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A NOTE ON DOMINATION PARAMETERS OF THE CONJUNCTION OF TWO SPECIAL GRAPHS

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Abstract

A dominating set D of G is called a split dominating set of G if the subgraph induced by the subset $V(G) - D$ is disconnected. The cardinality of a minimum split dominating set is called the minimum split domination number of G. Such subset and such number was introduced in [4]. In [2], [3] the authors estimated the domination number of products of graphs. More precisely, they were study products of paths. Inspired by those results we give another estimation of the domination number of the conjunction (the cross product) $P_n \wedge G$. The split domination number of $P_n \wedge G$ also is determined. To estimate this number we use the minimum connected domination number $\gamma_c(G)$.

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1. Definitions and Notations

In this paper we discuss finite connected, undirected simple graphs. For any graph G we denote $V(G)$ and $E(G)$, the vertex set of G and the edge set of G, respectively. We say that G is of *order* n if n is a cardinality of $V(G)$. By $\langle X\rangle_G$ we denote a subgraph of G which is induced by a subset $X \subset V(G)$. A *hanging vertex* is a vertex of G adjacent to exactly one vertex in G. The complement of G is denoted by $\overline{G}$. A subset $D \subseteq V(G)$ is a *dominating set* of G if for every $x \in V(G) - D$ there is a vertex $y \in D$ such that $xy \in E(G)$. We will also write that x is dominated by D or by y in G.

In [4] it was introduced the notion of split dominating set of a graph. We say that a dominating set $D \subseteq V(G)$ is a *split dominating set* of G if the induced subgraph $\langle V(G) - D\rangle_G$ is disconnected. A dominating set $D \subseteq V(G)$ is a *connected dominating set* of G, (see [5]) if the induced subgraph $\langle D\rangle_G$ is connected. The *domination number* [the *split domination number*, the *connected domination number*] of a graph G, denoted by $\gamma(G)$, $[\gamma_s(G), \gamma_c(G)]$ is the cardinality of a minimum dominating [a minimum split dominating, a minimum connected dominating] set of G. It is easy to see that $\gamma(G) \leq \gamma_s(G)$ and also $\gamma(G) \leq \gamma_c(G)$. A dominating set D is called a $\gamma(G)$-set [$\gamma_s(G)$-set, $\gamma_c(G)$-set] if D realizes the domination [split domination, connected domination] number, respectively. Note that there exists a $\gamma_c(G)$-set if and only if G is connected. The *conjunction* of two graphs G and H is a graph $G \wedge H$, with $V(G \wedge H) = V(G) \times V(H)$ and $(g_1, h_1)(g_2, h_2) \in E(G \wedge H)$ if and only if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. By P_n we denote an induced path on $n \geq 2$ vertices meant as a graph with $V(P_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(P_n) = \{x_ix_{i+1} : i = 1, 2, \ldots, n-1\}$. If $V(G) = \{y_1, y_2, \ldots, y_m\}$, then the *copy* G^* of G is the graph with the vertex set $V(G^*) = \{y_1^*, y_2^*, \ldots, y_n^*\}$ and $y_i^*y_j^* \in E(G^*)$ if and only if $y_iy_j \in E(G)$, where y_i^* corresponds to y_i. Further, let $D = \{y_1, y_2, \ldots, y_r\} \subset V(G)$, then the subset $D^* = \{y_1^*, y_2^*, \ldots, y_m^*\} \subset V(G^*)$ is called a *duplication* of D into the vertex set $V(G^*)$ of the copy G^* or shorter into G^*.

We consider the conjunction $P_n \wedge G$, for $n \geq 2$ with a special graph G. Before proceeding we introduce some notation with respect to $P_n \wedge G$. If $y_j \in V(G)$, then the vertex (x_i, y_j) of the conjunction of $P_n \wedge G$ is simply written as x_j^i. For a fixed integer i we put $X_i = \{x_j^i : 1 \leq j \leq |V(G)|\}$. A set B of all vertices belonging to k consecutive sets $X_{i+1}, \ldots, X_{i+k}$ is called a *block* of a graph $P_n \wedge G$ of size $k \times |V(G)|$. For a convenience, the set X_i we will call the i-th *column* of a graph $P_n \wedge G$. Any other terms not defined in this paper can be found in [1].

2. Introduction

In this section we introduce some basic facts which will be useful in further investigations. It was proved in [4], that

Theorem 1 [4]**.** $\gamma_s(P_n) = \left\lceil \frac{n}{3} \right\rceil$, *for* $n \geq 3$.

Theorem 2 [4]**.** *For any noncomplete graph G with at least one hanging vertex*

$$\gamma_s(G) = \gamma(G).$$

Next, it is easy to check that

Proposition 3. *There is no a split dominating set of $\overline{P_n}$, for $i = 1, 2, 3$.*

Proposition 4. $\gamma_s(\overline{P_4}) = 2$, *since* $\overline{P_4} \cong P_4$.

Now, we calculate a split domination number of $\overline{P_n}$ if $n \geq 5$.

Theorem 5. $\gamma_s(\overline{P_n}) = n - 3$, *for* $n \geq 5$.

Proof. Let $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, such that $d_{P_n}(x_1) = d_{P_n}(x_n) = 1$ and $d_{P_n}(x_i) = 2$, for $i = 2, 3, \ldots, n-1$. At the beginning we can observe that $d_{\overline{P_n}}(x_1) = d_{\overline{P_n}}(x_n) = n - 2$ and $d_{\overline{P_n}}(x_i) = n - 3$, for $i = 2, 3, \ldots, n-1$. Now, we show that the induced subgraph $H = \langle \{x_{n_1}, x_{n_2}, \ldots, x_{n_k}\} \rangle_{\overline{P_n}}$ is connected, when $n_1 < n_2 < \ldots < n_k$, for $k \geq 4$. Since $n_3 - n_1 \geq 2$, $n_4 - n_1 \geq 2, \ldots, n_k - n_1 \geq 2$, then x_{n_1} is adjacent to x_{n_s} in $\overline{P_n}$, for $s = 3, 4, \ldots, k$. Hence $H_1 = \langle \{x_{n_1}, x_{n_3}, x_{n_4}, \ldots, x_{n_k}\} \rangle_{\overline{P_n}}$ is a connected subgraph. Arguing as above we prove that $H_2 = \langle \{x_{n_2}, x_{n_4}, x_{n_5}, \ldots, x_{n_k}\} \rangle_{\overline{P_n}}$ also is connected. Since $k \geq 4$, then $V(H_1) \cap V(H_2) \neq \emptyset$ and $H = \langle V(H_1) \cup V(H_2) \rangle_{\overline{P_n}}$ is connected. This means that there is no a disconnected subgraph of $\overline{P_n}$ of order at least $n - 4$. To complete the proof we construct a split dominating set D of $\overline{P_n}$, such that $|D| = n - 3$. Let D consists of vertices x_i, for $i = 4, 5, \ldots, n$. Since $n \geq 5$, thus $D \neq \emptyset$ and $V(\overline{P_n}) - D = \{x_1, x_2, x_3\}$. Moreover, vertices x_1, x_2 are adjacent to $x_4 \in D$ in $\overline{P_n}$ and x_3 is adjacent to $x_5 \in D$ in $\overline{P_n}$. Furthermore, x_2 is an isolated vertex in $\left\langle V(\overline{P_n}) - D \right\rangle_{\overline{P_n}}$. All this together gives that D is the minimum split dominating set of $\overline{P_n}$ of order $n - 3$, as required. ■

From the structure of P_n, $\overline{P_n}$ and from the definition of the connected domination number it follows immediately

Proposition 6.

$$\gamma_c(P_n) = n - 2, \quad \textit{for } n \geq 3 \textit{ and}$$

$$\gamma_c(\overline{P_n}) = 2, \quad \textit{for } n \geq 4.$$

From Theorem 1, Theorem 5 and Proposition 6 it follows the Nordhaus-Gaddum type result

Theorem 7.

$$\gamma_s(P_n) + \gamma_s(\overline{P_n}) = \left\lceil \tfrac{n}{3} \right\rceil + n - 3, \quad \textit{for } n \geq 5,$$
$$\gamma_c(P_n) + \gamma_c(\overline{P_n}) = n, \quad \textit{for } n \geq 4.$$

3. Main Results

Proposition 8. *For any graph G, $\gamma(P_2 \wedge G) \leq 2\gamma(G)$.*

Proof. Let $D = \{x_1, x_2, \ldots, x_s\}$ be a minimum dominating set of G. Duplicating D into two columns $P_2 \wedge G$ we obtain a subset $A_2 = \{x_1^1, x_2^1, \ldots, x_s^1, x_1^2, x_2^2, \ldots, x_s^2\} \subset V(P_2 \wedge G)$. We show that A_2 is a dominating set of $P_2 \wedge G$. Let $x_j^1 \in (V(P_2 \wedge G) - A_2)$. Since D is a dominating set of G, then there exists a vertex x_k of D in G, such that $x_k x_j \in E(G)$. Further, by the definition of $P_2 \wedge G$ and by a construction of the subset A_2 we have that $x_j^1 x_k^2 \in E(P_2 \wedge G)$ and $x_k^2 \in A_2$, respectively. Hence x_j^1 is dominated by A_2 in $P_2 \wedge G$. Similarly, we can show that the vertex $x_j^2 \in (V(P_2 \wedge G) - A_2)$ is dominated by A_2 in $P_2 \wedge G$. All this together gives that A_2 is a dominating set of $P_2 \wedge G$ and $\gamma(P_2 \wedge G) \leq |A_2| = 2\gamma(G)$, as required. ■

It follows immediately from the obvious inequality $\gamma(G) \leq \gamma_c(G)$ and from the above proposition that

Corollary 9. *For any connected graph G, $\gamma(P_2 \wedge G) \leq 2\gamma_c(G)$.*

Proposition 10. *For any graph G with $\gamma_c(G) \geq 2$,*

$$\gamma(P_3 \wedge G) \leq 2\gamma_c(G).$$

Proof. Let D be a $\gamma_c(G)$-set. Put $A_3 = \{x_j^2, x_j^3 : \text{for all } x_j \in D\}$. Now we show that A_3 is a dominating set of $P_3 \wedge G$. Arguing as in a proof of Proposition 8, we see that A_3 dominates vertices $x_j^2, x_j^3 \in (V(P_3 \wedge G) - A_3)$ in $P_3 \wedge G$. To complete the proof we must show that any vertex from X_1 is dominated by A_3 in $P_3 \wedge G$. We recall that X_1 is the first column of the graph $P_3 \wedge G$ as it was mentioned earlier. Let $x_j^1 \in X_1$. If $x_j \in V(G) - D$, then it is dominated by a vertex $x_k \in D$ and in a consequence x_j^1 is dominated by $x_k^2 \in A_3$. Assume that $x_j \in D$. Since $\langle D \rangle_G$ is connected and $|D| = \gamma_c(G) \geq 2$, thus there exists a vertex $x_k \in D$ different from x_j, such that $x_j x_k \in E(G)$. Moreover, $x_j^1 x_k^2 \in E(P_3 \wedge G)$. This means that x_j^1 is dominated by A_3 in $P_3 \wedge G$ because of $x_k^2 \in A_3$. Hence A_3 is a dominating set of $P_3 \wedge G$. Since $\gamma(P_3 \wedge G) \leq |A_3| = 2\gamma_c(G)$, thus the theorem is true. ■

Remark 1. It is easy to see that A_3 also is a dominating set of $P_4 \wedge G$, where G is a graph with $\gamma_c(G) \geq 2$. Hence $\gamma(P_4 \wedge G) \leq 2\gamma_c(G)$ with $\gamma_c(G) \geq 2$.

Proposition 11. *For any graph G with $\gamma_c(G) \geq 2$*

$$\gamma(P_5 \wedge G) \leq 3\gamma_c(G).$$

Proof. Let $D = \{x_1, \ldots, x_m\}$ be a minimum connected dominating set of G. Duplicating D into 2-nd, 3-rd, 4-th column of $P_5 \wedge G$ we obtain a subset

$$A_5 = \{x_j^i : i = 2, 3, 4 \text{ and } j = 1, 2, \ldots m\} \subset V(P_5 \wedge G).$$

Simple observation shows that A_5 is a dominating set of $P_5 \wedge G$. Thus $\gamma(P_5 \wedge G) \leq |A_5| = 3\gamma_c(G)$ and proof is complete. ■

In [2] it was presented the following result

Proposition 12 [2]. *For $n > 1$ and every graph G we have*

$$\gamma(P_n \wedge G) \leq 2\gamma(G) \left(\left\lfloor \frac{n}{4} \right\rfloor + 1 \right).$$

Counterexample. Let $P_n = P_3$ and $G = P_5$, then $P_n \wedge G = P_3 \wedge P_5$ has two connected components, say Y_1 and Y_2. Further, this must be that $\gamma(P_3 \wedge P_5) = \gamma(Y_1) + \gamma(Y_2)$. It is easy to observe that $\gamma(Y_1) = 2$ and $\gamma(Y_2) = 3$, thus $\gamma(P_3 \wedge P_5) = 5$. Now, using the estimation from Proposition 12 we obtain $\gamma(P_3 \wedge P_5) \leq 4(\lfloor \frac{3}{4} \rfloor + 1) = 4$, since $\gamma(G) = \gamma(P_5) = 2$. It is not true, since as we noticed $\gamma(P_3 \wedge P_5) = 5$.

Using above facts we give another estimation for $\gamma(P_n \wedge G)$.

Theorem 13. *Let G be a graph with $\gamma_c(G) \geq 2$. Then, for $n \geq 2$ we have*

$$\gamma(P_n \wedge G) \leq \begin{cases} \gamma_c(G)(2\left\lfloor \frac{n-1}{4} \right\rfloor + 1), & \text{if } n \equiv 1 (\text{mod } 4), \\ \gamma_c(G)(2\left\lfloor \frac{n-1}{4} \right\rfloor + 2), & \text{otherwise.} \end{cases}$$

Proof. Let $n = 4q + r$, $q \geq 1$, $0 \leq r < 4$, $r \neq 1$. Partition the set $V(P_n \wedge G)$ into q blocks $B_1, \ldots, B_q$ of size $4 \times |V(G)|$ and one block B_{q+1} of size $r \times |V(G)|$ (it can be that $B_{q+1} = \emptyset$). Put A_j^i be a duplication of A_j into the block B_i, for $i = 1, 2, \ldots, q$ and $j = 2, 3, 5$, where A_j is the subset defined in the proofs of above propositions.

If $n = 4q$, then $D = \bigcup_{i=1}^{q} A_3^i$ is a dominating set of $P_n \wedge G$ and

$$|D| = 2q\gamma_c(G) = 2\left(\left\lfloor \frac{4q-1}{4} \right\rfloor + 1\right)\gamma_c(G) = \gamma_c(G)\left(2\left\lfloor \frac{n-1}{4} \right\rfloor + 2\right).$$

If $n = 4q + 2$, then $D = \bigcup_{i=1}^{q} A_3^i \cup A_2^{q+1}$ is a dominating set of $P_n \wedge G$ and

$$|D| = 2q\gamma_c(G) + 2\gamma_c(G) = \left(2\left\lfloor \frac{4q+1}{4} \right\rfloor + 2\right)\gamma_c(G) = \gamma_c(G)\left(2\left\lfloor \frac{n-1}{4} \right\rfloor + 2\right).$$

If $n = 4q + 3$, then $D = \bigcup_{i=1}^{q+1} A_3^i$ is a dominating set of $P_n \wedge G$ and

$$|D| = 2(q+1)\gamma_c(G) = 2\left(\left\lfloor \frac{4q+2}{4} \right\rfloor + 1\right)\gamma_c(G) = \gamma_c(G)\left(2\left\lfloor \frac{n-1}{4} \right\rfloor + 2\right).$$

Assume that $n = 4q + 1$. Thus we create $q - 1$ blocks of size $4 \times |V(G)|$, say $B_1, \ldots, B_{q-1}$ and one block B_q of size $5 \times |V(G)|$. Let $D = \bigcup_{i=1}^{q-1} A_3^i \cup A_5^q$, then D is a dominating set of $P_n \wedge G$ with

$$\begin{aligned} |D| &= 2(q-1)\gamma_c(G) + 3\gamma_c(G) = (2q+1)\gamma_c(G) \\ &= \left(2\left\lfloor \frac{4q}{4} \right\rfloor + 1\right)\gamma_c(G) = \left(2\left\lfloor \frac{n-1}{4} \right\rfloor + 1\right)\gamma_c(G). \end{aligned}$$

Therefore, since $\gamma(P_n \wedge G) \leq |D|$, the result holds, for $n \geq 4$ as it was assumed at the beginning of the proof.

Since $2\gamma_c(G) = (2\left\lfloor \frac{n-1}{4} \right\rfloor + 2)\gamma_c(G)$, for $n = 2, 3, 4$ and $3\gamma_c(G) = (2\left\lfloor \frac{5-1}{4} \right\rfloor + 1)\gamma_c(G)$, then Theorem 13 was proved for any $n \geq 2$. ■

Moreover, since $\gamma_c(\overline{P_k}) = 2$, for $k \geq 4$, then the last result and a simple calculation lead to the following conclusion.

Corollary 14 [2]. *For $n \geq 2$ and $k \geq 4$,*

$$\gamma(P_n \wedge \overline{P_k}) \leq \begin{cases} n, & \text{if } n \equiv 0 (\text{mod } 4), \\ n+1, & \text{if } n \equiv 1 (\text{mod } 4) \text{ and } n \equiv 3 (\text{mod } 4), \\ n+2, & \text{if } n \equiv 2 (\text{mod } 4). \end{cases}$$

Mention that for the graph $P_3 \wedge P_5$, considered after Proposition 12, using the estimation from Theorem 13 we have $5 = \gamma(P_3 \wedge P_5) \leq \gamma_c(P_5)(2 \left\lfloor \frac{3-1}{4} \right\rfloor + 2) = 6$.

At the end, we consider the minimum split domination number of the conjunction of P_n and a graph G with a special property. First, we assume that G has at least two hanging vertices, then we have

Proposition 15. *Let G be a graph with at least one hanging vertex. Then*

$$\gamma_s(P_n \wedge G) = \gamma(P_n \wedge G), \quad \text{for } n \geq 2.$$

Proof. Let G be a graph as in the statement of the corollary. Since G has at least one hanging vertex, thus by the definition of $P_n \wedge G$, we obtain that $P_n \wedge G$ has at least one hanging vertex (note that it has at least two hanging vertices, since $n \geq 2$). Then according to Theorem 2 we have that $\gamma_s(P_n \wedge G) = \gamma(P_n \wedge G)$, as desired. ■

Further, we assume that G is a connected graph with the minimum domination number equal to half its order.

The following result was given in [3].

Theorem 16 [3]. *A connected graph G of order $2n \geq 4$ has $\gamma(G) = n$ if and only if either $G \cong C_4$ or G satisfies: the vertex set of a graph G can be partitioned into two sets V_1 and V_2, such that $|V_1| = |V_2| = n$ with only matching between V_1 and V_2 and satisfying $\langle V_1 \rangle_G \cong \overline{K_n}$ and $\langle V_2 \rangle_G$ is connected.*

From the above theorem it follows that the graph G different from C_4 has at least two hanging vertices. Moreover, according to Proposition 15, we observe that $\gamma_s(P_n \wedge G) = \gamma(P_n \wedge G)$, for G mentioned in Theorem 16. Now, we give the estimation for the split domination number with respect to the conjunction of P_n and a graph G with the minimum domination number equal to half its order. But first we find a relationship between domination parameters in G.

Theorem 17. *Let G be a connected graph of order $2n \geq 4$ with $\gamma(G) = n$. Then $\gamma_s(G) = \gamma_c(G) = \gamma(G)$.*

Proof. Assume that $G \cong C_4$. The subset containing exactly two adjacent [not adjacent] vertices realizes $\gamma(G) = 2$ and it is a minimum connected [a minimum split dominating] set of C_4. Thus the result holds, for C_4. Now, assume that G is different from C_4. By Theorem 16 we have that $V(G)$ can be partitioned into two sets V_1 and V_2 of order n, such that $\langle V_2 \rangle_G$ is connected and $\langle V_1 \rangle_G \cong \overline{K_n}$. This means that the subset V_1 is a set of all hanging vertices of G. Let $D = V_2$, since there is a matching between $V_1 = V(G) - D$ and D in G. It means that D is a minimum dominating set of G. To complete this theorem, we show that D is a $\gamma_c(G)$-set and also a $\gamma_s(G)$-set. Because of $\langle D \rangle_G$ is connected, as it was stated in Theorem 13, then D is a $\gamma_c(G)$-set. Moreover, since $\langle V(G) - D \rangle_G \cong \overline{K_n}$, $n \geq 2$ is disconnected, thus we D is a $\gamma_s(G)$-set, proving the theorem. ■

Finally, using this theorem, Theorem 13 and Proposition 15 we obtain the following estimation for a split dominating number of $P_n \wedge G$.

Corollary 18. *Let G be a connected graph of order $2m \geq 4$ with $\gamma(G) = m$. Then*

$$\gamma_s(P_n \wedge G) = \gamma(P_n \wedge G) \leq \begin{cases} \gamma(G)(2\left\lfloor \frac{n-1}{4} \right\rfloor + 1), & \text{if } n \equiv 1 (\text{mod } 4), \\ \gamma(G)(2\left\lfloor \frac{n-1}{4} \right\rfloor + 2), & \text{otherwise.} \end{cases}$$

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