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WEAKLY $\mathcal{P}$-SATURATED GRAPHS

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Abstract

For a hereditary property $\mathcal{P}$ let $k_{\mathcal{P}}(G)$ denote the number of forbidden subgraphs contained in G. A graph G is said to be *weakly $\mathcal{P}$-saturated*, if G has the property $\mathcal{P}$ and there is a sequence of edges of $\overline{G}$, say $e_1, e_2, \ldots, e_l$, such that the chain of graphs $G = G_0 \subset G_0 + e_1 \subset G_1 + e_2 \subset \ldots \subset G_{l-1} + e_l = G_l = K_n$ $(G_{i+1} = G_i + e_{i+1})$ has the following property: $k_{\mathcal{P}}(G_{i+1}) > k_{\mathcal{P}}(G_i)$, $0 \le i \le l-1$.

In this paper we shall investigate some properties of weakly saturated graphs. We will find upper bound for the minimum number of edges of weakly $\mathcal{D}_k$-saturated graphs of order n. We shall determine the number $\mathrm{wsat}(n, \mathcal{P})$ for some hereditary properties.

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1. Introduction and Notation

We consider finite undirected graphs without loops or multiple edges. A graph G has a vertex set $V(G)$ and an edge set $E(G)$. Let $v(G)$, $e(G)$ denote the number of vertices and the number of edges of G, respectively. We say that G *contains* H whenever G contains a subgraph isomorphic to H.

The degree of $v \in V(G)$ is denoted by $d_G(v)$. The number of edges of a path is called the *length* of the path.
Let $\mathcal{I}$ denote the class of all graphs with isomorphic graphs being regarded as equal. If $\mathcal{P}$ is a proper nonempty subclass of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property of being in $\mathcal{P}$. We shall use the terms *class of graphs* and *property of graphs* interchangeably.

A property $\mathcal{P}$ is called *hereditary* if every subgraph of a graph G with property $\mathcal{P}$ also has property $\mathcal{P}$.

We list some properties to introduce the necessary notation which will be used in the paper. Let k be a non-negative integer.

$\mathcal{O} = \{G \in \mathcal{I} : G \text{ is totally disconnected}\}$,

$\mathcal{O}_k = \{G \in \mathcal{I} : \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\}$,

$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ contains no subgraph isomorphic to } K_{k+2}\}$,

$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,

$\mathcal{D}_k = \{G \in \mathcal{I} : G \text{ is } k\text{-degenerated, i.e., } \delta(H) \leq k \text{ for any } H \leq G\}$,

$\mathcal{W}_k = \{G \in \mathcal{I} : \text{the length of the longest path in } G \text{ is at most } k\}$.

Let $\mathcal{P}$ be a nontrivial hereditary property. Then there is a nonnegative integer $c(\mathcal{P})$, called the *completeness* of $\mathcal{P}$, such that $K_{c(p)+1} \in \mathcal{P}$ but $K_{c(p)+2} \notin \mathcal{P}$. Obviously

$$c(\mathcal{O}_k) = c(\mathcal{I}_k) = c(\mathcal{S}_k) = c(\mathcal{D}_k) = c(\mathcal{W}_k) = k.$$

For a hereditary property $\mathcal{P}$ the set of all *minimal forbidden subgraphs* of $\mathcal{P}$ is defined by

$\mathrm{F}(\mathcal{P}) = \{G \in I : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}$.

A graph is called $\mathcal{P}$-*maximal* if it does not contain any forbidden subgraph but it will contain a forbidden subgraph when any new edge is added to the graph. Let $\mathrm{M}(\mathcal{P})$ be the set of all $\mathcal{P}$-maximal graphs. The set of $\mathcal{P}$-maximal graphs of order n is denoted by $\mathrm{M}(n, \mathcal{P})$.

Many problems of extremal graph theory can be formulated as follows: What is the maximum (minimum) number of edges in a $\mathcal{P}$-maximal graph of order n? For a given hereditary property $\mathcal{P}$ we define those two numbers in the following manner:

$$\mathrm{ex}(n,\mathcal{P}) = \max\{e(G) : G \in \mathrm{M}(n,\mathcal{P})\},$$
$$\mathrm{sat}(n,\mathcal{P}) = \min\{e(G) : G \in \mathrm{M}(n,\mathcal{P})\}.$$

The set of all $\mathcal{P}$-maximal graphs of order n with exactly $\mathrm{ex}(n,\mathcal{P})$ edges is denoted by $\mathrm{Ex}(n,\mathcal{P})$. The members of $\mathrm{Ex}(n,\mathcal{P})$ are called *$\mathcal{P}$-extremal* graphs. By the symbol $\mathrm{Sat}(n,\mathcal{P})$ is denoted the set of all $\mathcal{P}$-maximal graphs of order n with $\mathrm{sat}(n,\mathcal{P})$ edges. These graphs are called *$\mathcal{P}$-saturated.*

The most famous Turán's Theorem [6] establishes the number of edges of $\mathcal{I}_k$-extremal graphs. On the other hand, Erdös, Hajnal and Moon [2] calculated the number $\mathrm{sat}(n,\mathcal{I}_k)$.

Bollobás [1] introduced the concept of a weakly k-saturated graph. Consider a graph of order n and add all those edges which are the only missing edge of complete graph of order k (i.e., we add the edge e if there are k such vertices of the graph, that the graph contains all the edges spanned by these k vertices, saving e). If by repeating this process a sufficient number of times the complete graph of order n is obtained, the original graph will be called *weakly k-saturated.*

Bollobás showed that if a graph G of order n is weakly k-saturated (for $3 \leq k \leq 7$) with the minimum number of edges then $e(G) = (k-2)n - \binom{k-1}{2}$. In the general case (i.e., for $k \geq 3$) the equality has been proved by Kalai [5].

Let $\mathcal{P}$ be a hereditary property and let $k_{\mathcal{P}}(G)$ denote the number of forbidden subgraphs contained in G. A graph G is said to be *weakly $\mathcal{P}$-saturated*, if G has the property $\mathcal{P}$ and there is a sequence of edges of $\overline{G}$, say $e_1, e_2, \ldots, e_l$, such that the chain of graphs $G = G_0 \subset G_0 + e_1 \subset G_1 + e_2 \subset \ldots \subset G_{l-1} + e_l = G_l = K_n$ $(G_{i+1} = G_i + e_{i+1})$ has the following property: $k_{\mathcal{P}}(G_{i+1}) > k_{\mathcal{P}}(G_i)$, $0 \leq i \leq l-1$. This sequence of edges will be called the *complementary sequence of G with respect to $\mathcal{P}$* or briefly the *complementary sequence* if it does not lead us to misunderstanding.

According to our terminology a weakly k-saturated graph is called weakly $\mathcal{I}_{k-2}$-saturated.

Let us denote a set of all weakly $\mathcal{P}$-saturated graphs of order n by $\mathrm{WSat}(n,\mathcal{P})$. Let the minimum and the maximum number of edges in a graph of $\mathrm{WSat}(n,\mathcal{P})$ be denoted by

$$\mathrm{wsat}(n,\mathcal{P}) = \min\{e(G) : G \in \mathrm{WSat}(n,\mathcal{P})\},$$

$$\mathrm{wex}(n,\mathcal{P}) = \max\{e(G) : G \in \mathrm{WSat}(n,\mathcal{P})\}.$$

From Theorem of Kalai and Theorem of Erdös, Hajnal, Moon it follows that $\mathrm{wsat}(n,\mathcal{I}_k) = \mathrm{sat}(n,\mathcal{I}_k)$. In Section 2 we shall describe a hereditary property $\mathcal{P}$ such that $\mathrm{wsat}(n,\mathcal{P}) < \mathrm{sat}(n,\mathcal{P})$. We will also investigate some properties of weakly saturated graphs. In Section 3 examples of weakly $\mathcal{D}_k$-saturated graphs and an upper bound for the number $\mathrm{wsat}(n,\mathcal{D}_k)$ will be given. In Section 4 we shall determine the number $\mathrm{wsat}(n,\mathcal{P})$ for some hereditary properties.

2. Some Properties of Weakly $\mathcal{P}$-Saturated Graphs

From the definition of weakly $\mathcal{P}$-saturated graphs it follows that any $\mathcal{P}$-maximal graph is weakly $\mathcal{P}$-saturated. First we prove that the maximum number of edges of weakly $\mathcal{P}$-saturated graphs is equal to the maximum number of edges of $\mathcal{P}$-maximal graphs.

Theorem 1. *Let $n \geq 1$. If $\mathcal{P}$ is a hereditary property, then* $\mathrm{wex}(n,\mathcal{P}) = \mathrm{ex}(n,\mathcal{P})$.

Proof. Every $\mathcal{P}$-maximal graph is weakly $\mathcal{P}$-saturated. Thus $\mathrm{wex}(n,\mathcal{P}) \geq \mathrm{ex}(n,\mathcal{P})$. On the other hand, if a graph of order n has more than $\mathrm{ex}(n,\mathcal{P})$ edges then it contains a forbidden subgraph. Hence $\mathrm{wex}(n,\mathcal{P}) \leq \mathrm{ex}(n,\mathcal{P})$. ■

Any non-negative integer valued function $f : \mathcal{I} \to N$ is called the *graph invariant* (*invariant*, for short). For a hereditary property $\mathcal{P}$ let us define the number

$$f(\mathcal{P}) = \min\{f(H) : H \in \mathrm{F}(\mathcal{P})\}.$$

Theorem 2. *Let $f(G)$ be an invariant satisfying:*

(1) *$f(H) \leq f(G)$ for $H \subseteq G$,*

(2) *$f(G+e) \leq f(G)+1$ for $e \in E(\overline{G})$.*

Then for any graph $G \in \mathrm{WSat}(n,\mathcal{P})$ with $n \geq c(\mathcal{P})+2$, we have

$$f(G) \geq f(\mathcal{P}) - 1.$$

Proof. From the definition of weakly $\mathcal{P}$-saturated graphs, it follows that there is an edge $e \in E(\overline{G})$ and a graph $F \in \mathrm{F}(\mathcal{P})$ such that $F \subseteq G + e$. Thus $f(\mathcal{P}) \leq f(F) \leq f(G+e) \leq f(G) + 1$. ■

The chromatic number and the clique number are examples of invariant satisfying assumptions of Theorem 2. The edge connectivity $\lambda(G)$ does not satisfy the assumption (1) of Theorem 2, but we shall prove that for $G \in \mathrm{WSat}(n, \mathcal{P})$ the inequality $\lambda(G) \geq \lambda(\mathcal{P}) - 1$ also holds.

Theorem 3. *Let $\lambda(\mathcal{P}) = \lambda > 0$ and $G \in \mathrm{WSat}(n, \mathcal{P})$. Then*

$$\lambda(G) \geq \lambda - 1.$$

Proof. Let S be an edge cutset of G such that $\lambda(G) = |S|$. Let G', G'' be two components of $G - S$. Since G is weakly $\mathcal{P}$-saturated, it follows that there is a complementary sequence $e_1, e_2, \ldots, e_l$ of G. Let e_i be the first edge of the sequence $e_1, e_2, \ldots, e_l$, which joins a vertex of G' with a vertex of G''. Let F denote a subgraph of $G_{i-1} + e_i$, which contains the edge e_i and is isomorphic with some graph of $\mathrm{F}(\mathcal{P})$. Then the set $S \cup \{e_i\}$ is an edge cutset of F. Thus $\lambda \leq \lambda(F) \leq |S| + 1 = \lambda(G) + 1$. ■

From the next theorem it follows that the behaviour of $\mathrm{wsat}(n, \mathcal{P})$ is not monotone in general.

Theorem 4. *Let $\mathcal{P}$ be the hereditary property such that $\mathrm{F}(\mathcal{P}) = \{2K_2\}$. Then*

$$\mathrm{wsat}(n, \mathcal{P}) = \begin{cases} 3, & \text{for } n = 4, \\ 1, & \text{for } n \geq 5\ . \end{cases}$$

Proof. It is easy to see that there is no weakly $\mathcal{P}$-saturated graph of order 4 with two edges. Since the graphs $K_{1,3}$ and $K_3 \cup K_1$ are weakly $\mathcal{P}$-saturated, we have $\mathrm{wsat}(4, \mathcal{P}) = 3$.

If $n \geq 5$ then $K_2 \cup (n-2)K_1$ is a weakly $\mathcal{P}$-saturated graph. By adding (as long as possible) an edge joining two vertices of $(n-2)K_1$ we obtain two independent edges, i.e., $2K_2$, and results in K_{n-2}. Since $n - 2 \geq 3$, it follows that every vertex of K_2 (in the original graph), we can join with every vertex of just obtained K_{n-2}. ■

From Theorem of Kalai and Theorem of Erdös, Hajnal and Moon, it follows that $\mathrm{wsat}(n, \mathcal{I}_k) = \mathrm{sat}(n, \mathcal{I}_k)$. Such equality also holds for the property $\mathcal{D}_1$.

Theorem 5. *Let $n \geq 1$. Then*

$$\mathrm{sat}(n, \mathcal{D}_1) = \mathrm{wsat}(n, \mathcal{D}_1) = n - 1.$$

Proof. Since $F(\mathcal{D}_1) = \{C_p : p \geq 3\}$, $\lambda(\mathcal{D}_1) = 2$ and every tree is weakly $\mathcal{D}_1$-saturated, it follows that $\mathrm{wsat}(n, \mathcal{D}_1) \leq n - 1$. From Theorem 3 we have $\lambda(G) \geq 1$ for $G \in \mathrm{WSat}(n, \mathcal{D}_1)$ then $\mathrm{wsat}(n, \mathcal{D}_1) \geq n - 1$. Thus $\mathrm{wsat}(n, \mathcal{D}_1) = n - 1$. Since the only $\mathcal{D}_1$-maximal graphs are trees, we have $\mathrm{sat}(n, \mathcal{D}_1) = n - 1$. ■

The next theorem describes a hereditary property $\mathcal{P}$ for which the minimum number of edges of weakly $\mathcal{P}$-saturated graphs of order n is less than the number of edges of $\mathcal{P}$-saturated graphs of order n.

Theorem 6. *Let $\mathcal{P}$ be the hereditary property such that $\mathrm{ex}(n, \mathcal{P}) = \mathrm{sat}(n, \mathcal{P})$, $\lambda(\mathcal{P}) = \lambda(H_0) = 1$, $H_0 \in \mathrm{F}(\mathcal{P})$ and every $\mathcal{P}$-maximal graph is connected. Then $\mathrm{wsat}(n, \mathcal{P}) < \mathrm{sat}(n, \mathcal{P})$, $n \geq v(H_0)$.*

Proof. Let $H_0 \in \mathrm{F}(\mathcal{P})$ with $\lambda(H_0) = 1$ and let e be a cutedge of H_0. Denote by H_1, H_2 components of $H_0 - e$. Let $v(H_1) = n_1$, $v(H_2) = n_2$. We define the graph $G = G_1 \cup G_2$ of order n assuming that $v(G_1) = n_1$, $v(G_2) = n - n_1$ and for $i = 1, 2$, G_i is $\mathcal{P}$-maximal. Obviously $n - n_1 \geq n_2$. Since all forbidden subgraphs are connected it follows that the graph G has property $\mathcal{P}$. Defined graph G is not connected, then by the assumption of the theorem, G is not $\mathcal{P}$-maximal. Thus $e(G) < \mathrm{ex}(n, \mathcal{P}) = \mathrm{sat}(n, \mathcal{P})$.

On the other hand, we will show that the graph G is weakly $\mathcal{P}$- saturated. Since each component of G is a $\mathcal{P}$-maximal graph, it follows that if we add any edge of $\overline{G}$ which joins two vertices of the same component we obtain a new forbidden subgraph containing the edge e. After adding all missing edges of each component we obtain the graph being a sum of complete graphs. Then each edge, which joins a vertex of the component of order n_1 with a vertex of the component of order $n - n_1$, belongs to a subgraph isomorphic to H_0. Thus the graph G is weakly $\mathcal{P}$-saturated and $e(G) \geq \mathrm{wsat}(n, \mathcal{P})$. Hence $\mathrm{wsat}(n, \mathcal{P}) < \mathrm{sat}(n, \mathcal{P})$. ■

In the next section we will show that the assumptions of Theorem 6 for the property $\mathcal{D}_k$ ($k \geq 2$) holds.

3. WEAKLY $\mathcal{D}_k$-SATURATED GRAPHS

The set of minimal forbidden subgraphs for property $\mathcal{D}_k$ was characterized by Mihók [4]. To describe the set $\mathrm{F}(\mathcal{D}_k)$ we need some more notations. For a nonnegative integer k and a graph G, we denote the set of all vertices of G of degree $k+1$ by $M(G)$. If $S \subseteq V(G)$ is a cutset of vertices of G and $G_1, \ldots, G_s,\ s \geq 2$ are the components of $G - S$, then the graph $G - V(G_i)$ is denoted by $H_i,\ i = 1, \ldots, s$.

Theorem 7. [4] *A graph G belongs to $\mathrm{F}(\mathcal{D}_k)$ if and only if G is connected, $\delta(G) \geq k+1,\ V(G) - M(G)$ is an independent set of vertices of G and for each cutset $S \subset V(G) - M(G)$ we have that $\delta(H_i) \leq k$ for each $i = 1, \ldots, s$.*

Let us present some useful examples of $\mathrm{F}(\mathcal{D}_k)$.

Example 1. Let $H_k,\ k \geq 2$, be the graph such that $V(H_k) = \{x_1, \ldots, x_k, y_1, \ldots, y_k, v_1, v_2, w_1, w_2\}$ with the following properties: vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ induce two complete graphs and $v_i w_i, v_i x_j, w_i y_j \in E(H_k)$ for $i = 1, 2,\ j = 1, \ldots, k$.

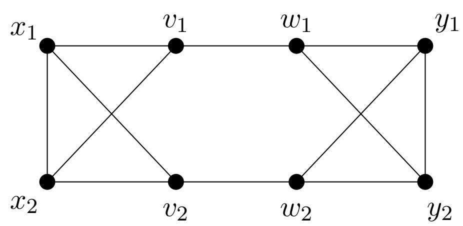

Figure 3.1. The graph H_k for $k = 2$

Example 2. Let $H'_k,\ k \geq 2$, be the graph such that $V(H'_k) = \{x_1, \ldots, x_k, y_1, \ldots, y_k, v_1, v_2, v_3, w_1, w_2, w_3\}$ with the following properties: verices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ induce two graphs obtained from K_k by removing $\lfloor \frac{k}{2} \rfloor$ independent edges and $v_i x_j,\ w_i y_j \in E(H'_k)$ for $i = 1, 2, 3, j = 1, \ldots, k$, and $v_1 w_1,\ v_2 v_3,\ w_2 w_3 \in E(H'_k)$.

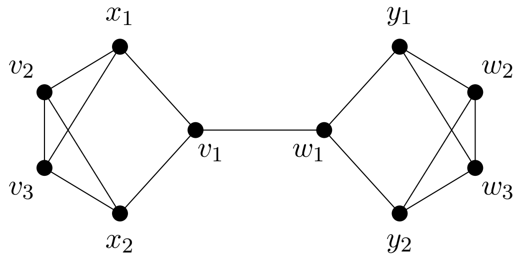

Figure 3.2. The graph H'_k for $k = 2$

By Example 2 we have that $\lambda(\mathcal{D}_k) = 1$ for $k \geq 2$. Since $\mathcal{D}_k$-maximal graphs are connected and $\mathrm{sat}(n, \mathcal{D}_k) = \mathrm{ex}(n, \mathcal{D}_k)$ (see e.g. [3]), it follows that the assumptions of Theorem 6 holds. Then we immediately have

Corollary 8. $\mathrm{wsat}(n, \mathcal{D}_k) < \mathrm{sat}(n, \mathcal{D}_k)$ *for* $n \geq 2(k+3),\ k \geq 2$.

To determine upper bound for the number $\mathrm{wsat}(n, \mathcal{D}_k)$ we need the following lemma.

Lemma 9. *Let* $k \geq 2$. *Then the graph* $H_k - v_2w_2$ *is weakly* $\mathcal{D}_k$*-saturated.*

Proof. Put $G = H_k - v_2w_2$. If the edge v_2w_2 is added to G then $G = H_k \in \mathrm{F}(\mathcal{D}_k)$ is obtained. If we add v_1v_2 or w_1w_2 to H_k then we obtain the graph K_{k+2} which belongs to $\mathrm{F}(\mathcal{D}_k)$. After adding the edge $x_iy_j,\ (1 \leq i, j \leq k)$, edges $(E(G) \cup \{v_1v_2, w_1w_2, x_iy_j\}) - \{v_1x_i, w_1y_j\}$ induce H_k. Now we can add the edge $v_1y_j,\ 1 \leq j \leq k$ since edges $(E(G) \cup \{v_2w_2, w_1w_2, v_1y_j\}) - \{w_2y_j, v_1w_1\}$ induce H_k. If we add the edge v_2w_j $(1 \leq j \leq k)$, we obtain the graph H_k induced by $(E(G) \cup \{w_1w_2, v_2y_j\}) - \{w_1y_j\}$. In a similar manner we can show that if we add edges x_iw_1 and x_iw_2 $(1 \leq i \leq k)$, a new forbidden subgraph appears. The last two edges $v_1w_2,\ v_2w_1$ we can add because edges $(E(G) \cup \{x_1y_1, v_1w_2, v_1v_2, w_1w_2\}) - \{x_1v_1, w_2y_1, v_1w_1\}$ and $(E(G) \cup \{x_1y_1, v_2w_1, v_1v_2, w_1w_2\}) - \{x_1v_2, w_1y_1, v_1w_1\}$ induce H_k. ■

Theorem 10. *Let* $k \geq 2$ *and* $n = 2(k+2)q + r$, *where* $q \geq 1,\ 0 \leq r \leq 2k+3$. *Then*

$$\mathrm{wsat}(n, \mathcal{D}_k) \leq \begin{cases} \frac{(k+2)(k+1)-1}{2(k+2)} n, & \text{for } r = 0, \\ \frac{(k+2)(k+1)-1}{2(k+2)} (n - r - (k+2)) + \\ \quad (r + k + 2)k - \binom{k+1}{2}, & \text{for } 0 < r < k+3, \\ \frac{(k+2)(k+1)-1}{2(k+2)} (n - r) + rk - \binom{k+1}{2}, & \text{for } r \geq k+3. \end{cases}$$

Proof. To prove the theorem it is enough to show that there is a weakly $\mathcal{D}_k$-saturated graph G of order n with such number of edges. Let $k \geq 2$ and $n = 2(k+2)q + r$, where $q \geq 1$, $0 \leq r \leq 2k+3$. Put $G' = H_k - v_2 w_2$. If $r \geq k+3$, then $G = qG' \cup H$, where $H \in \mathrm{M}(r, \mathcal{D}_k)$. If $0 \leq r < k+3$, then $G = (q-1)G' \cup H$, where $H \in \mathrm{M}(2(k+2)+r, \mathcal{D}_k)$. If $r = 0$, then $G = qG'$. By Lemma 9 it follows that each component of G is a weakly $\mathcal{D}_k$-saturated graph. Then we can add edges in each component of G to obtain a complete graph. After having added those edges we can join any vertices of two different components. ■

4. THE NUMBER $\mathrm{wsat}(n, \mathcal{P})$ FOR SOME HEREDITARY PROPERTIES

In this section we will calculate the minimum number of edges of weakly saturated graphs for some hereditary properties.

Theorem 11. *Let $k \geq 1$ and $n \geq k+2$. Then*

$$\mathrm{WSat}(n, \mathcal{O}_k) \supseteq \{T_r \cup T_s \cup tT_1 : r+s = k+2,\ r+s+t = n \text{ and } T_i \text{ is an abitrary tree of order } i\}$$

and

$$\mathrm{wsat}(n, \mathcal{O}_k) = k.$$

Proof. First we prove that the graph $G = T_r \cup T_s \cup tT_1$, where $r + s = k+2$, $r+s+t = n$ is weakly $\mathcal{O}_k$-saturated. If we add an edge of $\overline{G}$, which joins a vertex of T_r and a vertex of T_s then we obtain a tree of order $k+2$, i.e., we obtain a forbidden subgraph for property $\mathcal{O}_k$. If we join a vertex of the subgraph tT_1 with a vertex of the obtained tree of order $k+2$ we have a connected graph of order $k+3$. Thus new edge belongs to a tree of order $k+2$. Repeating this process we obtain a connected graph of order n in which each vertex of tT_1 is adjacent with any vertex of the tree of order $k+2$. Since for each edge of the complement of a connected graph there is a spanning tree which contains this edge, it follows that G is weakly $\mathcal{O}_k$-saturated. Hence $\mathrm{wsat}(n, \mathcal{O}_k) \leq e(G) = k$.

On the other hand, let G be a graph such that $G \in \mathrm{WSat}(n, \mathcal{O}_k)$ and $e(G) = \mathrm{wsat}(n, \mathcal{O}_k)$. Let e_1 be the first edge such that $G + e_1$ contains a forbidden subgraph, i.e., the graph $G + e_1$ contains a tree of order $k+2$. Thus $\mathrm{wsat}(n, \mathcal{O}_k) = e(G) \geq k$. ■

The proof of the next theorem is very similar to the proof of Theorem 11, then it is omitted.

Theorem 12. *Let $k \geq 1$ and $n \geq k+2$. Then*

$$\mathrm{WSat}(n, \mathcal{W}_k) \supseteq \{P_r \cup P_s \cup tP_1 : r+s = k+2,\ r+s+t = n\}$$

and

$$\mathrm{wsat}(n, \mathcal{W}_k) = k.$$

It is easy to see that the graphs $K_{k+1} + tK_1$, where $k+1+t = n$ are weakly $\mathcal{S}_k$-saturated. There are some other weakly $\mathcal{S}_k$-saturated graphs of order n. For example the graph G_1 (Figure 4.1) is weakly $\mathcal{S}_2$-saturated and the graph G_2 (Figure 4.1) is weakly $\mathcal{S}_3$-saturated.

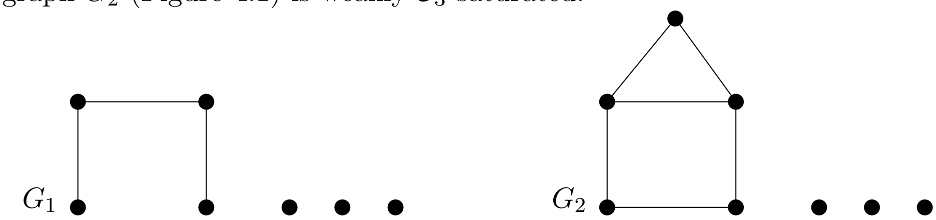

Figure 4.1. The graphs G_1 and G_2

Theorem 13. *Let $n \geq k+2 \geq 4$. Then*

$$\mathrm{wsat}(n, \mathcal{S}_k) = \binom{k+1}{2}.$$

Proof. Let G be a weakly $\mathcal{S}_k$-saturated graph of order n with the minimum number of edges. Then there is a complementary sequence $e_1, e_2, \ldots, e_l$ of G. Let $e_1 = u_1v_1$ and $d_G(u_1) = k$. Let $e_{f(1)}, \ldots, e_{f(t_1)}$ be the subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(1 \leq i \leq t_1)$ is adjacent with the vertex u_1. If in the graph $G' = ((G + e_{f(1)}) + e_{f(2)}) + \ldots + e_{f(t_1)}$ there is no vertex of degree less than k then let $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ be the new sequence of edges of $E(\overline{G})$ with the following property: $e_{f(1)}, \ldots, e_{f(t_1)}$ is the subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(1 \leq i \leq t_1)$ is adjacent with the vertex u_1 and $e_{f(t_1)+1}, \ldots, e_{f(l)}$ is the subsequence of $e_1, e_2, \ldots, e_l$ such that any edge $e_{f(i)}$, $(t_1 \leq i \leq l)$ is not adjacent with the vertex u_1. If in the graph G' there is a vertex of degree less than k

then let $e_{f(t_1+1)}$ be the first edge of $e_1, e_2, \ldots, e_l$, which is not adjacent with the vertex u_1. Let $e_{f(t_1+1)} = u_2v_2$ and u_2 be a vertex of G' such that $d_{G'}(u_2) \geq k$ and $u_1 \neq u_2$. Let $e_{f(t_1+1)}, \ldots, e_{f(t_2)}$ denote edges of $\{e_1, e_2, \ldots, e_l\} - \{e_{f(1)}, \ldots, e_{f(t_1)}, e_{f(t_1)+1}\}$ which are adjacent with the vertex u_2. If in the graph $G'' = ((G' + e_{f(t_1)+1}) + e_{f(t_1)+2}) + \ldots + e_{f(t_2)}$ there is no vertex of degree less than k we form a new sequence of edges of $E(\overline{G})$, $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ with the following property: $e_{f(1)}, \ldots, e_{f(t_1)}$ is a subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(1 \leq i \leq t_1)$ is adjacent with the vertex u_1 and $e_{f(t_1)+1}, \ldots, e_{f(t_2)}$ is a subsequence of $e_1, e_2, \ldots, e_l$ such that every edge $e_{f(i)}$, $(t_1 < i \leq t_2)$ is adjacent with the vertex u_2 and $e_{f(t_2)+1}, \ldots, e_{f(l)}$ is the subsequence of $e_1, e_2, \ldots, e_l$ such that any edge $e_{f(i)}$, $(t_2 < i \leq l)$ is not adjacent with the vertex u_1 and u_2. If in the graph G'' there is a vertex of degree less than k, we will repeat this steps until we will obtain a new sequence $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ of edges of $\overline{G}$. With this sequence of edges $e_{f(1)}, e_{f(2)}, \ldots, e_{f(l)}$ is related a sequence of vertices $u_1, u_2, \ldots, u_r$. It is easy to see that $r \leq k$, because after k steps there is no vertex of degree less than k. Then for the vertex $u_t \in \{u_1, \ldots, u_r\}$ we have

$$d_G(u_t) + t - 1 - |N_G(u_t) \cap \{u_1, \ldots, u_{t-1}\}| \geq k, \tag{1}$$

for the vertex $x \in V(G) - \{u_1, \ldots, u_r\}$ we have

$$d_G(x) + r - |N_G(x) \cap \{u_1, \ldots, u_r\}| \geq k. \tag{2}$$

Thus
$e(G) \geq \sum_{1 \leq t \leq r}(d_G(u_t) - |N_G(u_t) \cap \{u_1, \ldots, u_{t-1}\}|)$
$+\frac{1}{2}\sum_{x \in V(G)-\{u_1,\ldots,u_r\}}(d_G(x) - |N_G(x) \cap \{u_1, \ldots, u_r\}|)$
$\geq \sum_{1 \leq t \leq r}(k + 1 - t) + \frac{1}{2}(n - r)(k - r)$.
The right side of inequality achieves the minimum for $r = k$. Thus
$e(G) \geq \sum_{1 \leq t \leq r}(k + 1 - t) = \frac{1}{2}(k + 1)k$.
On the other hand, the graph $K_{k+1} \cup (n - k - 1)K_1$ is weakly $\mathcal{S}_k$-saturated. Thus $\mathrm{wsat}(n, \mathcal{S}_k) \leq \binom{k+1}{2}$. ■

In the next theorem we determine the number $\mathrm{wsat}(n, \mathcal{P})$ for a hereditary property with one forbidden subgraph which is a cycle of odd length.

Theorem 14. *Let $k \geq 1$ and $n \geq 2k + 2$. If $\mathcal{P}$ is the hereditary property such that $\mathrm{F}(\mathcal{P}) = \{C_{2k+1}\}$, then $\mathrm{wsat}(n, \mathcal{P}) = n - 1$.*

Proof. Since $\lambda(\mathcal{P}) = 2$, by Theorem 3 it follows that every weakly $\mathcal{P}$-saturated graph is connected. Then $\mathrm{wsat}(n, \mathcal{P}) \geq n - 1$. To prove that the inequality $\mathrm{wsat}(n, \mathcal{P}) \leq n - 1$ holds it is sufficient to show that there is a weakly $\mathcal{P}$-saturated graph of order n with $n - 1$ edges.

Let us show first that P_{2k+2} is a weakly $\mathcal{P}$-saturated graph. Let $V(P_{2k+2}) = \{v_1, \ldots, v_{2k+2}\}$ and $d(v_1) = d(v_{2k+2}) = 1$. It is easy to see that if we add the edge v_1v_{2k+1} then we obtain a cycle of order $2k + 1$. Similarly if we add the edge v_2v_{2k+2} a new cycle of order $2k + 1$ appears. Now we can add the edge v_1v_4. The edge v_1v_4 belongs to the cycle $v_1, v_2, v_{2k+2}, v_{2k+1}, \ldots, v_4, v_1$. To prove that if we add any edge v_1v_{2t} then a new cycles of order $2k + 1$ appears we will use induction on t. This is true for $t = 1, 2$. When the edges v_1v_{2i} for $i < t$ are added the vertices $v_1, v_{2t-2}, v_{2t-3}, \ldots, v_2, v_{2k+2}, v_{2k+1}, \ldots, v_{2t}, v_1$ induce a cycle of order $2k+1$ which contains the edge v_1v_{2t}. In the same manner, after having added edges v_1v_{2i+1} for $k \geq i > t$ we can add the edge v_1v_{2t+1}. A new cycle $v_1, v_{2t+3}, \ldots, v_{2k+2}, v_2, v_3, \ldots v_{2t+1}, v_1$ of order $2k + 1$ appears. Finally the vertex v_1 with all vertices of P_{2k+2} is joined. Similarly we can join each vertex v_t $(2 \leq t \leq 2k + 2)$ with all vertices of P_{2k+2}. Thus we obtain a graph K_{2k+2}. Hence P_{2k+2} is a weakly $\mathcal{P}$-saturated graph.

Let G be the graph of order $n \geq 2k+2$ with the following properties: G contains an induced path of order $2k + 2$, the remaining vertices of G form an independent set and each vertex of this set is adjacent with exactly one vertex of the path. Since the path of order $2k + 2$ is weakly $\mathcal{P}$-saturated, it follows that the graph G is weakly $\mathcal{P}$-saturated. Hence $\mathrm{wsat}(n, \mathcal{P}) \leq n - 1$. ■

In order to determine the number $\mathrm{wsat}(n, \mathcal{P})$ for hereditary property such that $\mathrm{F}(\mathcal{P}) = \{C_{2k}\}$ we need the folowing lemma.

Lemma 15. *Let $k \geq 2$ and $\mathcal{P}$ be the hereditary property such that $F(\mathcal{P}) = \{C_{2k}\}$, and G be a bipartite graph of order $n \geq 2k + 1$. Then $G \notin \mathrm{WSat}(n, \mathcal{P})$.*

Proof. On the contrary, suppose that there is a weakly $\mathcal{P}$-saturated bipartite graph G of order n. Let $e_1, e_2, \ldots, e_l$ be a complementary sequence of G. Let $e_i = xy$ be the first edge of the sequence $e_1, e_2, \ldots, e_l$ such that its ends x, y belong to the same colour class of G. (Notice, that the colour classes of G are uniquely determined because of connectivity of G.) Since the edge e_i belongs to an even cycle C_{2k} then there is an edge e_j, $j < i$ of this cycle (and the sequence given above) with both ends in one colour class which is impossible. ■

Theorem 16. *Let $k \geq 2$ and $n \geq 2k+1$. Let $\mathcal{P}$ be the hereditary property such that* $\mathrm{F}(\mathcal{P}) = \{C_{2k}\}$*. Then*

$$\mathrm{wsat}(n, \mathcal{P}) = n.$$

Proof. Let $G \in$WSat$(n, \mathcal{P})$. By Theorem 3 and Lemma 15 it follows that G is connected and contains an odd cycle. Thus $\mathrm{wsat}(n, \mathcal{P}) \geq n$.

To prove that the inequality $\mathrm{wsat}(n, \mathcal{P}) \leq n$ holds it is sufficient to show that there is a weakly $\mathcal{P}$-saturated graph of order n with n edges. First we prove that C_{2k+1} is a weakly $\mathcal{P}$-saturated graph. Let $V(C_{2k+1}) = \{v_1, v_2, \ldots, v_{2k+1}\}$. It is easy to see that if we add the edge v_1v_3 or the edge v_2v_{2k+1}, a cycle (containing this edge) of order $2k$ appears. To prove that if we add any edge v_1v_t $(3 \leq t \leq 2k)$ then we obtain a new cycle of order $2k$ we use induction on t. This is true for $t = 3$. After adding edges v_1v_i for $3 \leq i < t$ the vertices $v_1, v_{t-2}, v_{t-3}, \ldots, v_2, v_{2k+1}, v_{2k}, \ldots, v_t, v_1$ induce a cycle of order $2k$ which contains the edge v_1v_t. Then the vertex v_1 can be joined with all vertices of C_{2k+1}. In the similar manner we can show that we can join any vertex $v_t \in V(C_{2k+1})$ with all vertices of C_{2k+1}. Hence C_{2k+1} is weakly $\mathcal{P}$-saturated.

Let G be the graph with the following properties: G contains an induced cycle of order $2k+1$, remaining vertices of G form an independent set and each vertex of this set is adjacent with exactly one vertex of the cycle. Since the cycle of order $2k+1$ is weakly $\mathcal{P}$- saturated (can be extended to K_{2k+1}), it follows that the graph G also has this property, i.e., G is weakly $\mathcal{P}$-saturated. Hence $\mathrm{wsat}(n, \mathcal{P}) \leq n$. ■

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