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A PROOF OF MENGER'S THEOREM BY CONTRACTION

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Abstract

A short proof of the classical theorem of Menger concerning the number of disjoint AB-paths of a finite graph for two subsets A and B of its vertex set is given. The main idea of the proof is to contract an edge of the graph.

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Proofs of Menger's Theorem are given in [7, 6, 4, 8, 2]. A short proof is given by T. Böhme, F. Göring and J. Harant in [1]; another short proof based on edge deletion is given by the author in [5]. The new idea here is to get a short proof by contracting an arbitrary edge of the original graph.

For terminology and notation not defined here we refer to [3]. A graph with no edges is denoted by its vertex set. Let G be a finite graph (loops and multiple edges being allowed). For an edge e of G let $G - e$ and G/e denote the graphs obtained from G by removing e and contracting e to one vertex v_e, respectively. For (possibly empty) sets of vertices A and B of G let an *AB-separator* be a set of vertices of G such that the graph obtained from G by deleting these vertices contains no path from A to B. Note that a single vertex of $A \cap B$ is considered as a path from A to B, too. An *AB-connector* is a subgraph of G such that each of its components is a path from A to B having only one vertex in common with A and B, respectively. In particular the empty graph is also an AB-connector. If we contract an egde incident with a vertex of A or B then the resulting vertex is considered to be a vertex of A or B, respectively.

Theorem (Menger, 1927)**.** *Let G be a finite graph, A and B sets of vertices of G, and s the minimum number of vertices forming an AB-separator. Then there is an AB-connector C with $|C \cap A| = s$.*

Proof. If G is edgeless then set $C = A \cap B$. Suppose, G is a counterexample with $|E(G)|$ minimal. Then G contains an edge e from x to y and G/e has an AB-separator S with $|S| < s$, otherwise we are done. Obviously, $v_e \in S$. Then $P = (S \setminus \{v_e\}) \cup \{x, y\}$ is an AB-separator of G with $|P| = |S| + 1 = s$. An AP-separator (as well as an PB-separator) of $G - e$ is an AB-separator of G. Consequently, $G - e$ has an AP-connector X and a PB-connector Y containing P. Since $X \cap Y = P$, the set $C = (X \cup Y)$ is an AB-connector of G with $|C \cap A| = s$, a contradiction. ■

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