Discussiones Mathematicae
Graph Theory 22 (2002) 159–172

ON WELL-COVERED GRAPHS OF ODD GIRTH 7 OR GREATER

Bert Randerath

Institut für Informatik
Universität zu Köln
D-50969 Köln, Germany

e-mail: randerath@informatik.uni-koeln.de

and

Preben Dahl Vestergaard

Mathematics Department
Aalborg University
DK-9220 Aalborg Ø, Denmark

e-mail: pdv@math.auc.dk

Abstract

A maximum independent set of vertices in a graph is a set of pairwise nonadjacent vertices of largest cardinality α. Plummer [14] defined a graph to be *well-covered*, if every independent set is contained in a maximum independent set of G. One of the most challenging problems in this area, posed in the survey of Plummer [15], is to find a good characterization of well-covered graphs of girth 4. We examine several subclasses of well-covered graphs of girth ≥ 4 with respect to the *odd girth* of the graph. We prove that every isolate-vertex-free well-covered graph G containing neither C_3, C_5 nor C_7 as a subgraph is even very well-covered. Here, a isolate-vertex-free well-covered graph G is called *very well-covered*, if G satisfies $\alpha(G) = n/2$. A vertex set D of G is dominating if every vertex not in D is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum order of a dominating set of G. Obviously, the inequality $\gamma(G) \leq \alpha(G)$ holds. The family $\mathcal{G}_{\gamma=\alpha}$ of graphs G with $\gamma(G) = \alpha(G)$ forms a subclass of well-covered graphs. We prove that every connected member G of $\mathcal{G}_{\gamma=\alpha}$ containing neither C_3 nor C_5 as a subgraph is a K_1, C_4, C_7 or a corona graph.

Keywords: well-covered, independence number, domination number, odd girth.

2000 Mathematics Subject Classification: 05C70, 05C75.

1. Introduction and Notation

We consider finite, undirected, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For $A \subseteq V(G)$ let $G[A]$ be the subgraph induced by A. $N(x) = N_G(x)$ denotes the set of vertices adjacent to the vertex x and $N[x] = N_G[x] = N(x) \cup \{x\}$. More generally, we define $N(X) = N_G(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N_G[X] = N(X) \cup X$ for a subset X of $V(G)$. The vertex v is called an end vertex if $d(v,G) = 1$, and an isolated vertex if $d(v,G) = 0$, where $d(x) = d(x,G) = |N(x)|$ is the degree of $x \in V(G)$. Let $\Omega = \Omega(G)$ be the set of end vertices of G. An edge incident to an end vertex is called a pendant edge. We denote by $n = n(G) = |V(G)|$ the order of G. We write C_n for a circuit of length n and K_n for the complete graph of order n. A subgraph $\mathcal{F}$ of G with $V(\mathcal{F}) = V(G)$ is called a factor of G. Furthermore, a factor $\mathcal{F}$ of G is a perfect $[1,2]$-factor if every component of $\mathcal{F}$ is either a circuit or a K_2. The corona $G \circ K_1$ of a graph G is the graph obtained from G by adding a pendant edge to each vertex of G. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest circuit in G. The girth is ∞ if G has no circuit. The *odd girth* of a graph G is the length of a shortest odd circuit in G, it is ∞ if G is bipartite.

A maximum independent set of vertices in a graph is a set of pairwise nonadjacent vertices of largest cardinality. The cardinality $\alpha(G)$ of a maximum independent set in a graph G is called the independence number of G. Plummer [14] defined a graph to be *well-covered*, if every independent set is contained in a maximum independent set of G. These graphs are of interest because, whereas the problem of finding the independence number of a general graph is NP-complete, the maximum independent set can be found easily for well-covered graphs by using a simple greedy algorithm. Chvátal, Slater [4] and Sankaranarayana, Steward [17] independently showed that the property of being not well-covered is NP-complete. Hence, it is unlikely that there exists a good characterization of well-covered graphs.

The work on well-covered graphs appearing in literature (see [15]) has focused on certain subclasses of well-covered graphs. Finbow, Hartnell and Nowakowski [7], [8] characterized the well-covered graphs G of girth ≥ 5, i.e., G contains neither C_3 nor C_4 as a subgraph, and also the well-covered graphs G containing neither C_4 nor C_5 as a subgraph. Both sets of forbidden subgraphs are subsets of the set $\{C_3, C_4, C_5, C_7\}$, which precisely are all well-covered circuits. One of the most challenging problems in this area, posed in the excellent survey of Plummer [15], is to find a good characterization of well-covered graphs of girth ≥ 4, (i.e., G contains no C_3 as a subgraph).

We investigate several subclasses of well-covered graphs of girth ≥ 4 with respect to the odd girth of a graph. Our main interest are the well-covered graphs of odd girth ≥ 7, (i.e., G contains neither C_3 nor C_5 as a subgraph). It is well known (e.g. see [3], [15]) that any well-covered graph with n vertices, none of which is isolated, has $\alpha(G) \leq n/2$. An isolate-vertex-free well-covered graph G of order n with $\alpha(G) = n/2$ is called *very well-covered*. Staples [18] and Favaron [5] independently characterized this subclass of well-covered graphs. We prove that every isolate-vertex-free well-covered graph G with odd girth ≥ 9 (i.e., G contains no well-covered odd circuit as a subgraph) is also very well-covered.

A vertex set D of G is dominating if every vertex not in D is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum order of a dominating set of G. Obviously, the inequality $\gamma(G) \leq \alpha(G)$ holds. The family $\mathcal{G}_{\gamma=\alpha}$ of graphs G with $\gamma(G) = \alpha(G)$ form a subclass of the class of well-covered graphs. In 1970, Szamkołowicz [19] posed the problem of characterizing graphs G of $\mathcal{G}_{\gamma=\alpha}$. Very little is known about a characterization of such graphs. E.g. Topp and Volkmann [20] characterized all bipartite members of $\mathcal{G}_{\gamma=\alpha}$. We generalize their result by showing that every connected member G of $\mathcal{G}_{\gamma=\alpha}$ with odd girth ≥ 7 satisfies $\gamma(G) = n/2$ or is a 7-circuit or is a K_1.

2. Preliminaries

The following two observations (e.g. see [15]) are very useful in the subsequent proofs.

Observation 1. *Let I, J be two vertex sets in a graph G such that I is independent, $|J| < |I|$ and $N_G[I] \subseteq N_G[J]$. Then G satisfies $\gamma(G) < \alpha(G)$. Moreover, if J is likewise independent, then G is not well-covered.*

Observation 2. *If G is a well-covered graph and I is an independent set of G, then $G' = G - N_G[I]$ is also well-covered and $\alpha(G') = \alpha(G) - |I|$.*

It is easy to see that Observation 1 and 2 remain true if the property *well-covered* is replaced by having the property that $\gamma(G) = \alpha(G)$. The next result provides a more powerful tool, if we have additional information.

Lemma 3. *Let G be a well-covered graph, I an independent set of vertices in G and $G' = G[N_G[I]]$. Then*

1. *if $\alpha(G') = |I|$, then G' is also well-covered;*
2. *if G has no isolated vertex and $|N_G(I)| \leq |I|$, then G' is very well-covered.*

Proof. (1). Since $\alpha(G') = |I|$ we only have to show that there exists no maximal independent set J of G' with $|J| < |I|$. Otherwise, assume J is a maximal independent set of G' with $|J| < |I|$, then $N_G[I] \subseteq N_G[J]$ and by Observation 1 we obtain that G is not well-covered, a contradiction.

(2). Suppose G has no isolated vertex and we have $|N_G(I)| \leq |I|$. Berge [3] showed that for every independent set I of a well-covered graph G without isolated vertice, $|N_G(I)| \geq |I|$. Hence, we have $|N_G(I)| = |I|$. By Observation 1 we deduce that there exists no maximal independent set J of G' with $|J| < |I|$. Now assume there exists a maximal independent set J of G' with $|J| > |I|$. Note that $\emptyset \neq I \cap J \neq I$, $\emptyset \neq N_G(I) \cap J \neq N_G(I)$ and $|I - \{I \cap J\}| < |N_G(I) \cap J|$. With J independent we obtain $N_G(I \cap J) \cap \{N_G(I) \cap J\} = \emptyset$, i.e., $N_G(I \cap J) \subseteq N_G(I) - \{N_G(I) \cap J\}$. By rearranging

$$\begin{aligned} |I \cap J| &= |I| - |I - \{I \cap J\}| \\ &> |N_G(I)| - |N_G(I) \cap J| \\ &\geq |N_G(I \cap J)| \end{aligned}$$

we get a contradiction to Berge's result. Thus every maximal independent set of G' has cardinality $\alpha(G') = |I| = n(G')/2$, i.e., G' is very well-covered. ∎

Observation 4. *Let G be a graph of odd girth at least $2l+1$ and x a vertex of G. Then all vertices having distance exactly i to x with $1 \leq i < l$ form an independent set of G.*

3. Well-Covered Graphs of odd Girth 7 or Greater

Staples [18] and Favaron [5] independently characterized the family of very well-covered graphs. Finbow and Hartnell [6] proved that a well-covered graph without isolated vertices and with girth at least 8 is very well-covered. In the next theorem we state that it is enough to demand odd girth ≥ 9, i.e., C_4, C_6 are not forbidden.

Theorem 5. *Let G be well-covered with no isolated vertex and odd girth ≥ 9, then G is very well-covered.*

Proof. Let G be a well-covered graph with no isolated vertex and odd girth ≥ 9, such that G is not very well-covered and every isolate-vertex-free subgraph $G' = G - N_G[I]$, with I being an independent vertex set of G, is very well-covered. Now let $x \in V(G)$ and $J_i(x)$ denote the set of all vertices having distance exactly i from x. Because of Observation 4 each of the sets $J_1(x), J_2(x)$ and $J_3(x)$ are independent. The graph $G' = G - N_G[J_3(x)]$ is because of Observation 2 well-covered. Let $J_2'(x)$ be the subset of $J_2(x)$ containing all vertices of distance 2 from x, which are not adjacent to any vertex of $J_3(x)$. One (well-covered !) component of $G' = G - N_G[J_3(x)]$ is $G_x = G[N[x] \cup J_2'(x)]$. Note that the well-covered bipartite graph G_x is by the choice of G very well-covered, i.e., $\alpha(G_x) = n(G_x)/2 = |J_1(x)| = |\{x\} \cup J_2'(x)|$. (Since G is not very well-covered that implies $J_3(x) \neq \emptyset$.) Likewise $G'' = G - N_G[\{x\} \cup J_2'(x)]$ is well-covered, contains no isolated vertex, has odd girth ≥ 9 and satisfies because of G's choice $\alpha(G'') = n(G'')/2$. Observe that $n(G) = n(G'') + n(G_x)$ and $J_3(x)$ is contained in a maximum independent set A of G''. Moreover, the set $I = A \cup \{x\} \cup J_2'(x)$ is a maximum independent set of G with $|I| = n(G'')/2 + n(G_x)/2$, i.e., $\alpha(G) = n(G)/2$, a contradiction. ■

We now examine the members of the family of connected, well-covered graphs with odd girth at least 7, which are not very well-covered. In order to have a self-contained proof for Theorem 5 we required the special choice of G to deduce that the subgraph G_x is very well-covered. Alternatively, as G_x is both well-covered and bipartite, it is also very well-covered. From the proof of Theorem 5 we see that

successive removals of G_x for vertices x belonging to no C_7 produce decreasingly smaller graphs which are well-covered but not very well-covered. Hence, in this case it is possible to 'reduce' G. Moreover, if there exists an independent set I of G with $|N_G(I)| = |I|$, then we can also apply Lemma 3 *in order to 'reduce' G.*

We now examine the members of the family of connected, well-covered graphs with odd girth at least 7, which are not very well-covered. Finbow, Hartnell and Nowakowski [7] proved that a connected, well-covered graph with girth at least 6 is very well-covered or is one of C_7, K_1. Hence, if 4-circuits are not permitted there are only two exceptional graphs. But we will outline that allowing 4-circuits enlarges this class of exceptional graphs, i.e., the members of the family of connected, well-covered graphs with odd girth at least 7, which are not very well-covered. The following observation is an easy consequence of two results due to Berge [3] and Tutte [21]. Berge showed that for every independent set I of an isolate-vertex-free well-covered graph G we have $|N_G(I)| \geq |I|$, but then also the König-Hall condition — $|N_G(S)| \geq |S|$ for all subsets S of $V(G)$ — holds. Finally it is due to Tutte that the König-Hall condition is equivalent to the existence of a perfect $[1,2]$-factor $\mathcal{F}$.

Observation 6. *If G is an isolate-vertex-free well-covered graph, then G contains a perfect $[1,2]$-factor $\mathcal{F}$.*

Note that then there exists also a perfect $[1,2]$-factor $\mathcal{F}$ of G, such that $\mathcal{F}$ only contains induced odd circuits and K_2's. A canonical problem now is to examine the family of isolate-vertex-free well-covered graphs G, such that there exists a perfect $[1,2]$-factor $\mathcal{F}$ of G with $\alpha(\mathcal{F}) = \alpha(\mathrm{G})$. The core of the following conjecture is that all isolate-vertex-free well-covered graphs of odd girth ≥ 7 are contained in this subclass of the well-covered graphs.

Conjecture 7. *Let G be an isolate-vertex-free graph of odd girth at least 7. Then G is well-covered if and only if*

- *there exists a perfect $[1,2]$-factor $\mathcal{F}$ of G, such that $\mathcal{F}$ only contains (induced) 7-circuits and K_2's. Furthermore, we have $\alpha(\mathcal{F}) = \alpha(G)$.*
- *if C_1 and C_2 are two vertex-disjoint 7-circuits of $\mathcal{F}$, then there are*
  1. *either $G[V(C_1) \cup V(C_2)] = C_1 \cup C_2$;*
  2. *or $G[V(C_1) \cup V(C_2)] = C_7[2K_1]$;*
  3. *or there exist two vertices x_1, x_2 of distance 2 of C_1 and two vertices y_1, y_2 of distance 2 of C_2, such that these vertices induce a 4-circuit and these are the only edges between the two circuits C_1 and C_2.*
- *the set of vertices of the K_2-components induces a very well-covered graph.*
- *there exists a well-covered, isolate-vertex-free graph G^* of odd girth at least 7 and an independent vertex set I of G^*, such that we have $G = G^* - N_{G^*}[I]$ and there exists a perfect $[1,2]$-factor $\mathcal{F}$ of G^* only containing induced 7-circuits.*

If Conjecture 7 is true, then every isolate-vertex-free well-covered graph G of odd girth ≥ 7 satisfies $\alpha(G) \geq \frac{3}{7}n(G)$. Note that this result also holds for the related family of graphs G of odd girth ≥ 7 with $\delta(G) > n(G)/4$ as shown by Albertson, Chan and Haas [1].

4. On the Subclass $\mathcal{G}_{\gamma=\alpha}$

4..1 On members of $\mathcal{G}_{=}$ with odd girth ≥ 7

For the next theorem we need a characterization of isolate-vertex-free graphs G with $\gamma(G) = n(G)/2$, which is due to Payan and Xuong [11] and independently Fink, Jacobson, Kinch and Roberts [9]. As a considerable extension of this result Randerath and Volkmann [16] (see also [2] for a different proof) characterized all extremal graphs in the well-known inequality of Ore [10], i.e., they determined the connected graphs with $\gamma(G) = \lfloor n(G)/2 \rfloor$.

Proposition 8 [11], [9]**.** *Let G be a connected graph of order $n = n(G) \geq 2$. Then $\gamma(G) = n/2$ if and only if $G = C_4$ or $G = H \circ K_1$ for some arbitrary connected graph H.*

In next theorem we study another subclass of the well-covered graphs of odd girth ≥ 7.

Theorem 9. *Let G be a connected graph of order $n = n(G) \geq 2$ and odd girth at least 7. Then $\gamma(G) = \alpha(G)$ if and only if $G = C_4, C_7$ or $G = H \circ K_1$ where H is a connected graph of odd girth at least 7.*

Proof. By inspection we see that the graphs $C_4, C_7, H \circ K_1$ have $\gamma = \alpha$. Conversely, assume G is a graph of smallest order $n = n(G) \geq 2$ such that G is connected, has odd girth ≥ 7, has $\gamma(G) = \alpha(G)$ and G is neither C_4, C_7 nor a corona graph, or equivalently $\gamma(G) = \alpha(G) < n(G)/2$ and $G \neq C_7$. We shall prove that no such G exists by deriving a contradiction. Thus, by the minimality of n, any graph G' with $n(G') = n'$ vertices, $2 \leq n' < n$, which is connected, has odd girth ≥ 7 and satisfies $\gamma(G') = \alpha(G')$ has this common value equal to $n'/2$ or is a C_7. Recall, that Observation 2 also holds for the subclass of well covered graphs fulfilling $\gamma = \alpha$. If $\delta(G) = 1$, let $y \in V(G)$ have degree one neighbours $I = \{x_1, x_2, \ldots, x_k\}, k \geq 1$. By Observation 2, $G' = G - N_G[I]$ fulfills $\gamma(G') = \alpha(G')$, thus each component G''_i, $1 \leq i \leq l$, of G' also has $\gamma(G''_i) = \alpha(G''_i)$. G is dominated by y together with a dominating set from each of the l components G''_i, $1 \leq i \leq l$, so $\gamma(G) \leq 1 + \sum_{i=1}^{l} \gamma(G''_i)$. From I together with l maximum independent sets of each G''_i we obtain that

$k + \sum_{i=1}^{l} \alpha(G_i'') \leq \alpha(G)$; combining this with $\gamma(G_i'') = \alpha(G_i'')$, for $1 \leq i \leq l$ and $\gamma(G) = \alpha(G)$ we find that $k = 1$. Next we obtain a contradiction to $\gamma(G) = \alpha(G) < n/2$. If $\gamma(G') = \alpha(G') = \frac{n(G')}{2}$, then with $n = n' + 2$ we deduce $\gamma(G) = \alpha(G) = \frac{n(G)}{2}$, a contradiction. Therefore, $\gamma(G') = \alpha(G') < \frac{n(G')}{2}$ and by the minimality of n, G' has to contain a 7-circuit C as a component. Note that at least one vertex $z_1 \in V(C)$ is adjacent to y. Now it is easy to see that there exists an independent set I containing the endvertex x_1 and three pairwise nonadjacent vertices of the 7-circuit C and a vertex set J containing the vertex y and two nonadjacent vertices of the 7-circuit C satisfying the property $N_G[I] \subseteq N_G[J]$. With $\gamma(G) = \alpha(G)$ we get a contradiction by Observation 1.

Therefore, we have $\delta(G) \geq 2$. Consider a vertex x of minimum degree $\delta = \delta(G)$ and denote by I_x the union of x and the set of isolated vertices of $G - N_G[x]$. If $G' = G - N_G[I_x]$ is the empty graph, G is bipartite with vertex classes I_x and $N(x)$. Since G is well covered we find that $|I_x| = |N(x)| = \delta$ and $G = K_{\delta,\delta}$. That implies $\gamma(G) = 2$, $\alpha(G) = \delta$, and hence $G = C_4$ against the hypothesis; consequently, G' is not the empty graph. Since G' contains no isolated vertex and for each i, $1 \leq i \leq l$, the G'-component G_i'' has odd girth ≥ 7 and $\gamma(G_i'') = \alpha(G_i'')$ we obtain that, by the minimality of n, G_i'' is a C_4, a C_7 or a corona graph. Now suppose G' contains a circuit-component, say G_1'' is a C_4 or a C_7. Then again applying Observation 1 a simple case by case analysis of the corresponding graph G''' produces a contradiction. Thus, each component of G' is a corona graph $G'' = H'' \circ K_1$. Because G has odd girth at least 7 and $\delta \geq 2$ we can drop the case that G' contains K_2 as a component. Thus, we also deduce $n' \geq 4$. Let uv be an edge in H'' and let u' and v' be their respective neighbours in $\Omega(G'')$. Each of u' and v' has exactly one neighbour in H'' and hence at least $\delta - 1$ neighbours in $N(x)$. If $\delta(G) \geq 3$, as $|N(x)| = \delta$, some vertex in $N(x)$ is adjacent to both u' and v', that creates a C_5 against the assumption that G has odd girth at least 7. So $\delta(G) = 2$ and G contains a C_7. We have $I_x = \{x\}$, otherwise we can deduce that $\gamma(G) < \alpha(G)$. In G the set $\{x\} \cup \Omega(G')$ is maximum independent with $\alpha(G) = \frac{n'}{2} + 1$ vertices. Let H' denote the union of all H''. Since H'' has no isolated vertex we have by a result of Ore that each $\gamma(H'') \leq \frac{1}{2}|V(H'')|$ and hence $\gamma(H') \leq \frac{1}{2}\frac{n'}{2}$. Let D' be a dominating set of H' with $|D'| \leq \frac{n'}{4}$. Each vertex in $\Omega(G')$ is joined to precisely one vertex in H' and to at least one vertex in $N(x)$. Thus, the two vertices in $N(x)$ together with D' dominate G and $\gamma(G) \leq 2 + \frac{n'}{4}$.

From $\alpha(G) = \gamma(G)$ we obtain $\frac{n'}{2} + 1 \leq \frac{n'}{4} + 2$ and by rearranging $\frac{n'}{4} \leq 1$, implying $n' = 4$. Moreover, we have $G' = P_4$ and $G = C_7$, a contradiction to our hypothesis. ■

4..2 On triangle-free members of $\mathcal{G}$ $_=$

An interesting subproblem of Szamkołowicz's $\mathcal{G}_{\gamma=\alpha}$-problem and Plummer's well-covered-girth-4-problem is to FIND A GOOD CHARACTERIZATION OF ALL GRAPHS OF $\mathcal{G}_{\gamma=\alpha}$ WITH GIRTH ≥ 4, i.e., find a good characterization of all triangle-free graphs G satisfying $\gamma(G) = \alpha(G)$.

Description of a family of triangle free well covered graphs $(G_j)_{j\in N}$:

For $j \geq 1$ let G_j be the j-regular graph on $3j-1$ vertices described by

$V(G) = \{v_1, v_2 \ldots, v_{3j-1}\}$,

$N(x_i) = \{v_{i+j}, v_{i+j+1}, \ldots, v_{i+2j-1}\}, 1 \leq i \leq 3j-1$, indices are added modulo $3j-1$, so that $v_{3j} = v_1, v_{3j+1} = v_2$ etc.

The first three graphs in this family are $G_1 = K_2, G_2 = C_5$ and $G_3 = ML_8$, the Möbius ladder on 8 vertices. Note that these graphs are circulants. We can easily establish that $\alpha(G_j) = j$ and that the maximal independent sets in G_j precisely are the $3j-1$ neighbourhood sets $N(x_i), 1 \leq i \leq 3j-1$, each consisting of j vertices, so G_j is well covered. For $j = 1, 2, 3$ we see that $\gamma(G_j) = \alpha(G_j)$ but for $j \geq 4$ we have that $\{v_1, v_{j+1}, v_{2j+1}\}$ dominates G_j and hence that $3 = \gamma(G_j) < \alpha(G_j) = j$. We shall use $G_j, j = 1, 2, 3$, in the construction of $\mathcal{H}^+$ below.

Szamkołowicz asked for a characterization of graphs with $\gamma = \alpha$. In Theorem 9 we gave an answer for graphs with odd girth ≥ 7.

In addition we shall now construct $\mathcal{H}^+$, a family of graphs in $\mathcal{G}_{\gamma=\alpha}$:

Let H be a graph with vertex set $V(H) = \{a_1, a_2, \ldots, a_k, b_1, b_1', b_2, b_2', \ldots, b_\ell, b_\ell'\}$, ($k = 0$ or $\ell = 0$ may occur), and $E(H)$ is any set of edges such that

(1) $b_i b_i' \notin E(H), 1 \leq i \leq \ell$, *and*

(2) $H \cup \{b_i b_i' | 1 \leq i \leq \ell\}$ *is a connected graph.*

Let H^+ be obtained from H by attaching to each $a_s, 1 \leq s \leq k$, either

1) *a new vertex x_s and a new edge $a_s x_s$ or*
2) *four new vertices x_s, y_s, z_s, w_s and five new edges such that $a_s x_s y_s z_s w_s$ is a C_5, or*
3) *seven new vertices $x_s^1, x_s^2, \ldots, x_s^7$ and 12 new edges such that $a_s x_s^1, x_s^2, \ldots, x_s^7$ is an 8-circuit plus 4 edges joining diametrically opposite vertices, i.e., $a_s x_s^1, x_s^2, \ldots, x_s^7$ spans a $ML_8 = G_3$.*

Further, to each pair of independent vertices $b_s, b_s', 1 \leq s \leq \ell$ we attach 3 new vertices x_s, y_s, z_s and 5 new edges producing a 5-circuit $b_s x_s y_s b_s' z_s$.

Each graph H^+ from this family $\mathcal{H}^+$ just constructed satisfies $\gamma(H^+) = \alpha(H^+)$. We note that $\mathcal{H}^+$ includes the family PC of well covered graphs from [7]. Furthermore, the exceptional graphs K_1, C_4, C_7, P_{10}, P_{13}, Q_{13} and P_{14} (see Figure 1) also determined in [7] are not only well covered, but they also satisfy $\gamma = \alpha$.

Figure 1

Observation 10. *The exceptional graphs* $K_1, C_4, C_7, P_{10}, P_{13}, Q_{13}, P_{14}$ *and the graphs of* $\mathcal{H}^+$ *all belong to* $\mathcal{G}_{\gamma=\alpha}$.

So far we obtained a large subclass of triangle-free members of $\mathcal{G}_{\gamma=\alpha}$. In the following we will enlarge this subclass.

Pinter constructed in [12, 13] families of W_2-graphs of girth 4, where a graph G is a W_2-graph, if G is well-covered and every vertex x of G is an extendable vertex, i.e., $G - x$ remains well-covered. One major issue of the concept of extendable vertices ([7, 8]) is that for two well-covered graphs G_1 and G_2 each having an extendable vertex x_i with $i = 1, 2$ the graph G obtained by G_1 and G_2 and the additional edge x_1x_2 remains well-covered. Observe that it is not very difficult to show that for the class $\mathcal{G}_{\gamma=\alpha}$ the concept of extendable vertices is also valid. Note that every vertex of the corresponding graph H of a graph H^+ from our family $\mathcal{H}^+$ is an extendable vertex. Now we describe Pinters 'stable' operations used in [12, 13].

Operation 1 [12]. Suppose G is a $\underline{W_2}$-graph (of girth 4) with adjacent degree two vertices x and y which are not on a triangle. Let $N_G(x) = \{u, y\}$ and $N_G(y) = \{v, x\}$ and a, b and c be new vertices. Form a new graph H with $V(H) = V(G) \cup \{a, b, c\}$ and $E(H) = E(G) \cup \{xa, ab, bc, cy, cu\}$. Then H is also a $\underline{W_2}$-graph (of girth 4) with $\alpha(G) = \alpha(H) + 1$.

Operation 2 [13]. Suppose H is a $\underline{W_2}$-graph of girth 4 and C is a 4-circuit in H such that $\alpha(H - C) = \alpha(H) - 1$ and $H - C$ is in $\underline{W_2}$. Let $C = abcd$ and let xy be a new line and $A = v_1v_2v_3v_4$ be a new 4-circuit. Form a new graph G with $V(G) = V(H) \cup V(A) \cup \{x, y\}$ and $E(G) = E(H) \cup E(A) \cup \{xy, v_1x, v_3y, v_2a, v_2b, v_4c, v_4d\}$. Then G is also a $\underline{W_2}$-graph of girth 4 with $\alpha(G) = \alpha(H) + 2$.

Operation 3 [13]. Suppose H is a $\underline{W_2}$-graph of girth 4 with disjoint 4-circuits C_1 and C_2 such that (i) $\alpha(H - C_i) = \alpha(H) - 1$ for $i = 1, 2$ and (ii) $H - C_i$ is in $\underline{W_2}$ for $i = 1, 2$. Also, H is connected or has exactly two components. In the disconnected case, each component contains exactly one of the 4-circuits C_i. Let $C_1 = u_1y_1v_1x_1$ and $C_2 = u_2y_2v_2x_2$ and let $A = abcd$ be a new 4-circuit. Form a new graph G with $V(G) = V(H) \cup V(A)$ and $E(G) = E(H) \cup E(A) \cup \{au_1, av_1, cx_1, cy_1, bx_2, by_2, du_2, dv_2\}$. Then G is also a $\underline{W_2}$-graph of girth 4 with $\alpha(G) = \alpha(H) + 1$.

Operation 1 requires the property that if v is an endvertex, then also u is an endvertex, i.e., either the graph in consideration is a path with four vertices or $d(u), d(x), d(y), d(v) \geq 2$. Since a W_2-graph G satisfies $\delta(G) \geq 2$, it is not necessary to require the extra condition. But if we want to replace W_2 by $\mathcal{G}_{\gamma=\alpha}$, we have to add the additional condition.

Two disjoint copies of the graph $H' \in \mathcal{G}_{\gamma=\alpha}$, obtained by a 4-circuit $uxyv$ and one carried out Operation 1, fullfils the conditions of Operation 3, if we consider an arbitrary 4-circuit of H'. If Operation 3 is carried out the resulting graph H^* satisfies $\gamma(H^*) = 6 < 7 = \alpha(H^*)$. Thus for the graph H in consideration we have to add the condition that there exists no minimum dominating D of H hitting for each of the two considered 4-circuits at least two adjacent vertices. In order to see that these conditions for the new Operation 3 can be satisfied by a graph of $\mathcal{G}_{\gamma=\alpha}$, we only have to consider two disjoint copies of the graph $H'' \in \mathcal{G}_{\gamma=\alpha}$, obtained by a 5-circuit $uxyvz$ and one carried out Operation 1.

Lemma 11. *The above operations are also valid for the class $\mathcal{G}_{\gamma=\alpha}$, i.e., we can replace $\underline{W_2}$ by $\underline{\mathcal{G}_{\gamma=\alpha}}$, we only have to add the before mentioned conditions.*

The proof of this lemma is tedious and not very difficult and therefore we omit here the proof. Moreover, we can relax the conditions of the Operation 1:

if we consider a $\mathcal{G}_{\gamma=\alpha}$-graph G (of girth 4) with three vertices u, x, y which induce a path $P = uxy$ in G such that $\alpha(G - P) = \alpha(G) - 1$. Let a, b and c be new vertices. Form a new graph H with $V(H) = V(G) \cup \{a, b, c\}$ and $E(H) = E(G) \cup \{xa, ab, bc, cy, cu\}$. Then H is also a $\mathcal{G}_{\gamma=\alpha}$-graph (of girth 4) with $\alpha(G) = \alpha(H) + 1$.

Operation 2 can also be relaxed:

Suppose H is a $\mathcal{G}_{\gamma=\alpha}$-graph of girth 4 and M is a 4-circuit or a K_2 in H such that $\alpha(H - M) = \alpha(H) - 1$. If $M = acbd$ let xy be a new line and $A = v_1v_2v_3v_4$ be a new 4-circuit. Form a new graph G with $V(G) = V(H) \cup V(A) \cup \{x, y\}$ and $E(G) = E(H) \cup E(A) \cup \{xy, v_1x, v_3y, v_2a, v_2b, v_4c, v_4d\}$. If $M = ac$ let xy be a new line and $A = v_1v_2v_3v_4$ be a new 4-circuit. Form a new graph G with $V(G) = V(H) \cup V(A) \cup \{x, y\}$ and $E(G) = E(H) \cup E(A) \cup \{xy, v_1x, v_3y, v_2a, v_4c\}$. Moreover, in the latter case we can add the additional edges ax and cy. Note that then the eight vertices $a, c, x, y, v_1, v_2, v_3, v_4$ induce the Moebius ladder $ML8$.

Recall that therefore we obtain one of our basic building blocks, the Moebius ladder $ML8$, in the construction of our class $\mathcal{H}^+$ by applying the modified operation 2 on a K_2 representing an endvertex and its unique neighbour.

4..3 Concluding Remark

The graph W obtained by identifying two paths with four vertices, where each path is contained in a 5-circuit, plays a central role in all of the above mentioned operations and surely also in a characterization of all triangle-free graphs G satisfying $\gamma(G) = \alpha(G)$. In this last section we started with a class $\mathcal{H}^+$ constructed by basic building blocks. Then we briefly summarized Pinters 'stable' operations used for W_2-graphs and adapted (modified) these operations for the class $\mathcal{G}_{\gamma=\alpha}$. A combination of $\mathcal{H}^+$ and the modified operations constructs a large class of graphs satisfying $\gamma = \alpha$, but we expect that there are further operations needed in order to characterize $\mathcal{G}_{\gamma=\alpha}$ $-\{K_1, C_4, C_7, P_{10}, P_{13}, Q_{13}, P_{14}\}$.

Acknowledgement

The authors thank Lutz Volkmann and Bert Hartnell for discussions on this topic and the referees for their helpful comments.

References

[1] M.O. Albertson, L. Chan and R. Haas, *Independence and graph homomorphisms*, J. Graph Theory **17** (1993) 581–588.

[2] X. Baogen, E. Cockayne, T.W. Haynes, S.T. Hedetniemi and Z. Shangchao, *Extremal graphs for inequalities involving domination parameters,* Discrete Math. **216** (2000) 1–10.

[3] C. Berge, *Regularizable graphs*, Ann. Discrete Math. **3** (1978) 11–19.

[4] V. Chvátal and P.J. Slater, *A note on well-covered graphs*, Ann. Discrete Math. **55** (1993) 179–182.

[5] O. Favaron, *Very well-covered graphs*, Discrete Math. **42** (1982) 177–187.

[6] A. Finbow and B. Hartnell, *A game related to covering by stars*, Ars Combin. **16** (A) (1983) 189–198.

[7] A. Finbow, B. Hartnell and R.J. Nowakowski, *A characterization of well-covered graphs of girth* 5 *or greater*, J. Combin. Theory (B) **57** (1993) 44–68.

[8] A. Finbow, B. Hartnell and R.J. Nowakowski, *A characterization of well-covered graphs which contain neither* 4- *nor* 5-*cycles*, J. Graph Theory **18** (1994) 713–721.

[9] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985) 287–293.

[10] O. Ore, Theory of Graphs (Amer. Math. Soc. Colloq. Publ. **38**, 1962).

[11] C. Payan and N.H. Xuong, *Domination-balanced graphs.* J. Graph Theory **6** (1982) 23–32.

[12] M.R. Pinter, *A class of planar well-covered graphs with girth four*, J. Graph Theory **19** (1995) 69–81.

[13] M.R. Pinter, *A class of well-covered graphs with girth four*, Ars Combin. **45** (1997) 241–255.

[14] M.D. Plummer, *Some covering concepts in graphs*, J. Combin. Theory **8** (1970) 91–98.

[15] M.D. Plummer, *Well-covered graphs: a survey*, Quaestiones Math. **16** (1993) 253–287.

[16] B. Randerath and L. Volkmann, *Characterization of graphs with equal domination and covering number*, Discrete Math. **191** (1998) 159–169.

[17] R.S. Sankaranarayanan and L.K. Stewart, *Complexity results for well-covered graphs*, Networks **22** (1992) 247–262.

[18] J. Staples, Ph. D. dissertation (Vanderbilt University, Nashville, TN, 1975).

[19] L. Szamkołowicz, *Sur la classification des graphes en vue des propriétés de leurs noyaux*, Prace Nauk. Inst. Mat. i Fiz. Teoret., Politechn. Wroclaw., Ser. Stud. Mater. **3** (1970) 15–21.

[20] J. Topp and L. Volkmann, *On domination and independence numbers of graphs*, Resultate Math. **17** (1990) 333–341.

[21] W.T. Tutte, *The* 1*-factors of oriented graphs*, Proc. Amer. Math. Soc. **4** (1953) 922–931.

Received 4 August 2000
Revised 23 December 2001