle.

Starting our construction with $l \geq 3$ cubes results in an infinite sequence of graphs satisfying Theorem 3. ■

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