Discussiones Mathematicae
Graph Theory 22 (2002) 123–148

CONDITIONS FOR β-PERFECTNESS

Judith Keijsper

University of Twente
Faculty of Mathematical Sciences
7500 AE Enschede, The Netherlands

and

Meike Tewes*

Freiberg University
Faculty of Mathematics and Computer Sciences
09596 Freiberg, Germany

Abstract

A β-perfect graph is a simple graph G such that $\chi(G') = \beta(G')$ for every induced subgraph G' of G, where $\chi(G')$ is the chromatic number of G', and $\beta(G')$ is defined as the maximum over all induced subgraphs H of G' of the minimum vertex degree in H plus 1 (i.e., $\delta(H) + 1$). The vertices of a β-perfect graph G can be coloured with $\chi(G)$ colours in polynomial time (greedily).

The main purpose of this paper is to give necessary and sufficient conditions, in terms of forbidden induced subgraphs, for a graph to be β-perfect. We give new sufficient conditions and make improvements to sufficient conditions previously given by others. We also mention a necessary condition which generalizes the fact that no β-perfect graph contains an even hole.

Keywords: chromatic number, colouring number, polynomial time.

2000 Mathematics Subject Classification: 05C15, 05C75.

*Research was supported by the DAAD within the program "Gemeinsames Hochschulsonderprogramm III von Bund und Ländern".

1. Introduction

Graphs in this paper are finite graphs without loops and multiple edges. The parameter $\beta(G)$ associated with a graph G, as well as the concept of a β-perfect graph were introduced in [9]. The definition of $\beta(G)$ is as follows.

$$\beta(G) := \max\{\delta(G') + 1 \mid G' \text{ is an induced subgraph of } G\}.$$

Here, $\delta(G)$ denotes the minimum vertex degree in the graph G. We just mention that $\beta(G)$ equals the colouring number $col(G)$ which was introduced by Erdős and Hajnal [5] to study, in particular, infinite graphs. See e.g. [8] for more information. It was proved by several authors (cf. for example [7]) that for any graph G the value $\beta(G)$ can be calculated in polynomial time.

Using a minimum degree sequence, the vertices of any graph G can be greedily coloured with at most $\beta(G)$ colours, and hence $\beta(G)$ forms a trivial upper bound for the chromatic number $\chi(G)$. A graph G is said to be *β-perfect* if $\beta(G') = \chi(G')$ for every induced subgraph G' of G. So the vertices of any induced subgraph G' of a β-perfect graph, can be coloured with $\chi(G')$ colours in polynomial time. We say that G is *β-imperfect* if it is not β-perfect.

Since a graph G isomorphic to an even induced cycle satisfies $2 = \chi(G) < \beta(G) = 3$, no β-perfect graph can contain an even hole (a graph is said to contain an *even* (*odd*) *hole* if it contains an even (odd) chordless cycle of length at least four). More generally, a β-perfect graph does not contain any regular induced subgraphs, except perhaps odd holes and cliques, as we observe in Section 5.

The necessary condition of being even hole-free is by no means sufficient: there are many examples of even hole-free graphs that are β-imperfect. In Section 5, we show that in fact every 3-regular connected even hole-free graph which is not the complete graph is minimally β-imperfect. The fact that a graph is even hole-free does give a performance guarantee for the greedy colouring algorithm, since it was shown in [9] that $\beta(G) \leq 2(\chi(G)-1)$ for such a graph G. Conforti, Cornuéjols, Kapoor, and Vušković [3] showed that one can check in polynomial time whether a graph is even hole-free.

A *short-chorded cycle* is a cycle of length at least four with exactly one chord which forms a triangle with two edges of the cycle. In particular, by a *diamond* we mean a short-chorded cycle on four vertices. If, besides even holes, short-chorded cycles are excluded as induced subgraphs, then this is sufficient to force β-perfectness, as was proved by Markossian, Gasparian, and Reed [9]. This result was improved in [6] to the following

Theorem 1.1 (Figueiredo, Vušković [6]). *If G is a graph that contains no even hole, no diamond and no short-chorded cycle on six vertices, then G is β-perfect.*

A graph G is said to *contain* the graph H, if H is an induced subgraph of G. If G does not contain a copy of H, then G is H*-free.*

It was conjectured that in fact no short-chorded cycles of order greater than four have to be excluded to obtain the same conclusion.

Conjecture 1.2 (Figueiredo, Vušković [6]). *If G is a graph that contains no even hole and no diamond, then G is β-perfect.*

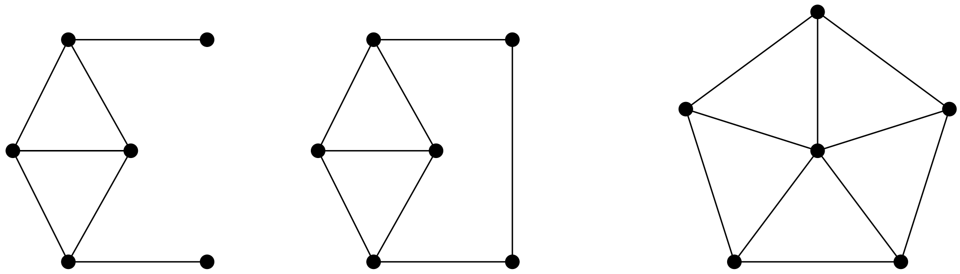
A *simplicial extreme* of a graph G is a vertex $v \in V(G)$ having one of the following two properties: either the degree of v in G is at most 2, or v is a *simplicial vertex* of G (that is, the neighbourhood of v in G induces a clique in G). Figueiredo and Vušković proved their result by proving the existence of a simplicial extreme in any graph satisfying the conditions of Theorem 1.1. In other words, they derived Theorem 1.1 from the following result (by using arguments similar to the ones in Lemma 1.6 below).

Theorem 1.3 (Figueiredo, Vušković [6]). *If G is a graph that contains no even hole, no diamond and no short-chorded cycle on six vertices, then G has a simplicial extreme.*

In the same way (by proving existence of a simplicial extreme), we will show in this paper that two more classes of graphs defined in terms of forbidden induced subgraphs, are β-perfect. Section 2 deals with claw-free graphs (a *claw-free* graph is a graph containing no $K_{1,3}$ as an induced subgraph). There, we will prove the following theorem.

Theorem 1.4. *Let G be a claw-free graph without even holes that contains none of the graphs in Figure* 1*. Then G is β-perfect.*

In Section 3, net-free graphs are considered. The *net* is a graph isomorphic to the graph with vertices a, a', b, b', c, c', and edges ab, bc, ca, aa', bb', and cc'. The following will be shown.



D_1 D_2 D_3

Figure 1. Forbidden induced subgraphs for claw-free even hole-free graphs

Theorem 1.5. *If G is a graph which contains no even hole, no diamond and no net, then G is β-perfect.*

In particular, Theorems 1.4 and 1.5 show that Conjecture 1.2 is valid for claw-free graphs and for net-free graphs.

As mentioned, β-perfectness of the graphs described in these two theorems will be derived from the fact that a graph G in either one of the given classes contains a simplicial extreme. In fact, we need the existence of this simplicial extreme only in an induced subgraph H of G where the β-value is attained (that is where $\beta(G) = \delta(H) + 1$). This is stated in the following lemma, which will be useful throughout the paper.

Lemma 1.6. *Let G be a graph without even holes and let H be an induced subgraph of G such that $\beta(G) = \delta(H)+1$. If H contains a simplicial extreme, then $\chi(G) = \beta(G)$.*

For the proof of the above lemma, we need the well-known theorem of Dirac on the existence of simplicial vertices in triangulated (or chordal) graphs.

Theorem 1.7 (Dirac [4])**.** *Every triangulated graph which is not a clique contains at least two nonadjacent simplicial vertices.*

For an induced subgraph G' of a graph G ($G' = G$ possibly) we denote the degree of a vertex v in G' (i.e., the number of vertices of $V(G')$ adjacent in G to v) by $d(v, G')$. Note that v might be a vertex of G' or not. We also use the short form $d(v, S)$ instead of $d(v, G[S])$, for $S \subseteq V(G)$.

***Proof of Lemma* 1.6** (essentially due to [6])**.** Assume first that H has a simplicial vertex x. Then it is obvious that $\chi(G) \leq \beta(G) = \delta(H) + 1 \leq$

$d(x, H) + 1 \leq \chi(H) \leq \chi(G)$. If H has no simplicial vertex, then there is a vertex $y \in V(H)$ with $d(y, H) \leq 2$ and it follows by Theorem 1.7 that H is not triangulated. Hence, H contains an odd hole implying that $\chi(H) \geq 3$. Altogether, we have $\chi(G) \leq \beta(G) = \delta(H) + 1 \leq d(y, H) + 1 \leq 3 \leq \chi(H) \leq \chi(G)$, which completes the proof. ■

So far, every sufficient condition for β-perfectness we have mentioned, was given in terms of forbidden induced subgraphs and implied existence of a simplicial extreme. This means that all the β-perfect graphs G that have been obtained so far have the following special property: for every induced subgraph G' of G, either $\chi(G') = \omega(G')$ or $\chi(G') = 3 > 2 = \omega(G')$ (where $\omega(G')$ denotes the size of a largest clique in G').

In Section 4, we prove the following extension of Theorem 1.1, which introduces a more general type of β-perfect graph. Indeed, graphs satisfying the conditions of this theorem need not have a simplicial extreme (the 5-wheel D_3 is an example) and in general do not have the special property described above.

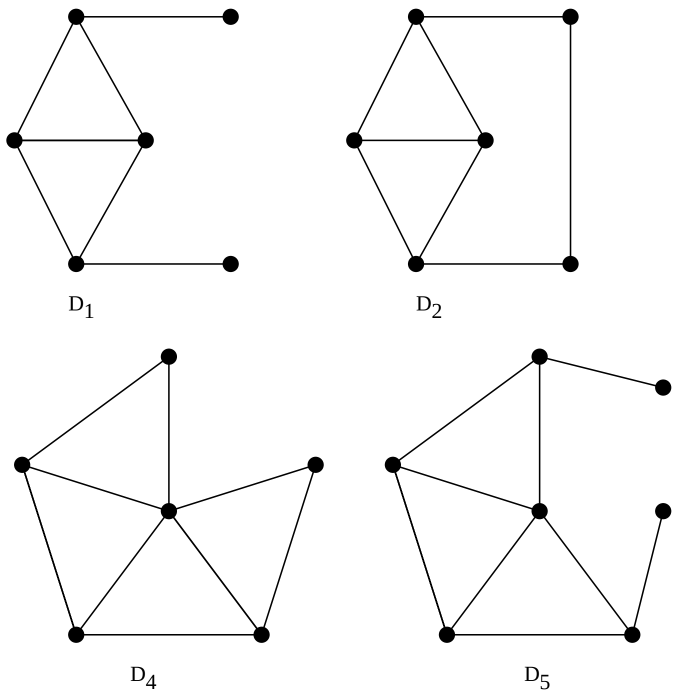


Figure 2. Forbidden induced subgraphs containing the diamond

Theorem 1.8. *Let G be an even hole-free graph containing none of the graphs in Figure* 2 *and none of the graphs in Figure* 3*. Then G is β-perfect.*

This theorem describes the possible neighbourhoods of a diamond (Figure 2) or a short-chorded 6-cycle (Figure 3) in a minimally β-imperfect graph (note that short-chorded cycles are β-perfect graphs). To prove this theorem in Section 4, we will apply Theorem 1.3, and furthermore we will exploit the observation (also used in Lemma 1.6) that to prove $\chi(G) = \beta(G)$ for a graph G, it suffices to prove $\chi(H) = \beta(H)$ for one induced subgraph H of G where the β-value is attained (we take a minimal such H).

Since all graphs in Figure 2 contain a diamond, and both graphs in Figure 3 contain a net, Theorem 1.5 can be viewed as a corollary of Theorem 1.8.

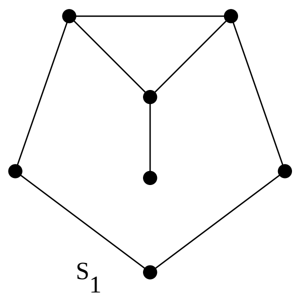


Figure 3. Forbidden induced subgraphs containing the short-chorded 6-cycle

With the results obtained in Section 4, we can also improve Theorem 1.4 and obtain the following stronger result for claw-free graphs.

Theorem 1.9. *Let G be a claw-free graph without even holes that contains no D_1 or D_2. Then G is β-perfect.*

Finally, note that by the results in [3] mentioned above, all sufficient conditions for β-perfectness we deal with in this paper can be checked in polynomial time.

2. Claw-Free Graphs

In this section, we prove Theorem 1.4, which states that for even hole-free graphs that are in addition claw-free, it suffices to exclude three supergraphs of the diamond to guarantee β-perfectness, namely the three graphs D_1, D_2, and D_3, depicted in Figure 1. This will imply a characterization of β-perfect line graphs.

Consider the following family of graphs $\mathcal{G} = \{G \text{ simple graph} \mid G \text{ is complete or has two nonadjacent simplicial extremes}\}$.

Lemma 2.1. *Let $G \notin \mathcal{G}$ such that $G' \in \mathcal{G}$ for every proper induced subgraph G' of G. Then G does not have a clique cutset.*

Proof. Suppose $G \notin \mathcal{G}$ is such a graph that does have a clique cutset C. It means that there are two proper induced subgraphs G_1 and G_2 of G with $C = V(G_1) \cap V(G_2)$, such that in G there are no edges between $V(G_1) \setminus C$ and $V(G_2) \setminus C$. By assumption, $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}$, so there are three possibilities:

G_1, G_2 are both complete graphs. In this case, since every vertex of a complete graph is simplicial, if we choose $v \in V(G_1) \setminus C$ and $w \in V(G_2) \setminus C$, then v, w are nonadjacent simplicial vertices in G. Hence $G \in \mathcal{G}$, a contradiction.

G_1, G_2 both contain two nonadjacent simplicial extremes. Then at least one simplicial extreme v of G_1 is in $V(G_1) \setminus C$, and at least one simplicial extreme w of G_2 is in $V(G_2) \setminus C$. Now v, w are two nonadjacent simplicial extremes in G. Hence $G \in \mathcal{G}$, a contradiction.

The case that G_1 is complete, and G_2 contains two nonadjacent simplicial extremes can be dealt with by a similar argument.

Hence G does not have a clique cutset. ∎

Analogously to the definition of $d(v, G')$ given in Section 1, where G' is some induced subgraph of a graph G ($G' = G$ possibly) and $v \in V(G)$, we denote the *neighbourhood* of a vertex v in G' (i.e., the set of vertices in $V(G')$ adjacent in G to v) by $N(v, G')$. Again, we use $N(v, S)$ instead of $N(v, G[S])$ for $S \subseteq V(G)$ and v might be a vertex of G' or not.

***Proof of Theorem* 1.4.** We claim that any graph G satisfying the conditions of the theorem is in $\mathcal{G}$. It suffices to prove this claim, because if G' is an induced subgraph of G, and if H is an induced subgraph of G' with $\beta(G') = \delta(H) + 1$, then $H \in \mathcal{G}$ (since H also satisfies the conditions of the theorem), so in particular H has a simplicial extreme, and hence $\chi(G') = \beta(G')$, by Lemma 1.6.

To prove the above claim, suppose that G is a minimal counterexample, i.e., $G \notin \mathcal{G}$ is a claw-free, even hole-free graph having none of the graphs in

Figure 1 as an induced subgraph, but any proper induced subgraph of G is a member of $\mathcal{G}$.

Then G is connected and, by Lemma 2.1, G does not have a clique cutset. Moreover, since $G \notin \mathcal{G}$, G is not triangulated (by Theorem 1.7). So G contains a hole $Q = G[z_1, z_2, \ldots, z_k]$ (where $z_1 z_2 \ldots z_k z_1$ is a cycle) which is an odd hole because G is even hole-free. Choose Q to be a shortest such hole. Since any odd hole contains two nonadjacent simplicial extremes, and since $G \notin \mathcal{G}$, G properly contains Q, i.e. $V(G-Q) \neq \emptyset$.

In the following, all indices concerning cycles should be taken modulo the cycle length.

Claim 1. No $x \in V(G-Q)$ is adjacent in G to more than two consecutive vertices on Q.
Suppose to the contrary that $xz_i, xz_{i+1}, xz_{i+2} \in E(G)$ and without loss of generality let $i = 1$. Since G does not contain D_1 and D_2, x is adjacent to z_4 or z_k, say $xz_4 \in E(G)$. To avoid the even hole $xz_4z_5 \ldots z_k z_1 x$, there is a further neighbour z_i of x for some $5 \leq i \leq k$. Since G does not contain the 5-wheel D_3, we know that $k > 5$. But this leads to $G[\{x, z_1, z_3, z_i\}] \cong K_{1,3}$ (if $i \neq k$) or $G[\{x, z_2, z_4, z_k\}] \cong K_{1,3}$ (if $i = k$), a contradiction.

Claim 2. If $d(x, Q) > 0$ for some $x \in V(G-Q)$, then $N(x, Q) = \{z_i, z_{i+1}\}$ for some $i \in \{1, 2, \ldots, k\}$.
Without loss of generality let $xz_1 \in E(G)$. Since G is claw-free, x is adjacent to z_2 or z_k, say $xz_2 \in E(G)$. If $N(x, Q) = \{z_1, z_2\}$, we are done. Hence suppose that $z_i \in N(x, Q)$ for some $3 \leq i \leq k$, where we may assume that i is chosen to be as large as possible. By Claim 1, we have $4 \leq i \leq k-1$. Then $xz_{i-1} \in E(G)$ and $xz_i z_{i+1} \ldots z_k z_1 x$ induces a hole. This leads to a contradiction to the minimality of Q, respectively to the hypothesis.

Since G is connected and $V(G-Q) \neq \emptyset$, there is a vertex $x \in V(G-Q)$ with $d(x, Q) > 0$ and by Claim 2 we may assume that $N(x, Q) = \{z_1, z_2\}$. Let G_x denote the component of $G - Q$ containing x.

Claim 3. If $d(y, Q) > 0$ for some $y \in G_x$, then $N(y, Q) = \{z_1, z_2\}$.
Suppose to the contrary that this does not hold and let P be a shortest path in G_x leading from a vertex $p_1 \in V(G_x)$ with $N(p_1, Q) = \{z_1, z_2\}$ to a vertex $p_s \in V(G_x)$ with $N(p_s, Q) > 0$ but $N(p_s, Q) \neq \{z_1, z_2\}$. Note that $d(p, Q) = 0$ for every $p \in P - \{p_1, p_s\}$. By Claim 2, $N(p_s, Q) = \{z_i, z_{i+1}\}$ for some $2 \leq i \leq k$. Since Q has an odd number of vertices, one of $Q[z_2, \ldots, z_i]$

and $Q[z_{i+1},\ldots,z_1]$ is odd and the other one is even. So one of these segments together with P forms an even hole, a contradiction.

By Claim 3, the edge z_1z_2 is a clique cutset in G and this contradiction completes the proof. ■

Beineke [1] gave a characterization of line graphs in terms of forbidden induced subgraphs. Note that the claw as well as D_1, D_2, and D_3 belong to Beinekes set of forbidden subgraphs. Hence, Theorem 1.4 implies the following.

Corollary 2.2. *A line graph G is β-perfect if and only if G contains no even holes.*

This means that only graphs without even cycles have β-perfect line graphs. For other graphs H, the upper bound $\Delta(H)+1$ on the edge chromatic number $\chi'(H)$ of H, given by Vizing's Theorem, is at least as good as the upper bound obtained by taking the β-value of the line graph of H.

Note that the graphs in Figure 1 are β-perfect. For D_1 and D_2, there are examples of claw-free graphs showing that it is not possible to delete either one of them from the list of forbidden subgraphs. The graph D_3 however can be removed from this list, as we will show in Section 4.

3. Net-Free Graphs

In this section, we will prove Theorem 1.5, which shows that Conjecture 1.2 is valid for net-free graphs. We use the following result from [9].

Theorem 3.1 (Markossian, Gasparian, Reed [9]). *Let G be a triangle-free graph without even holes. Then for every $x \in V(G)$ either $d(x,G) = |V(G)|-1$ or there exists a vertex $y \in V(G) \setminus N(x,G)$ with $d(y,G) \leq 2$.*

From this, we can derive the next theorem in an elementary way.

Theorem 3.2. *If G is a graph which contains no even hole, no diamond and no net, then G has a simplicial extreme.*

Proof. We suppose to the contrary that there is no simplicial extreme in G, which implies that G is not complete. Define C to be the largest clique in G. If there is no triangle in G, then it follows from Theorem 3.1 that there is a vertex $y \in V(G)$ such that $d(y, G) \leq 2$, a contradiction. Hence, $|V(C)| \geq 3$. The maximality of C implies moreover that for every vertex $u \in V(G - C)$ there is a vertex $z_u \in V(C) \setminus N(u, G)$. It follows that u has at most one neighbour in C, since otherwise two neighbours $z_1, z_2 \in V(C)$ together with u and z_u would form a diamond.

Since, by our assumption, no vertex of C is a simplicial vertex, we have $zu_z \in E(G)$ for some $u_z \in V(G-C)$ for every $z \in V(C)$. Since $d(u_z, C) = 1$ and $|V(C)| \geq 3$, this leads to a net in G, a contradiction. ∎

***Proof of Theorem* 1.5.** We obtain the desired result directly from Theorem 3.2 and Lemma 1.6. ∎

4. Enlarging the Forbidden Subgraphs

A minimal induced subgraph H of a graph G that satisfies $\beta(G) = \delta(H) + 1$ has the property that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. For graphs H with this property that contain no short even holes, the following holds.

Lemma 4.1. *Let H be a graph without* 4- *and* 6-*holes such that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. Then H contains a diamond if and only if H contains D_3* (*see Figure* 1), *D_6* (*see Figure* 4) *or one of the graphs in Figure* 2.

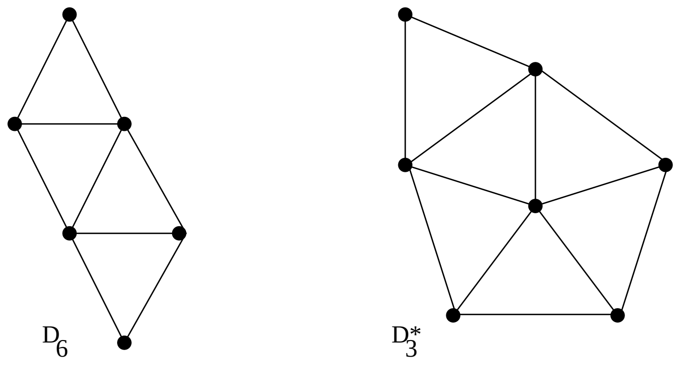


Figure 4. The graphs D_6 and D_3^*

Proof. To prove that the existence of a diamond in H implies the existence of one of the six supergraphs listed above, we proceed in two steps. In the following we denote by T the graph obtained from the 5-wheel D_3 by deleting one vertex on the rim.

Step 1. If H contains a diamond, then H contains D_1 or D_2 or T as an induced subgraph.

Proof. Consider a counterexample H. Then there exist sets $A, B, Z \subseteq V(H)$ satisfying

$$A, B \neq \emptyset, |Z| \geq 2,\ A \cap B = A \cap Z = B \cap Z = \emptyset,$$
$$az, bz \in E(H),\ \forall a \in A,\ \forall b \in B, \forall z \in Z,$$
$$ab \notin E(H),\ \forall a \in A,\ \forall b \in B.$$

Indeed, since H contains a diamond, say with vertices a, z_1, b, z_2 in that order on a 4-cycle, and with chord z_1z_2, the sets $A := \{a\}$, $B := \{b\}$, and $Z := \{z_1, z_2\}$ fulfill these requirements. Now, in addition we assume that A, B, Z are chosen in such a way that

1. $|Z|$ is maximal,
2. $|A \cup B|$ is maximal for this choice of Z.

Now, since $A^+ := H[A \cup Z]$ and $B^+ := H[B \cup Z]$ are proper induced subgraphs of H, by assumption it holds that $\delta(A^+) < \delta(H)$, and $\delta(B^+) < \delta(H)$. Moreover, because H is 4-hole free, Z induces a clique in H, and therefore

$$d(a, A^+) \leq |Z| + |A| - 1 = d(z, A^+),\ \forall a \in A, \forall z \in Z,$$
$$d(b, B^+) \leq |Z| + |B| - 1 = d(z, B^+),\ \forall b \in B, \forall z \in Z.$$

It follows that there is an $a^* \in A$ satisfying $d(a^*, A^+) = \delta(A^+)$, and (because $\delta(A^+) < \delta(H)$) that there is an $a' \in V(H - A^+)$ with $a^*a' \in E(H)$. Similarly there are $b^* \in B$ and $b' \in V(H - B^+)$ with $b^*b' \in E(H)$. Because there are no edges between A and B,

$$a' \notin B \text{ and } b' \notin A.$$

So both a' and b' are vertices of H not contained in $A \cup B \cup Z$. Furthermore,

$$a'y \notin E(H), \forall y \in B \text{ and } b'x \notin E(H), \forall x \in A. \tag{1}$$

Indeed, if (say) $a'y$ were an edge, for some $y \in B$, then $A' := \{a^*\}$, $B' := \{y\}$, and $Z' := Z \cup \{a'\}$ would contradict the maximal choice of Z. Next, we prove that

$$a'b' \notin E(H). \tag{2}$$

Indeed, suppose that $a'b' \in E(H)$. By (1), the edges $a'b^*$, and $b'a^*$ do not exist in H. So, for some $z \in Z$, either $a'z$ or $b'z \in E(H)$, since otherwise H would contain an induced D_2 (recall that $|Z| \geq 2$). But if $a'z \in E(H)$ then also $b'z \in E(H)$ (and vice versa), since otherwise $a'zb^*b'a'$ would be an induced 4-hole in H. But now we have obtained an induced T (on a^*, a', z, b^*, b') in H, a contradiction.

Finally, we claim that

$$\begin{gathered} N(a', Z) \in \{\emptyset, Z\},\ N(b', Z) \in \{\emptyset, Z\}, \\ \text{least one of them is nonempty.} \end{gathered} \tag{3}$$

To prove this, assume that $a'z \in E(H)$ and $a'z' \notin E(H)$ for some $z, z' \in Z$, $z \neq z'$. Then (since $a'b^* \notin E(H)$ by (1)) the induced subgraph $H[a^*, a', z, z', b^*]$ is isomorphic to T, a contradiction. Similarly for b'. Moreover, if we suppose that $N(a', Z) = \emptyset = N(b', Z)$, then (using (1) and (2)) the vertices a^*, a', z, z', b^*, b' induce a D_1 in H, which is also a contradiction. This proves (3).

Now without loss of generality we may assume that $N(a', Z) = Z$. But then $A' := A \cup \{a'\}$, $B' := B$, $Z' := Z$ contradict the maximal choice of $A \cup B$. This completes the proof of Step 1.

Step 2. If T is a subgraph of H, then H contains D_3, D_4, D_5, or D_6 as an induced subgraph.

Proof. Again we consider a counterexample H containing T but none of the other four graphs. Let now $C \subseteq V(H)$ such that C induces a clique and there are vertex sets $A, B \subseteq V(H)$ and a partition $C = C_1 \cup C_2 \cup C_3$ with $C_i \neq \emptyset$ $(i = 1, 2, 3)$ of C having the following properties:

A and B induce connected subgraphs of H,

$C_1 \cup C_3 \subseteq N(a, H)$, $C_2 \cup C_3 \subseteq N(b, H)$, $\forall a \in A, \forall b \in B$,

$C_2 \cap N(a, H) = C_1 \cap N(b, H) = \emptyset$, $\forall a \in A, \forall b \in B$,

$ab \notin E(H)$, $\forall a \in A, \forall b \in B$.

Note that $A \cap B = A \cap C = B \cap C = \emptyset$. Since T is a subgraph of H, there exist vertex sets A, B and C with the desired properties. In the following, we may assume that A, B and C are chosen such that

1. $|C|$ is maximal,
2. $|C_1 \cup C_2|$ is minimal for this choice of C,
3. $|A \cup B|$ is maximal for this choice of C, C_1, C_2.

Consider the proper subgraphs $A^+ = H[A \cup C_1 \cup C_3]$ and $B^+ = H[B \cup C_2 \cup C_3]$ of H. Since $H[C]$ is a clique, we deduce analogously to Step 1

$$\begin{aligned} d(a, A^+) &\leq |C_1| + |C_3| + |A| - 1 = d(c, A^+)\ \forall a \in A, \forall c \in C_1 \cup C_3, \\ d(b, B^+) &\leq |C_2| + |C_3| + |B| - 1 = d(c', B^+)\ \forall b \in B, \forall c \in C_2 \cup C_3 \end{aligned}$$

and hence there is a vertex $a^* \in A$, and a vertex $b^* \in B$ with $d(a^*, A^+) = \delta(A^+)$ and $d(b^*, B^+) = \delta(B^+)$. By the properties of A, B, C and since $\delta(A^+), \delta(B^+) < \delta(H)$, it follows that $A' = N(a^*, H - (A \cup B \cup C)) \neq \emptyset$ and $B' = N(b^*, H - (A \cup B \cup C)) \neq \emptyset$.

We proceed by proving five claims.

Claim 1. For every $x \in A'$ and $y \in B'$, we have $d(x, C_2) < |C_2|$ and $d(y, C_1) < |C_1|$.
We verify the claim for the set A' (then the analogous result holds for B' by symmetry). Suppose to the contrary that $C_2 \subseteq N(x, H)$ for some $x \in A'$. Then $cx \in E(H)$ for every $c \in C_1 \cup C_3$, since otherwise acc_2xa induces a 4-hole, where $c_2 \in C_2$ is an arbitrary vertex. Hence, $C \cup \{x\}$ induces a clique. If xb for some $b \in B$, then the choice $C^* := C_1 \cup C_2 \cup (C_3 \cup \{x\})$, $A^* := \{a^*\}$ and $B^* := \{b\}$ contradicts the maximality of C. If $d(x, B) = 0$, then we derive the same contradiction by choosing $C^* := (C_1 \cup \{x\}) \cup C_2 \cup C_3$, $A^* := \{a^*\}$, and $B^* := B$.

Claim 2. For every $x \in A'$ ($y \in B'$) with $d(x, C_1) > 0$ ($d(y, C_2) > 0$), we have $d(x, B) = 0$ ($d(y, A) = 0$).
Let $z_1 \in C_1$ such that $xz_1 \in E(H)$. Assume that $xb \in E(H)$ for some $x \in A'$, $b \in B$ and consider the cycle xz_1z_2bx, where $z_2 \in C_2$ is an arbitrary vertex. Since H contains no 4-hole and $z_1b \notin E(H)$, it follows that xz_2 for every $z_2 \in C_2$, contradicting Claim 1.

Analogously, the result follows for every $y \in B'$.

Claim 3. There exist $a' \in A'$ and $b' \in B'$ with $d(a', C_1 \cup C_3) < |C_1 \cup C_3|$ and $d(b', C_2 \cup C_3) < |C_2 \cup C_3|$.

Again, for symmetry reasons, it is enough to show the claim for the set A'. Suppose to the contrary that $d(x, C_1 \cup C_3) = |C_1 \cup C_3|$ for every $x \in A'$. The maximality of $|A \cup B|$ implies that $xu_x \in E(H)$ for some $u_x \in B \cup C_2$. By Claim 2, $u_x \in C_2$. On the other hand, Claim 1 implies that $xv_x \notin E(H)$ for some $v_x \in C_2$, $v_x \neq u_x$. For the rest of the proof of Claim 3, fix u_x and v_x for every $x \in A'$. We show

$$ax \in E(H)\ \forall a \in A, \forall x \in A'. \tag{4}$$

Let $x \in A'$ be an arbitrary vertex with corresponding vertices $u_x, v_x \in C_2$ described above. By the definition of A', there is nothing to show for $a = a^*$.

Now, let $a \in N(a^*, A)$ and consider the subgraph $F = H[\{a, a^*, x, u_x, v_x, z_1\}]$, where z_1 is some vertex of C_1. Note that $\{aa^*, a^*x, xu_x, u_xv_x, z_1a, z_1a^*, z_1x, z_1u_x, z_1v_x\} \subseteq E(F)$. By assumption, H does not contain D_4 as an induced subgraph, and hence it follows that $ax \in E(F) \subseteq E(H)$.

Since A is connected, we successively obtain the result for every $a \in A$. This proves (4).

Next, we show

$$H[A'] \text{ is a clique.} \tag{5}$$

Assume that $xx' \notin E(H)$ for some distinct vertices $x, x' \in A'$.

Note first that no vertex $z_2 \in C_2$ is adjacent to both x and x', since otherwise $a^*xz_2x'a^*$ is a 4-hole. Furthermore, for every $z_2 \in C_2$, we have either $z_2x \in E(H)$ or $z_2x' \in E(H)$. To see this, assume that $z_2x, z_2x' \notin E(H)$. Then, again because D_4 is not induced in H, the subgraph $F = H[\{x', a^*, x, u_x, z_2, z_1\}]$ (for some $z_1 \in C_1$) implies that $x'u_x \in E(H)$ leading to the 4-hole $x'a^*xu_xx'$.

Hence, the set C_2 can be partitioned into $C_2 = C_2' \cup C_2''$ such that for $z_2 \in C_2$, we have $z_2 \in C_2'$ if and only if $z_2x' \in E(H)$ and $z_2 \in C_2''$ if and only if $z_2x \in E(H)$. Then the choice $C^* := C$ with $C_1^* := C_2'$, $C_2^* := C_2''$, $C_3^* := C_1 \cup C_3$, $A^* := \{x'\}$ and $B^* := \{x\}$ contradicts the minimality of $|C_1 \cup C_2|$. This proves (5).

Now, define $A^{++} := H[A \cup A' \cup C_1 \cup C_3]$. It follows from (4), (5), together with our assumption $(d(x, C_1 \cup C_3) = |C_1 \cup C_3|, \forall x \in A')$ that

$$d(u, A^{++}) = d(u, A^+) + |A'| \leq |A| - 1 + |C_1| + |C_3| + |A'|,\ \forall u \in A \cup C_1 \cup C_3,$$
$$d(x, A^{++}) = |A| + |A'| - 1 + |C_1| + |C_3|,\ \forall x \in A'.$$

Since $a^* \in A$ was chosen to be a vertex with $d(a^*, A^+) = \delta(A^+)$, we also have $d(a^*, A^{++}) = \delta(A^{++})$. Note that A^{++} is a proper subgraph of H and hence $d(a^*, H - A^{++}) = d(a^*, H - (A \cup B \cup C \cup A')) > 0$, which contradicts the definition of A'. This completes the proof of Claim 3.

In the following, let $a' \in A'$ and $b' \in B'$ be such that they have the property described in Claim 3.

Claim 4. $d(a', C_2) = d(b', C_1) = 0$.
Suppose that $a'c_2 \in E(H)$ for some $c_2 \in C_2$. Let $c \in C_1 \cup C_3$ be an arbitrary vertex. Now the cycle $a'a^*cc_2a'$ forces that $a'c \in E(H)$. Hence, $d(a', C_1 \cup C_3) = |C_1 \cup C_3|$, contradicting Claim 3.

Analogously, we conclude that $d(b', C_1) = 0$.

Claim 5. $d(a', B) = d(b', A) = 0$.
Again by symmetry, we verify the claim only for a'. Assume that $a'b \in E(H)$ for some $b \in B$ and let $z_3 \in C_3$ be an arbitrary vertex. Then the cycle $a'a^*z_3ba'$ implies that $d(a', C_3) = |C_3|$ and hence there exists some $z_1 \in C_1$ such that $a'z_1 \notin E(H)$ by Claim 3. Moreover, it follows from Claim 4 that $a'z_2 \notin C_2$ for every $z_2 \in C_2$. The contradiction $H[\{a', a^*, z_1, z_2, z_3, b\}] \cong D_3$ finishes the proof of Claim 5.

Finally, we analyse the neighbourhood of a' and b' in $C_1 \cup C_3$ and $C_2 \cup C_3$, respectively.

Let $c_2 \in C_2$ be an arbitrary vertex. If $d(a', C_1 \cup C_3) > 0$ it follows by Claim 3 that there are vertices $c_1 \in C_1$ and $c_3 \in C_3$ such that either $a'c_1 \in E(H)$ and $a'c_3 \notin E(H)$ or $a'c_1 \notin E(H)$ and $a'c_3 \in E(H)$. In the first case, $H[\{a', a^*, c_1, c_2, c_3, b^*\}]$ is isomorphic to D_6, in the second case, it is isomorphic to D_4. We derive the same contradiction if $d(b', C_2 \cup C_3) > 0$.

Hence, assume that $d(a', C_1 \cup C_3) = d(b', C_2 \cup C_3) = 0$ and let $z_i \in C_i$ $(i = 1, 2, 3)$. Since we obtain the 6-hole $a'a^*c_1c_2b^*b'a'$ if $a'b' \in E(H)$, the vertices a' and b' are not adjacent. Thus $H[\{a', a^*, c_1, c_2, c_3, b^*, b'\}]$ is isomorphic to D_5, a contradiction.

This completes the proof of Step 2 and hence of the whole lemma. ■

To show that $\chi(G) = \beta(G)$ for a given graph G, it is enough to show the existence of a simplicial extreme in a subgraph H of G where the β-value of G is attained, by Lemma 1.6. If we choose such an H to be as small as possible with respect to inclusion, i.e., $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H, then by Lemma 4.1 every condition formulated in the preceeding sections involving diamonds can be replaced by a weaker condition, excluding not the diamond itself but D_3, D_6, and the graphs in Figure 2. This leads to better conditions since any supergraph of H is now allowed to contain diamonds which are embedded in G in a way different from the possibilities indicated by the six graphs $D_1, D_2, \ldots, D_6$.

Having a closer look to the list of possible embeddings of the diamond given by Lemma 4.1, it turns out that the graph D_6 is redundant. This will be shown in two steps.

Lemma 4.2. *Let H be a graph without 4-holes such that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. If H contains the subgraph D_6, then it also contains D_1, D_2, D_4, (see Figure 2), or D_3^* (see Figure 4).*

Proof. Suppose that the statement does not hold and let H be a counterexample. Hence, D_6 is an induced subgraph of H, but H does not contain D_1, D_2,D_4, or D_3^*.

Let $C, X, Y, Z \subseteq V(H)$ be disjoint vertex sets of H such that there is a partition $C = C_1 \cup C_2 \cup C_3 \cup C_4$ of C with the following properties:

$C_i \neq \emptyset$, $i \in \{1, 2, 3, 4\}$,

$c_1c_2, c_1c_3, c_2c_3, c_2c_4, c_3c_4 \in E(H)$ and $c_1c_4 \notin E(H)$, $\forall c_i \in C_i$, $i \in \{1, 2, 3, 4\}$,

$N(x, C \cup Y) = C_1 \cup C_2$ and $N(y, C \cup X) = C_3 \cup C_4$, $\forall x \in X$, $\forall y \in Y$,

$zu \in E(H)$, $\forall z \in Z$, $u \in C \cup X \cup Y$,

$X, Y \neq \emptyset$ ($Z = \emptyset$ is possible),

$H[X]$ and $H[Y]$ are connected subgraphs of H.

At least one such collection of vertex sets C, X, Y, Z exists, since H contains D_6. Note that the 4-hole-freeness of H implies that $H[C_i]$ ($i = 1, 2, 3, 4$) and $H[Z]$ are cliques. In the following, we may assume that the sets C, X, Y, and Z are chosen such that

1. $|C|$ is maximal,
2. $|X \cup Y|$ is maximal for this choice of C,
3. $|Z|$ is maximal for this choice of C, X, and Y.

Consider the proper subgraphs $X^+ = H[X \cup C_1 \cup C_2 \cup Z]$ and $Y^+ = H[Y \cup C_3 \cup C_4 \cup Z]$ of H. Then

$$d(x, X^+) = d(x, X) + |C_1| + |C_2| + |Z| \le |X| + |C_1| + |C_2| + |Z| - 1, \ \forall x \in X,$$

$$d(u, X^+) = X| + |C_1| + |C_2| + |Z| - 1, \ \forall u \in C_1 \cup C_2 \cup Z,$$

$$d(y, Y^+) = d(y, Y) + |C_3| + |C_4| + |Z| \le |Y| + |C_3| + |C_4| + |Z| - 1, \ \forall y \in Y,$$

$$d(v, Y^+) = |Y| + |C_3| + |C_4| + |Z| - 1, \ \forall v \in C_3 \cup C_4 \cup Z,$$

and hence there are vertices $x_0 \in X$ and $y_0 \in Y$ with $d(x_0, X^+) = \delta(X^+)$ and $d(y_0, Y^+) = \delta(Y^+)$. Since $\delta(X^+), \delta(Y^+) < \delta(H)$ and by the properties of C, X, Y, and Z described above, it follows that $X' := N(x_0, H - (C \cup X \cup Y \cup Z)) \neq \emptyset$ and $Y' := N(y_0, H - (C \cup X \cup Y \cup Z)) \neq \emptyset$.

If not otherwise specified, then in the following, c_i, $i \in \{1, 2, 3, 4\}$, x, y, and z are arbitrary vertices of C_i, X, Y, and Z, respectively. We proceed by proving four claims.

Claim 1. $d(x', C_1) = |C_1|$ and $d(y', C_4) = |C_4|$ for every $x' \in X', y' \in Y'$.
By symmetry, it suffices to show that $d(y', C_4) = |C_4|$ for an arbitrary $y' \in Y'$. Suppose to the contrary that there is a vertex $c_4^* \in C_4$ which is not adjacent to y'. Since we obtain the 4-hole $y'y_0c_4^*c_2y'$, if $c_2y' \in E(H)$ for some $c_2 \in C_2$, it follows that $d(y', C_2) = 0$. If $y'c_3 \notin E(H)$, then $H[\{x, c_2, c_3, c_4^*, y_0, y'\}]$ is therefore isomorphic to D_1 or D_2, and we conclude that $d(y', C_3) = |C_3|$. To avoid now $H[\{c_3, y', y_0, c_4^*, c_2, c_1\}] \cong D_4$, we obtain $d(y', C_1) = |C_1|$ and, since $H[\{c_3, y', y_0, c_4^*, c_2, c_1, x\}] \not\cong D_3^*$, this implies $d(y', X) = |X|$. Then the 4-hole $y'xc_2c_3y'$ completes the proof of the claim.

Claim 2. $d(x', C_3) > 0$ for every $x' \in X'$, or $d(y', C_2) > 0$ for every $y' \in Y'$.
Suppose to the contrary that there exist vertices $x' \in X'$ and $y' \in Y'$ such that $d(x', C_3) = 0$ and $d(y', C_2) = 0$. By Claim 1 and since there are no 4-holes in H, we deduce that then $d(x', C_4 \cup Y) = 0$ and $d(y', C_1 \cup X) = 0$.

If $d(x', C_2) = |C_2|$, then because $d(x', C_1) = |C_1|$ by Claim 1, we have $N(x', C \cup Y) = C_1 \cup C_2$. Hence, $\hat{X} = X \cup \{x'\}$ and $\hat{Y} = Y$ contradict the maximal choice of $|X \cup Y|$ (note that $\hat{X}$ is connected). Analogously, we can add y' to Y, if $d(y', C_3) = |C_3|$. Hence, there exist $c_2^* \in C_2$ and $c_3^* \in C_3$ such that $x'c_2^*, y'c_3^* \notin E(H)$. Then $H[\{x', y', c_1, c_2^*, c_3^*, c_4\}]$ is isomorphic to D_1 or D_2 and this contradiction completes the proof of Claim 2.

From now on, we assume without loss of generality that

$$d(y', C_2) > 0, \quad \forall y' \in Y'. \tag{6}$$

To avoid a 4-hole consisting of y', y_0 and some $c_2^* \in C_2$ with $y'c_2^* \in E(H)$, together with some $c_3 \in C_3$ or $z \in Z$, we conclude

$$d(y', C_3) = |C_3| \text{ and } d(y', Z) = |Z|, \quad \forall y' \in Y'. \tag{7}$$

Claim 3. $d(y', C_1) < |C_1|$ for every $y' \in Y'$.
If $d(y', C_1) = |C_1|$ for some $y' \in Y'$, then we have $d(y', C_4) = |C_4|$ by Claim 1, and we derive $d(y', C_2) = |C_2|$ as well (since $c_1c_2c_4y'c_1$ is not a 4-hole). Now, with (7), we obtain $d(y', C) = |C|$.

If $d(y', X) = |X|$, then by the maximal choice of $|Z|$, it follows that $y'y^* \notin E(H)$ for some $y^* \in Y$. But now the sets $\hat{C}_2 = C_2 \cup \{y'\}$, $\hat{C}_i = C_i$ $(i = 1, 3, 4)$, $\hat{X} = X$, and $\hat{Y} = \{y^*\}$ contradict the maximal choice of $|C|$.

If $y'x^* \notin E(H)$ for some $x^* \in X$, then we derive the same contradiction with the sets $\hat{C}_3 = C_3 \cup \{y'\}$, $\hat{C}_i = C_i$ $(i = 1, 2, 4)$, $\hat{X} = \{x^*\}$, and $\hat{Y} = \{y_0\}$.

Claim 4. $d(y', Y) = |Y|$ for every $y' \in Y'$ and $H[Y']$ is a clique.
To show the first statement, let $y' \in Y'$ be an arbitrary vertex. By the definition of Y', we have $y'y_0 \in E(H)$. Now let $y_1 \in N(y_0, Y)$. By Claim 3 and (6), there are $c_1^* \in C_1$ and $c_2^* \in C_2$ such that $y'c_1^* \notin E(H)$ and $y'c_2^* \in E(H)$. Since $H[\{c_3, c_1^*, c_2^*, y', y_0, y_1\}] \ncong D_4$, it follows that $y'y_1 \in E(H)$. Since Y is connected, the desired result follows inductively.

Now suppose to the contrary that Y' does not induce a clique, say $y_1'y_2' \notin E(H)$ for some $y_1', y_2' \in Y'$. Then no vertex $c_2 \in C_2$ is adjacent to both y_1' and y_2', since otherwise $H[\{y_1', c_2, y_2', y_0\}]$ is a 4-hole. By (6), this implies $d(y_1', C_2), d(y_2', C_2) < |C_2|$ and therefore $d(y_1', C_1) = d(y_2', C_1) = 0$, since otherwise $y_1'c_1\hat{c}_2c_4y_1'$ respectively $y_2'c_1\hat{c}_2c_4y_2'$ would be a 4-hole (where $\hat{c}_2$ is such that $y_i'\hat{c}_2 \notin E(H)$, $i = 1, 2$). Let $c_2^* \in C_2$ such that $y_1'c_2^* \in E(H)$. Then $y_2'c_2^* \notin E(H)$ and with (7) we derive the contradiction $H[\{c_3, c_1, c_2^*, y_1', y_0, y_2'\}] \cong D_4$.

Consider the proper subgraph $Y^{++} = H[Y \cup C_3 \cup C_4 \cup Z \cup Y']$ of H. By (7) and Claims 1 and 4, it follows that

$$\begin{aligned} d(y, Y^{++}) &= d(y, Y) + |C_3| + |C_4| + |Z| + |Y'|, \quad \forall y \in Y, \\ d(v, Y^{++}) &= |Y| + |C_3| + |C_4| + |Z| + |Y'| - 1, \quad \forall v \in C_3 \cup C_4 \cup Z \cup Y'. \end{aligned}$$

Hence, since $y_0 \in Y$ has minimal degree in Y^+, it also has minimal degree vertex in Y^{++}. Since $d(y_0, X \cup C_1 \cup C_2) = 0$, this implies that there is a vertex in $V(H - (C \cup X \cup Y \cup Z \cup Y'))$ that is adjacent to y_0. This contradiction to the definition of Y' finishes the proof of the lemma. ■

Lemma 4.3. *Let H be a graph without 4- and 6-holes such that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. If D_3^* is contained in H, then H also contains D_1, D_2, or D_4.*

Proof. Assume that the statement does not hold, i.e., there is a graph H with the desired degree property that contains D_3^* but none of D_1, D_2, and D_4. Consider disjoint vertex sets $A, U, V, Z \subseteq V(H)$ with the following properties:

$U = \{u_0, u_1, u_2, u_3, u_4\}$ such that $H[U]$ is a 5-hole with cycle $u_0u_1u_2u_3u_4u_0$,

$Z \neq \emptyset$ with $d(z, U) = |U|$, $\forall z \in Z$,

$A \neq \emptyset$ with $N(a, U \cup Z) = \{u_0, u_1\}$, $\forall a \in A$,

$d(v, U \cup Z \cup A) = |U \cup Z \cup A|$, $\forall v \in V$ ($V = \emptyset$ is possible).

Note that such a collection of vertex sets exists since H contains D_3^*. In the following let the sets A, U, V, and Z be chosen such that

1. $|U \cup Z|$ is maximal,
2. $|A|$ is maximal for this choice of U and Z,
3. $|V|$ is maximal for this choice of U, Z and A.

Consider the proper subgraph $A^+ = H[\{u_0, u_1\} \cup A \cup V]$ of H. Since $H[V]$ is a clique (two non-adjacent vertices of V together with u_0 and u_2 induce a 4-hole) and by the properties above, it follows that

$$\begin{aligned} d(u_0, A^+) &= d(u_1, A^+) = |A| + |V| + 1, \\ d(a, A^+) &= d(a, A) + 2 + |V| \leq |A| + |V| + 1, \quad \forall a \in A, \\ d(v, A^+) &= |A| + |V| + 1, \quad \forall v \in V. \end{aligned}$$

Hence, there is a vertex $a^* \in A$ with $d(a^*, A^+) = \delta(A^+)$ and by the hypothesis, $a^*a' \in E(H)$ for some $a' \in V(H - A^+)$. In fact, $N(a^*, U \cup Z) = \{u_0, u_1\}$ implies that $a' \in V(H - (A \cup U \cup V \cup Z))$. In the following, let z denote an arbitrary vertex in Z.

Claim 1. $d(a', Z) = 0$.
Suppose to the contrary that $a'z_0 \in E(H)$ for some $z_0 \in Z$. Then $a'u_0, a'u_1 \in E(H)$, since there is no 4-hole in H. We show next that $a'u_2 \notin E(H)$ and $a'u_4 \notin E(H)$.

By symmetry it suffices to verify the first statement. Assume that $a'u_2 \in E(H)$. Then $H[\{u_0, u_1, u_2, u_3, u_4, a'\}] \ncong D_2$ implies that $a'u_3 \in E(H)$ or $a'u_4 \in E(H)$. Indeed, since there is no 4-hole in H, *both* u_3 and u_4 are neighbors of a' and we conclude that $d(a', U) = |U|$. To avoid a 4-hole consisting of a', u_1, u_3 and z, we furthermore have $d(a', Z) = |Z|$. Now the maximality of $|V|$ implies that there exists a vertex $a_0 \in A$ with $a'a_0 \notin E(H)$ and hence the sets $\hat{U} = U$, $\hat{Z} = Z \cup \{a'\}$, $\hat{A} = \{a_0\}$ contradict the maximal choice of $|U \cup Z|$.

Since $a'u_2, a'u_4 \notin E(H)$, we also have $a'u_3 \notin E(H)$ and we derive the contradiction $H[\{a', u_1, u_2, u_3, u_4, z_0\}] \cong D_4$.

Claim 2. $a'u_0 \notin E(H)$ and $a'u_1 \notin E(H)$.
By symmetry, we only show the second statement. Assume that $a'u_1 \in E(H)$. Then $a'u_3 \notin E(H)$ and $a'u_4 \notin E(H)$, since otherwise $a'u_1zu_3a'$ would be a 4-hole (using $a'z \notin E(H)$ by Claim 1). Moreover, $a'u_2 \notin E(H)$, since otherwise either $H[\{a^*, a', u_2, u_3, u_4, u_0\}]$ is a 6-hole (if $a'u_0 \notin E(H)$) or $H[\{a', u_2, z, u_0\}]$ is a 4-hole (if $a'u_0 \in E(H)$).

But now $H[\{a', a^*, u_0, z, u_2, u_1\}] \ncong D_4$ leads to $a'u_0 \in E(H)$. Recalling Claim 1, the set $\hat{A} = A \cup \{a'\}$ contradicts the maximality of $|A|$.

By Claim 1 and 2, the subgraph $H[\{a', a^*, u_0, u_1, z, u_3\}]$ is isomorphic to D_1 or D_2 which completes the proof of the lemma. ■

Lemmas 4.2 and 4.3 can be combined as follows.

Corollary 4.4. *Let H be a graph without 4- and 6-holes such that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. If H contains D_6, then it also contains D_1, D_2, or D_4.*

The following lemma presents another refinement of the set of forbidden induced subgraphs for β-perfect graphs. It states that the 5-wheel D_3 can be deleted from the list of supergraphs of the diamond given in Lemma 4.1, since it is either redundant or yields the desired equality $\chi(G) = \beta(G)$. Note that if we further exclude a 6-hole in the graph H in consideration, the subgraph D_6 is also redundant by Corollary 4.4.

Lemma 4.5. *Let H be a graph without 4-holes such that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. If H contains D_3, then H also contains D_1, D_2, D_4, or D_6 (see Figure 2), or $\chi(H) = \beta(H)$.*

Proof. Consider a counterexample H. So H contains D_3, it satisfies $\chi(H) < \beta(H)$, and it does not contain D_1, D_2, D_4 or D_6.

Since D_3 is a subgraph of H, there exist vertex sets $U, Z \subseteq V(H)$ such that

$$U = \{u_0, u_1, u_2, u_3, u_4\},\ Z \neq \emptyset,\ U \cap Z = \emptyset,$$

$$H[U] \text{ is a 5-hole with cycle } u_0u_1u_2u_3u_4u_0,$$

$$zu_i \in E(H), \quad \forall z \in Z,\ u_i \in U.$$

Let Z be chosen such that $|Z|$ is maximal for the given U, and define $W = H[U \cup Z]$. Since H does not contain a 4-hole, Z induces a clique. Hence, $\chi(W) = |Z| + 3$ and

$$(8) \qquad d(u_i, W) = |Z| + 2, \quad \forall u_i \in U \quad \text{and} \quad d(z, W) = |Z| + 4, \quad \forall z \in Z.$$

We claim that W is a proper subgraph of H. Indeed, if $W = H$ then $\delta(H) = |Z| + 2$ by (8) and we derive the contradiction $\chi(H) = |Z| + 3 = \delta(H) + 1 = \beta(H)$.

Hence, $\delta(W) < \delta(H)$ and, by (8), we have $N_i := N(u_i, H - W) \neq \emptyset$ for $i \in \{0, 1, \ldots, 4\}$. In the following, all indices appearing in connection with some N_i should be taken modulo 5. We proceed by proving three claims.

Claim 1. $N_i \cap (N_{i-2} \cup N_{i+2}) = \emptyset$ for every $0 \leq i \leq 4$.
Suppose to the contrary that $N_i \cap (N_{i-2} \cup N_{i+2}) \neq \emptyset$ for some $i = 0, 1, \ldots, 4$. Without loss of generality, let $i = 0$ and $v \in N_0 \cap N_2$. To avoid a 4-hole $vu_0u_1u_2v$, we have $u_1v \in E(H)$. Consider the subgraph $H[U \cup \{v\}]$ of H. Since H does not contain D_2, it follows that v is adjacent to u_3 or to u_4. In fact, since H is 4-hole-free, both vu_3 and vu_4 are in $E(H)$. But now the choice $Z' := Z \cup \{v\}$ contradicts the maximality of $|Z|$.

Claim 2. $N_i \cap (N_{i-1} \cup N_{i+1}) = \emptyset$ for every $0 \leq i \leq 4$.
Again we assume that there exists a vertex $v \in N_i \cap (N_{i+1} \cup N_{i-1})$ for some $0 \leq i \leq 4$, where we suppose $i = 0$ and $v \in N_0 \cap N_1$. By Claim 1, $v \notin N_2 \cup N_3 \cup N_4$. For some $z \in Z$, consider the subgraph $H[\{z, v, u_0, u_1, u_2, u_3\}]$ of H. Since there is no D_6 in H, it follows that $vz \in E(H)$. Then $H[\{z, v, u_0, u_4, u_3, u_2\}] \cong D_4$, a contradiction.

Claim 3. $vz \notin E(H)$ for every $v \in N_0 \cup \ldots \cup N_4$, $z \in Z$.

If $vz \in E(H)$ for some $z \in Z$ and $v \in N_0 \cup \ldots \cup N_4$, say $v \in N_1$, then Claim 1 and 2 imply that $vu_j \notin E(H)$ for $j = 2, 3, 4$ and the contradiction $H[\{z, v, u_1, u_2, u_3, u_4\}] \cong D_4$ proves the claim.

Let $v_0 \in N_0$ and $v_2 \in N_2$. By Claim 1, 2, and 3, $v_0 \neq v_2$ and the subgraph $H[\{v_0, v_2, u_0, u_1, u_2, z\}]$, where $z \in Z$ is an arbitrary vertex, is isomorphic to D_1 or D_2. This contradiction completes the proof of the lemma. ■

Note that the graph W mentioned in the proof of Lemma 4.5 fulfils $\chi(W) = \beta(W)$ without having a simplicial extreme.

By applying the technique from the proofs of Lemmas 4.1 up to 4.5 to diamond-free graphs, we can argue next that also the condition in Theorem 1.1 involving the short-chorded 6-cycle can be replaced by a weaker condition involving two supergraphs of the short-chorded 6-cycle. This is summarized in the following lemma.

Lemma 4.6. *Let H be a diamond-free graph without 4-holes, such that $\delta(H') < \delta(H)$ for every proper induced subgraph H' of H. Then H contains a short-chorded 6-cycle if and only if it contains one of the graphs in Figure 3 (S_1 or S_2) as an induced subgraph.*

Proof. Consider a counterexample H. Since H contains a short-chorded 6-cycle, it contains a 5-hole $Q = H[x_1, x_2, u, v, w]$ (where $x_1x_2uvwx_1$ is a cycle), such that the set A defined as

$$A := \{a \in V(H) \mid ax_1, ax_2 \in E(H), au, av, aw \notin E(H)\}$$

is nonempty. Now, in addition assume that $|A|$ is maximal.

Because $A^+ := H[A \cup \{x_1, x_2\}]$ is a proper induced subgraph of H, by assumption it holds that $\delta(A^+) < \delta(H)$. Moreover,

$$d(a, A^+) \leq |A| + 1 = d(x_i, A^+),\ \forall a \in A, i \in \{1, 2\}.$$

It follows that there is an $a^* \in A$ satisfying $d(a^*, A^+) = \delta(A^+)$, and (because $\delta(A^+) < \delta(H)$) that there is an $a' \in V(H - A^+)$ with $a^*a' \in E(H)$. Since a^* is not adjacent to u, v, or w, the vertex a' is not in Q.

We claim that

$$a'u \notin E(H), a'w \notin E(H).$$

Indeed, suppose without loss of generality that $a'u \in E(H)$, then since H does not contain a 4-hole, also $a'x_2 \in E(H)$. But this means that $H[a^*, a', u, x_2]$ is a diamond, a contradiction.

If $a'x_1 \in E(H)$, then also $a'x_2 \in E(H)$ since otherwise $H[a^*, a', x_1, x_2]$ would be a diamond. Thus by symmetry, either both $a'x_1$ and $a'x_2$ are edges of H, or both are not.

Moreover, if $a'x_1 \in E(H)$, then $a'v \notin E(H)$ because H is 4-hole-free and $a'w \notin E(H)$. Hence, if $a'x_1$ and $a'x_2$ are edges of H, then there are no other edges between a' and Q in H, and $A' := A \cup \{a'\}$ contradicts the maximality of A.

So there are only two possibilities: either a' is not adjacent to any vertex of Q, in which case H contains S_1 as an induced subgraph, or the only neighbour of a' in Q is v, in which case H contains S_2 as an induced subgraph. This contradiction finishes the proof. ∎

These results together imply Theorem 1.8.

***Proof of Theorem* 1.8.** Let G be a graph not containing any of the graphs depicted in Figure 2 and Figure 3 as induced subgraphs. For showing that G is β-perfect, it suffices to prove $\chi(G) = \beta(G)$.

Let H be a minimal induced subgraph of G satisfying $\beta(G) = \delta(H) + 1$. If H is diamond-free, then H does not contain any short-chorded 6-cycles, by Lemma 4.6. Thus, by Theorem 1.3, H contains a simplicial extreme, which by Lemma 1.6 leads to $\chi(G) = \beta(G)$.

If H contains a diamond then, by Lemma 4.1, Corollary 4.4, and the hypothesis, H contains D_3. Now Lemma 4.5 implies

$$\chi(G) \leq \beta(G) = \delta(H) + 1 = \beta(H) = \chi(H) \leq \chi(G). \qquad ∎$$

Note that both graphs in Figure 3 contain a net. Therefore, the above theorem implies Theorem 1.5, which was derived in another way in Section 3 (there we did not use Theorem 1.3, but explicitly showed existence of a simplicial extreme for the graph class in consideration in Theorem 3.2).

All graphs in Figures 2 and 3 are β-perfect. For D_1, D_2, and D_4, we have examples showing that it is not possible to delete either one of them from the list of forbidden subgraphs. For the other graphs (D_5, S_1, and S_2) we have no such examples, and we believe that they are superfluous in this list.

Lemma 4.5 and Corollary 4.4 also easily allow us to prove Theorem 1.9 from the weaker Theorem 1.4.

***Proof of Theorem* 1.9.** Let G be a claw-free, even hole-free, D_1-free, D_2-free graph. Let H be a minimal induced subgraph satisfying $\beta(G) = \delta(H) + 1$. By Theorem 1.4 we may assume that H contains D_3. By Lemma 4.5 and Corollary 4.4, H contains D_4 or satisfies $\chi(H) = \beta(H)$. But since D_4 is not claw-free, we must have $\chi(H) = \beta(H)$, which implies (as before) that G is β-perfect. ■

5. Regular Graphs

In this section, we observe that β-perfect graphs are not only even hole-free but that in fact they do not contain any induced regular subgraphs, except perhaps odd holes and cliques. For graphs with maximum degree at most three, we also show the converse.

Regular graphs are examples of graphs G for which $\delta(H) < \delta(G)$ for all proper induced subgraphs H of G, as is stated (and generalized) in the following lemma. Note that $\beta(G) = \delta(G) + 1$ for those graphs.

Lemma 5.1. *Let G be a regular, connected graph, then $\delta(H) < \delta(G)$ for every proper induced subgraph H of G. More generally, G has the property that $\delta(H) < \delta(G)$ for every proper induced subgraph H of G with $|V(H)| > k$, if for some nonnegative integer k, G contains at least $|V(G)| - k$ vertices of degree $\delta(G)$ and it is $(k+1)$-connected.*

Proof. We prove the general assertion. Let G be a $(k+1)$-connected graph containing at most k vertices of degree greater than $\delta(G)$. If $\delta(H) \geq \delta(G)$ for some proper induced subgraph H of G with $|V(H)| > k$, then no vertex in $V(G-H)$ is adjacent to a vertex in $V(H)$ of degree $\delta(G)$ in G. In other words, the neighbours in $V(H)$ of every vertex $v \in V(G-H)$ are contained in the set of vertices in $V(H)$ of degree greater than $\delta(G)$ in G, which has size at most k. But this means that G has a vertex cut of size at most k, a contradiction. ■

Complete graphs and odd cycles are easily seen to be β-perfect. But these are the only regular β-perfect graphs, as follows from the following lemma (see also [8, section 4.1] for the connection between Brooks' Theorem and $\beta(G)$).

Lemma 5.2. *Let G be an r-regular, connected graph, for some nonnegative integer r. Then $\chi(G) < \beta(G)$, unless $G = K_{r+1}$, or $r = 2$ and G is an odd cycle.*

Proof. If G is r-regular, then $\beta(G) = r + 1$, because $\delta(G) = r$, and no proper subgraph H of G has $\delta(H) > \delta(G)$ by Lemma 5.1. Now Brooks' Theorem [2] states that $\chi(G) < \Delta(G) + 1 = r + 1 = \beta(G)$, unless G is a complete graph on $r + 1$ nodes, or $r = 2$ and G is an odd cycle. ∎

The next lemma states that the converse statement is true for even hole-free graphs with maximum degree at most 3.

Lemma 5.3. *Let G be a connected even hole-free graph, with $\delta(G) < \Delta(G) \leq 3$. Then G is β-perfect.*

Proof. Let G' be any induced subgraph of G. Let H be a minimal induced subgraph of G' such that $\beta(G') = \delta(H) + 1$. Clearly $\delta(H) \leq 3$. Moreover, $\delta(H) = 3$ is impossible, since G is not regular, and can not contain a proper $\Delta(G)$-regular subgraph. If $\delta(H) \leq 2$, then H contains a simplicial extreme, and hence $\chi(G') = \beta(G')$, by Lemma 1.6. ∎

Together, these statements imply the following.

Theorem 5.4 *Let G be a 3-regular, connected, even hole-free graph not equal to K_4. Then G is minimally β-imperfect.*

Proof. Directly from Lemma 5.2 and Lemma 5.3. ∎

The corresponding statement for 4-regular graphs is false: below we present a 4-regular graph, not containing any 2- or 3-regular subgraphs except odd holes and cliques, which is not minimally β-imperfect.

Another immediate corollary of Lemma 5.2 is the following.

Theorem 5.5. *Let G be a β-perfect graph. Then G does not contain any induced regular subgraphs, except perhaps odd holes and cliques.*

Proof. Directly from Lemma 5.2. ∎

Note that the above theorem generalizes the fact that β-perfect graphs are even hole-free. However, the condition in Theorem 5.5 is still not strong enough to imply β-perfectness of a graph. This is illustrated by the following example.

Define R as the graph on 15 vertices obtained from a triangle $x_1x_2x_3$ by adding three copies of the graph D_1 in Figure 2, where the two vertices of degree 1 are identified with x_1 and x_2, x_2 and x_3, x_3 and x_1, respectively.

It is easy to see that the only regular induced subgraphs of R are cliques on three vertices and 5-holes. Also, $3 = \chi(R) < \beta(R) = 4$.

Moreover, if we add three vertices to R, each of which is adjacent to the four vertices of a diamond in R, we obtain a connected 4-regular graph with only cliques and 5-holes as induced regular subgraphs and which is not minimally β-imperfect (since it contains R).

Acknowledgements

Research for this paper was carried out while the second author was visiting the University of Twente. She would like to thank the people there, in particular Prof. C. Hoede, for the hospitality and for making this stay possible. Both authors thank H. Broersma, C. Hoede, J. Hurink, and I. Schiermeyer for many fruitful discussions.

References

[1] L.W. Beineke, *Characterizations of derived graphs*, J. Combin. Theory **9** (1970) 129–135.

[2] R.L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. **37** (1941) 194–197.

[3] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, *Finding an even hole in a graph*, in: Proceedings of the 38^{th} Annual Symposium on Foundations of Computer Science (1997) 480–485.

[4] G.A. Dirac, *On rigid circuit graphs*, Abh. Math. Univ. Hamburg **25** (1961) 71–76.

[5] P. Erdős and A. Hajnal, *On the chromatic number of graphs and set-systems*, Acta Math. Acad. Sci. Hungar. **17** (1966) 61–99.

[6] C. Figueiredo and K. Vušković, *A class of β-perfect graphs*, Discrete Math. **216** (2000) 169–193.

[7] H.-J. Finck and H. Sachs, *Über eine von H.S. Wilf angegebene Schranke für die chromatische Zahl endlicher Graphen*, Math. Nachr. **39** (1969) 373–386.

[8] T.R. Jensen and B. Toft, Graph colouring problems (Wiley, New York, 1995).

[9] S.E. Markossian, G.S. Gasparian and B.A. Reed, *β-perfect graphs*, J. Combin. Theory (B) **67** (1996) 1–11.

Received 10 August 2000
Revised 3 July 2001