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DETOUR CHROMATIC NUMBERS

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Abstract

The nth *detour chromatic number*, $\chi_n(G)$ of a graph G is the minimum number of colours required to colour the vertices of G such that no path with more than n vertices is monocoloured. The number of vertices in a longest path of G is denoted by $\tau(G)$. We conjecture that $\chi_n(G) \leq \lceil \frac{\tau(G)}{n} \rceil$ for every graph G and every $n \geq 1$ and we prove results that support the conjecture. We also present some sufficient conditions for a graph to have nth chromatic number at most 2.

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1. Introduction

A longest path in a graph G is called a *detour* of G. The number of vertices in a detour of G is called the *detour order* of G and is denoted by $\tau(G)$. The *girth* $g(G)$ and the *circumference* $c(G)$ are, respectively, the order of a shortest and a longest cycle in G. The *odd girth* $g_o(G)$ of a non-bipartite graph G is the order of a smallest odd cycle of G.

An n-*detour colouring* of G is a colouring of the vertices of G such that no path of order greater than n is monocoloured. The nth *detour-chromatic number of* G, denoted by $\chi_n(G)$, is the minimum number of colours required for an n-detour colouring of G. These chromatic numbers were introduced by Chartrand, Geller and Hedetniemi in 1968 (see [4]).

The path of order n is denoted by P_n. We say that a set W of vertices in G is P_{n+1}-*free* if $G[W]$ (the subgraph of G induced by W) has detour order at most n. Thus an n-detour colouring of G corresponds to a partition of the vertex set of G into P_{n+1}-free sets.

A partition of the vertex set of G into two sets, A and B, such that $\tau(G[A]) \leq a$ and $\tau(G[B]) \leq b$ is called an (a,b)-*partition* of G. If G has an (a,b)-partition for every pair (a,b) of positive integers such that $a+b=\tau(G)$, then we say that G is τ-*partitionable.* The following conjecture, known as the Path Partition Conjecture, is stated in [1], [9] and [3] and studied in [2], [6] and [7].

Conjecture 1. *Every graph is* τ*-partitionable.*

We shall show that, if the Path Partition Conjecture is true, the following conjecture would also be true.

Conjecture 2. $\chi_n(G) \leq \lceil \frac{\tau(G)}{n} \rceil$ *for every graph* G *and every* $n \geq 1$.

In Section 2 we prove that $\chi_n(G) \leq \left\lceil \frac{\tau(G)-n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1$ for every graph G, if $2 \leq n \leq \tau(G)$. This bound is significantly smaller than the one given in [4]. Using results from [6] and [7] (proved in support of Conjecture 1) we also show that Conjecture 2 holds for several classes of graphs.

In Section 3 we show that graphs with large enough odd girth, as well as graphs with small enough bipartite index, have nth chromatic number at most 2, indicating that having nth detour chromatic number at most 2 is a natural generalization of the property of being bipartite.

2. Bounds for χ_n in Terms of τ

The first detour-chromatic number, χ_1, is the ordinary chromatic number χ. The following bound for χ is well-known (see for example [5], Corollary 8.8, on page 226).

Theorem 2.1. $\chi(G) \leq \tau(G)$ *for every graph* G.

The above result follows from the observation that $\tau(G-M) \le \tau(G)-1$ for every maximal independent set M of G.

The following bound for χ_n appears in [4].

Theorem 2.2 (Chartrand, Geller and Hedetniemi)**.** *If G is any graph and $2 \le n \le \tau(G)-1$, then*

$$\chi_n(G) \le \left\lfloor \frac{1}{2}(\tau(G)-n-1) \right\rfloor + 2.$$

The proof of Theorem 2.2 relies on the observation that $\tau(G-M) \le \tau(G)-2$ for every maximum P_{n+1}-free subset M of G, if $2 \le n \le \tau(G)$. In [7] we proved the following stronger result.

Theorem 2.3. *Let G be a graph and n an integer such that $2 \le n \le \tau(G)$. If M is a maximal P_{n+1}-free subset of $V(G)$, then*

$$\tau(G-M) \le \tau(G) - \frac{2n+2}{3}.$$

This result enables us to prove the following:

Theorem 2.4. *If G is any graph, then*

$$\chi_n(G) \le \begin{cases} \left\lceil \frac{\tau(G)-n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1 & \text{if } 2 \le n \le \tau(G), \\ 1 & \text{if } n > \tau(G). \end{cases}$$

Proof. We use induction on $\tau(G)$. The result obviously holds for all graphs K with $\tau(K)=2$. Suppose the result is true for all graphs H with $\tau(H) < k$ for some $k > 2$. Let G be an arbitrary graph with $\tau(G)=k$. If $n \ge k$ the result holds for G, so we may suppose that $n < k$. Let M be a maximal P_{n+1}-free subset of $V(G)$. By Theorem 2.3

$$\tau(G-M) \le k - \left\lceil \frac{2n+2}{3} \right\rceil < k$$

and therefore, by the induction assumption,

$$\chi_n(G-M) \le \begin{cases} \left\lceil \frac{\tau(G)-\lceil \frac{2n+2}{3} \rceil - n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1 & \text{if } 2 \le n \le \tau(G-M), \\ 1 & \text{if } n > \tau(G-M). \end{cases}$$

By including the subset M in any P_{n+1}-free partition of $G - M$ we get a P_{n+1}-free partition of G. Hence

$$\chi_n(G) \leq \chi_n(G - M) + 1.$$

We now verify that the inequality for $\chi_n(G)$ holds for all the possible values for n. First, if $2 \leq n \leq \tau(G - M)$ then

$$\begin{aligned}\chi_n(G) &\leq \left\lceil \frac{\tau(G) - \lceil \frac{2n+2}{3} \rceil - n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1 + 1 \\ &= \left\lceil \frac{\tau(G) - n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1.\end{aligned}$$

Next, if $\tau(G - M) < n \leq \tau(G) - \lceil (2n+2)/3 \rceil$ then

$$\begin{aligned}\chi_n(G) &\leq 1 + 1 \\ &\leq \left\lceil \frac{\tau(G) - n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1\end{aligned}$$

because in this case

$$\frac{\tau(G) - n}{\lceil (2n+2)/3 \rceil} \geq 1.$$

Finally, if $\tau(G) - \lceil (2n+2)/3 \rceil < n < k$ then

$$\begin{aligned}\chi_n(G) &\leq 1 + 1 \\ &= \left\lceil \frac{\tau(G) - n}{\lceil (2n+2)/3 \rceil} \right\rceil + 1\end{aligned}$$

because in this case

$$0 < \frac{\tau(G) - n}{\lceil (2n+2)/3 \rceil} < 1.$$

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The Path Partition Conjecture can also be stated as follows:

Conjecture 1′. *For any graph G and any positive integer $n < \tau(G)$, there exists a P_{n+1}-free set H in G such that $\tau(G - H) \leq \tau(G) - n$.*

In [6] and [7] it is shown that the Path Partition Conjecture is true for several hereditary classes of graphs. In order to apply those results to detour chromatic numbers, we first prove:

Theorem 2.5. *Let $\mathcal{P}$ be a hereditary class of graphs and n a positive integer. If $\mathcal{P}$ has the property that every graph $G \in \mathcal{P}$ with $\tau(G) > n$ has a P_{n+1}-free set W such that $\tau(G - W) \leq \tau(G) - n$, then $\mathcal{P}$ also has the property that $\chi_n(G) \leq \lceil \tau(G)/n \rceil$ for every graph G in $\mathcal{P}$.*

Proof. The proof is by induction on the detour order. The result is obviously true for graphs with detour order at most n. Let G be a graph in $\mathcal{P}$ with $\tau(G) = k > n$. Then G has a P_{n+1}-free subset W such that $\tau(G - W) \leq k - n$. Since $\mathcal{P}$ is a hereditary class, $G - W \in \mathcal{P}$ and therefore, by our induction hypothesis, $\chi_n(G - W) \leq \lceil \frac{\tau(G-W)}{n} \rceil$. Now

$$\begin{aligned} \chi_n(G) &\leq \chi_n(G - W) + 1 \\ &\leq \left\lceil \frac{\tau(G - W)}{n} \right\rceil + 1 \\ &\leq \left\lceil \frac{k - n}{n} \right\rceil + 1 \\ &= \left\lceil \frac{k}{n} \right\rceil . \end{aligned}$$

■

Applying Theorem 2.5 to the class of all graphs, we note that if Conjecture 1 is true, then Conjecture 2 will also be true.

Corollary 4.7 of [6] implies:

Theorem 2.6. *If $g(G) \geq n - 1$ and $n < \tau(G)$, then G has a P_{n+1}-free set W such that $\tau(G - W) \leq \tau(G) - n$.*

Since the class of all graphs with girth at least $n - 1$ is a hereditary class, Theorem 2.6 together with Theorem 2.5 imply:

Corollary 2.7. *For any graph G, $\chi_n(G) \leq \lceil \frac{\tau(G)}{n} \rceil$ for every $n \leq g(G) + 1$.*

The class of 2-degenerate graphs is another hereditary class for which the Path Partition Conjecture holds. (A graph G is *r-degenerate* if every induced subgraph H of G has minimum degree at most r.)

Theorem 2.8. *Let G be a 2-degenerate graph and let (a, b) be any pair of positive integers such that $\tau(G) \leq a + b$. Then G has an (a, b)-partition.*

Proof. The proof is by induction on the order of G. Let v be a vertex of G of degree at most 2. By the induction hypothesis, $G - v$ has an (a, b)-partition (A, B). If v has no neighbours in A, then $(A \cup \{v\}, B)$ is an (a, b)-partition of G. We may therefore assume that v has one neighbour, say x, in A and the other one, say y, in B. If x is not an end-vertex of a P_a in A, then $(A \cup \{v\}, B)$ is an (a, b)-partition of G. If x is an end-vertex of a P_a in A, then y is not an end-vertex of a P_b in B (otherwise G would have a path of order $a + b + 1$), and then $(A, B \cup \{v\})$ is an (a, b)-partition of G. ■

Corollary 2.9. *If G is a 2-degenerate graph, then $\chi_n(G) \leq \lceil \frac{\tau(G)}{n} \rceil$ for all $n \geq 1$.*

The following is implied by Theorem 5.1 of [6] and Theorem 4.2 of [2].

Theorem 2.10.
(i) *For every graph G, $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$ for every $n \leq 6$.*
(ii) *If $\tau(G) \leq 13$, then $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$ for every $n \geq 1$.*

3. Nearly Bipartite Graphs

The *bipartite index* of a graph G is the minimum number of vertices whose removal from G results in a bipartite graph. In [8] Györi, Kostochka and Łuczak, showed that graphs without small odd cycles are "nearly bipartite", in the sense that their bipartite index is relatively small. By adapting the proof of the Lemma in [8], we prove that graphs without small odd cycles are "nearly bipartite" in the sense that their nth detour chromatic number is at most 2, for relatively small n.

Theorem 3.1. *If $n \geq 1$ and G contains no odd cycles of order less than $\tau(G) - n + 2$, then $\chi_n(G) \leq 2$.*

Proof. Without loss of generality we may assume that G is a connected graph. If G contains no odd cycles, then G is bipartite and hence $\chi_1(G) \leq 2$, so the result holds in that case. Now suppose G contains an odd cycle and let $g_o(G) = m$. Let C be an odd cycle in G with m vertices $v_0, v_1, \ldots, v_{m-1}$ and put

$$N_0 = \{v_0\} \text{ and } N_i = \{x \in V(G) | d(x, v_0) = i \text{ for } i \geq 1\}.$$

Then

$$N_i \cap V(C) = \{v_i, v_{m-i}\} \text{ if } i = 1, \dots, \frac{m-1}{2}$$

and

$$N_i \cap V(C) = \emptyset \text{ if } i \geq \frac{m+1}{2}.$$

If some N_i contains two adjacent vertices, then G contains an odd cycle with exactly one vertex in N_j for some $j \leq i-1$, and exactly two vertices in each of $N_{j+1}, \dots, N_i$. The order of this odd cycle is at most $2i+1$; hence $2i+1 \geq m$. Therefore

$$\tau(G[N_i]) = 1 \text{ if } i < \frac{m-1}{2}.$$

Let P be a path in $G[N_{\frac{m-1}{2}}]$ and let x be an end-vertex of P. Then there is an $x - v_0$-path Q of order $\frac{m+1}{2}$ with one vertex in each of the sets $N_0, N_1, \dots, N_{\frac{m-1}{2}}$. Let R_1 be the path $v_1, v_2, \dots, v_{\frac{m-3}{2}}$ and let R_2 be the path $v_{m-1}, v_{m-2}, \dots, v_{\frac{m+3}{2}}$. Since G has no odd cycles of order less than m, at least one of R_1 and R_2, say R_1, is disjoint from Q. Thus P, followed by Q, followed by R_1 is a path of order

$$v(P) + \frac{m-1}{2} + \frac{m-3}{2} = v(P) + m - 2.$$

Since $m \geq \tau(G) - n + 2$, by assumption, it follows that $v(P) \leq n$, and hence

$$\tau(G[N_{\frac{m-1}{2}}]) \leq n.$$

If $i > \frac{m-1}{2}$ and L is a path in $G[N_i]$, then L is disjoint from C and there is a path from an end-vertex of L to a vertex on C; hence

$$\tau(G[N_i]) \leq \tau(G) - m \leq n - 2.$$

Now put

$$A = \bigcup_{i\ even} N_i \text{ and } B = \bigcup_{i\ odd} N_i.$$

Since there are no edges between non-consecutive N_i and we have proved that each N_i has detour number at most n, it follows that $\tau(G[A]) \leq n$ and $\tau(G[B]) \leq n$, and hence (A, B) is an (n, n)-partition of G. ∎

We now give another sufficient condition for a graph to have nth chromatic number at most 2.

Theorem 3.2. *Let G be a graph with bipartite index r. If $n \geq r+1$ and r is even or $n \geq r+2$ and r is odd, then $\chi_n(G) \leq 2$.*

Proof. Let R be a set of r vertices in G such that $G-R$ is bipartite. Let (S_1, S_2) be a vertex partition of $G-R$ into independent sets and let (R_1, R_2) be a partition of R with $|R_1| = \lfloor \frac{r}{2} \rfloor$ and $|R_2| = \lceil \frac{r}{2} \rceil$. Put

$$V_i = S_i \cup R_i, \text{ for } i = 1, 2.$$

Then (V_1, V_2) is a partition of the vertex set of G. Since S_i is an independent set, no path in V_i has more than $2|R_i| + 1$ vertices for $i = 1, 2$. Hence

$$\tau(V_i) \leq \begin{cases} r+2 & \text{if } r \text{ is odd,} \\ r+1 & \text{if } r \text{ is even.} \end{cases}$$

Thus $\tau(G[V_i]) \leq n$ and hence $\chi_n(G) \leq 2$. ■

The results in this section can also be stated in terms of (a,b)-partitions. By setting $n = a$ and $b = \tau(G) - a$, Theorem 3.1 translates to:

Theorem 3.3. *Let G be a graph with $\tau(G) = a + b$; $1 \leq a \leq b$. If $g_o(G) \geq b+2$, then G has an (a,a)-partition.*

By slightly adapting the proof of Theorem 3.2, we also obtain:

Theorem 3.4. *If G is a graph with bipartite index at most $\lfloor \frac{\tau(G)-3}{2} \rfloor$, then G is τ-partitionable.*

Proof. Let R, S_1 and S_2 be as in the proof of Theorem 3.2, but let R_1 consist of $\lfloor \frac{a-1}{2} \rfloor$ vertices of R and put $R_2 = R - R_1$. Put $A = R_1 \cup S_1$ and $B = R_2 \cup S_2$. Then (A, B) is a partition of the vertex set of G, and

$$\tau(G[A]) \leq 2 \left\lfloor \frac{a-1}{2} \right\rfloor + 1 \leq a - 1 + 1 = a$$

and

$$\begin{aligned} \tau(B) &\leq 2\left(r - \left\lfloor \frac{a-1}{2} \right\rfloor\right) + 1 \\ &\leq 2\left\lfloor \frac{\tau(G)-3}{2} \right\rfloor - 2\left\lfloor \frac{a-1}{2} \right\rfloor + 1 \\ &\leq (a+b) - 3 - (a-2) + 1 \\ &= b. \end{aligned}$$

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Corollary 3.5. *If G is a tripartite graph with n_1 vertices in the smallest part and $\tau(G) \geq 2n_1 + 3$, then G is τ-partitionable.*

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