# COLOURING GRAPHS WITH PRESCRIBED INDUCED CYCLE LENGTHS 

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#### Abstract

In this paper we study the chromatic number of graphs with two prescribed induced cycle lengths. It is due to Sumner that trianglefree and $P_{5}$-free or triangle-free, $P_{6}$-free and $C_{6}$-free graphs are 3 colourable. A canonical extension of these graph classes is $\mathcal{G}^{I}(4,5)$, the class of all graphs whose induced cycle lengths are 4 or 5 . Our main result states that all graphs of $\mathcal{G}^{I}(4,5)$ are 3 -colourable. Moreover, we present polynomial time algorithms to 3 -colour all triangle-free graphs $G$ of this kind, i.e., we have polynomial time algorithms to 3 -colour every $G \in \mathcal{G}^{I}\left(n_{1}, n_{2}\right)$ with $n_{1}, n_{2} \geq 4$ (see Table 1). Furthermore, we consider the related problem of finding a $\chi$-binding function for the class $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$. Here we obtain the surprising result that there exists no linear $\chi$-binding function for $\mathcal{G}^{I}(3,4)$.


Keywords: colouring, chromatic number, induced subgraphs, graph algorithms, $\chi$-binding function.
2000 Mathematics Subject Classification: 05C15, 05C75, 05C85.

## 1. Introduction and Results

We consider finite undirected simple graphs. For terminology and notation not defined here we refer to [1]. As introduced by Gyárfás [6], a family $\mathcal{G}$ of graphs is called $\chi$-bound with $\chi$-binding function $f$, if $\chi\left(G^{\prime}\right) \leq f\left(\omega\left(G^{\prime}\right)\right)$ holds whenever $G^{\prime}$ is an induced subgraph of $G \in \mathcal{G}$.

Erdös [4] showed that for each pair $g, k$ with $g, k \geq 4$ there exist graphs with girth $g$ and chromatic number $k$. Hence, triangle-free graphs can have arbitrary large chromatic number. Sumner [13] showed that triangle-free and $P_{5}$-free or triangle-free, $P_{6}$-free and $C_{6}$-free graphs are 3 -colourable.

For $t \geq 5$ define $\mathcal{G}_{t}$ as the class of all triangle-free graphs which are $P_{t}$-free and $C_{i}$-free for $6 \leq i \leq t$. For $k \geq 1$ and $3 \leq n_{1}<n_{2}<\cdots<n_{k}$ let $\mathcal{G}^{I}\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ be the class of all graphs whose induced cycle lengths are equal to one of $n_{1}, n_{2}, \ldots, n_{k}$. Thus

$$
\mathcal{G}_{5} \subset \mathcal{G}_{6} \subset \mathcal{G}_{7} \subset \cdots \subset \mathcal{G}^{I}(4,5)
$$

and all graphs $G$ of $\mathcal{G}_{5}$ and $\mathcal{G}_{6}$ are 3 -colourable by Sumners result. Note that all graphs of $\mathcal{G}_{t}$ have diameter at most $t-2$ whereas graphs of $\mathcal{G}^{I}(4,5)$ can have arbitrary diameter.

Our research was motivated by the question whether 3-colourability still holds for a superclass $\mathcal{G}_{t}\left(\right.$ of $\mathcal{G}_{5}$ and $\mathcal{G}_{6}$ ) for some $t \geq 7$. Theorem 1 states that all graphs of $\mathcal{G}^{I}(4,5)$ are 3 -colourable. Hence, the answer is affirmative for each $t \geq 7$. Moreover, we can guarantee a 3 -colouring with some additional properties. For a fixed integer $p \geq 2$ we call a graph $G \in \mathcal{G}^{I}(4,2 p+1)$ $3^{*}$-colourable with root $v$, if there is a 3-colouring of $G$ such that $G\left[N_{G}^{p}(v)\right]$ is coloured with two colours, where $N_{G}^{p}(v)$ is the set of vertices having distance $p$ from $v$. Observe that this definition implies the following useful property: If $G$ is $3^{*}$-colourable with root $v$, then we can choose a 3 -colouring such that $G\left[N_{G}^{i}(v)\right]$ is coloured monochromatic for every $1 \leq i<p$ and $G\left[N_{G}^{p}(v)\right]$ is coloured with two colours. Furthermore, if this property holds for every vertex of $G \in \mathcal{G}^{I}(4,2 p+1)$, then we call $G 3^{*}$-colourable. This definition is motivated by the following observation.

If $G_{1}, G_{2} \in \mathcal{G}^{I}(4,2 p+1)$ and $v_{i} \in G_{i}$ for $i=1,2$, then the new graph $G^{*}$ with vertex set $V\left(G^{*}\right)=V\left(G_{1}-v_{1}\right) \cup V\left(G_{2}-v_{2}\right)$ and edge set $E\left(G^{*}\right)=$ $E\left(G_{1}-v_{1}\right) \cup E\left(G_{2}-v_{2}\right) \cup\left\{u_{1} u_{2} \mid u_{i} \in N_{G_{i}}\left(v_{i}\right)\right.$ for $\left.i=1,2\right\}$ is likewise a member of $\mathcal{G}^{I}(4,2 p+1)$. The invariance of $\mathcal{G}^{I}(4,2 p+1)$ concerning this graph operation reasons the equivalence of $3^{*}$ - and 3 -colourability for the class $\mathcal{G}^{I}(4,2 p+1)$.

Theorem 1. Every graph of $\mathcal{G}^{I}(4,2 p+1)$ with $p \geq 2$ is $3^{*}$-colourable.
The proof of Theorem 1 is based on decomposition and provides a polynomial time algorithm to $3^{*}$-colour a given graph $G \in \mathcal{G}^{I}(4,2 p+1)$. Note that the class $\mathcal{G}^{I}(4,5)$ is a canonical extension of $\mathcal{G}^{I}(4)$, which are the well-known chordal bipartite graphs (e.g. see [2]). Very recently Brandt [3] examined the maximal (with respect to edge addition) triangle-free members of the class $\mathcal{G}^{I}(4,5)$ with emphasis on graph homomorphisms. Brandt also observed that the class $\mathcal{G}^{I}(4,5)$ is a natural extension of $\mathcal{G}^{I}(4)$ - the chordal bipartite graphs - and he introduced for members of $\mathcal{G}^{I}(4,5)$ the terminology of chordal triangle-free graphs.

Motivated by the first theorem we consider next the classes $\mathcal{G}^{I}(2 q, 2 p+1)$ and $\mathcal{G}^{I}\left(2 p^{\prime}+1,2 q^{\prime}\right)$ for $q, q^{\prime} \geq 3$ and $p, p^{\prime} \geq 2$. But first we will examine the larger class $\mathcal{G}^{I}\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ with $n_{1} \geq 5$. A graph $G$ is $r$-degenerate, if there exists an ordering $\left(v_{1}, \ldots, v_{n}\right)$ of $V(G)$ such that $d_{G\left[\left\{v_{i}, \ldots, v_{n}\right\}\right]}\left(v_{i}\right) \leq r$ for all $1 \leq i \leq n$.

Theorem 2. Every graph of $\mathcal{G}^{I}\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ with $k \geq 1$ and $n_{1} \geq 5$ is ( $k+1$ )-degenerate. Especially, every vertex $v$ of $G$ being an endvertex of $a$ longest induced path of $G$ satisfies $d_{G}(v) \leq k+1$.

Corollary 3. Every graph of $\mathcal{G}^{I}\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ with $k \geq 1$ and $n_{1} \geq 5$ is ( $k+2$ )-colourable.

The last result reveals an interesting relation to the colouring properties of graphs of the class $\mathcal{G}\left(n_{1}, n_{2}, \ldots, n_{k}\right)$, the class of all graphs whose (not necessarily induced) cycle lengths are equal to one of $n_{1}, n_{2}, \ldots, n_{k}$. Now let $G$ be a graph with $r$ different odd and $s$ different even cycle lengths (which need not to be induced). In [10] Mihók and Schiermeyer presented a polynomial time colouring algorithm, called MAXBIP, which recursively constructs maximal bipartite subgraphs. Based on MAXBIP they proved the following theorem, answering thereby a question of B. Bollobás and P. Erdős [5].

Theorem 4 (Mihók and Schiermeyer [10], 1997). $\chi(G) \leq \min \{2 r+2$, $2 s+3\}$.

With $k=r+s$ this also implies $\chi(G) \leq k+2$. The question of B. Bollobás and P. Erdős [5] only concerned $(2 r+2)$-colourability of graphs with $r$ different odd cycle lengths (which need not to be induced). This question was
first answered affirmative by A. Gyárfás [7]. Additional informations and a related conjecture can be found in the excellent book [8] of T. Jensen and B. Toft on graph colouring problems.

Obviously, Corollary 3 is best possible for $k=1$. But for $k=2$ we are able to improve Corollary 3. For the next theorem we need to recall the definition of the famous Petersen graph $P^{*}$. This 3-regular, non-bipartite graph $P^{*}$ of order 10 is a member of the class $\mathcal{G}^{I}(5,6)$. The Petersen graph $P^{*}$ consists of two disjoint induced 5-cycles $C^{1}=a_{0} a_{1} a_{2} a_{3} a_{4} a_{0}$ and $C^{2}=b_{0} b_{1} b_{2} b_{3} b_{4} b_{0}$ and the additional edges $a_{0} b_{0}, a_{1} b_{3}, a_{2} b_{1}, a_{3} b_{4}$ and $a_{4} b_{2}$. Obviously $P^{*}$ is 3 -colourable.

Theorem 5. Every graph $G$ of $\mathcal{G}^{I}(2 q, 2 p+1)$ or $\mathcal{G}^{I}\left(2 p^{\prime}+1,2 q^{\prime}\right)$ with $q, q^{\prime} \geq 3$ and $p, p^{\prime} \geq 2$ fulfills at least one of the following properties:

1. $G$ is bipartite;
2. $G$ satisfies $\delta(G) \leq 2$;
3. $G \in \mathcal{G}^{I}(5,6)$ and one of the following properties holds:
(a) $G \cong P^{*}$;
(b) $G$ contains a clique cutset, i.e., $K_{1}$ or $K_{2}$ clique cutset.

Every graph $G$ of $\mathcal{G}^{I}(2 q, 2 p+1)$ or $\mathcal{G}^{I}\left(2 p^{\prime}+1,2 q^{\prime}\right)$ with $q, q^{\prime} \geq 3$ and $p, p^{\prime} \geq 2$ not isomorphic to $P^{*}$ fulfills at least one of the three properties because of Theorem 5. Testing whether $G$ is bipartite, has minimal degree two or contains a complete cutset of size at most two can be done very efficiently. Moreover, if $G \in \mathcal{G}^{I}(5,6)$ is non-bipartite, $\delta(G) \geq 3$ and contains no complete cutset, then $G \cong P^{*}$, which obviously is 3 -colourable. Hence, Theorem 5 provides a polynomial time algorithm to 3 -colour a given graph $G \in \mathcal{G}^{I}(2 q, 2 p+1)$ or $G \in \mathcal{G}^{I}\left(2 p^{\prime}+1,2 q^{\prime}\right)$ with $q, q^{\prime} \geq 3$ and $p, p^{\prime} \geq 2$. This algorithm (recursively) makes use of the fact that the graph (in question) is bipartite, has a vertex of degree at most two, is isomorphic to the Petersen graph or the graph can be decomposed into two smaller graphs according to a complete cutset of size at most two.

Corollary 6. Every graph $G$ of $\mathcal{G}^{I}(2 q, 2 p+1)$ or $\mathcal{G}^{I}\left(2 p^{\prime}+1,2 q^{\prime}\right)$ with $q, q^{\prime} \geq 3$ and $p, p^{\prime} \geq 2$ is 3 -colourable.

Now we consider the related problem of finding a (best possible) $\chi$-binding function $f^{*}$ for $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$ and for completeness also for its subclasses $\mathcal{G}^{I}\left(n_{1}\right)$.

Recall that a graph is perfect if for each induced subgraph $H$ of $G$ the chromatic number $\chi(H)$ equals the corresponding clique number $\omega(H)$. Furthermore the lexicographic product $G_{1}\left[G_{2}\right]$ of two graphs $G_{1}$ and $G_{2}$ contains the vertex set $V\left(G_{1}\left[G_{2}\right]\right)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and two vertices $(a, b)$ and $(c, d)$ are adjacent in $G_{1}\left[G_{2}\right]$ if $a$ is adjacent to $c$ in $G_{1}$ or $a=c$ and $b$ is adjacent to $d$ in $G_{2}$.

For convenience we drop the condition that $n_{1}$ is always smaller than $n_{2}$ in the definition of $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$.
(I) $n_{1}, n_{2}$ are even:

For even $n_{1}$ and $n_{2}$ all graphs of $\mathcal{G}^{I}\left(n_{1}\right)$ and $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$ are bipartite and thus perfect with $f^{*}(\omega)=\omega \leq 2$.
(II) $n_{1}$ is even, $n_{2}$ is odd: (A) $n_{2} \geq 5$ :

By our results (Theorem 1, Corollary 6) every graph of $\mathcal{G}^{I}\left(n_{2}\right)$ and $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$ is 3 -colourable, i.e., with $\omega \leq 2$ we have $f^{*}(\omega)=\omega+1 \leq 3$.
(II) $n_{1}$ is even, $n_{2}$ is odd: (B1) $n_{2}=3$ and $n_{1} \geq 6$ :

Recently, Rusu [11] proved that all members of a superclass of $\mathcal{G}^{I}(3,2 q)$ are perfect for any $q \geq 3$. Hence, we also have $f(\omega)=\omega$. A well-known subclass of $\mathcal{G}^{I}(3,2 q)$ is $\mathcal{G}^{I}(3)$ containing the chordal graphs.
(II) $n_{1}$ is even, $n_{2}$ is odd: (B2) $n_{2}=3$ and $n_{1}=4$ :

In 1987 Gyárfás [6] conjectured (motivated by the Strong Perfect Graph Conjecture) that there exists a $\chi$-binding function for $\mathcal{G}^{I}(3,4)$. But this Conjecture is still open. In [6] Gyárfás also suggested to examine whether there exists a linear $\chi$-binding function for hereditary classes of graphs. For $\mathcal{G}^{I}(3,4)$ we have constructed the following sequence of graphs $\left(H_{i}\right)$. Starting with $H_{1}=\bar{C}_{7}$, the complement of the 7 -cycle, we define $H_{i+1}=\bar{C}_{7}\left[H_{i}\right]$, the lexicographic product of the graphs $\bar{C}_{7}$ and $H_{i}$. Note that $\omega\left(H_{i+1}\right)=$ $3 \omega\left(H_{i}\right)$. Furthermore, in any colouring of $H_{i+1}$ we need for each copy of $H_{i}$ at least $\chi\left(H_{i}\right)$ different colours. With $\alpha\left(\bar{C}_{7}\right)=2$ we then observe that every colour of a colouring of $H_{i+1}$ appears in at most two different copies of $H_{i}$. Hence, $H_{i}$ has the order $n\left(H_{i}\right)=7^{i}$, the independence number $\alpha\left(H_{i}\right)=2^{i}$ and the clique number $\omega\left(H_{i}\right)=3^{i}$. Therefore, its chromatic number $\chi\left(H_{i}\right)$ is at least $(7 / 2)^{i}$. Thus, the $\chi$-binding function $f^{*}$ for $\mathcal{G}^{I}(3,4)$ satisfies $f^{*}(\omega) \geq(7 / 6)^{i} \omega$ for every integer $i$. Hence, we obtain the following surprising result:

Theorem 7. There exists no linear $\chi$-binding function for $\mathcal{G}^{I}(3,4)$.

It is noteworthy to mention that $\mathcal{G}^{I}(3,4)$ contains all weakly triangulated graphs. Recently, Scott [12] achieved some related results.
(III) $n_{1}, n_{2}$ are odd: (A1) $n_{1}, n_{2} \geq 5$ :

Markossian, Gasparian and Reed [9] showed that all triangle-free and even-hole-free graphs are 2-degenerate and thus are 3 -colourable. Hence, $f^{*}(\omega)=$ $\omega+1 \leq 3$ is a $\chi$-binding function for $\mathcal{G}^{I}\left(n_{1}\right)$ and $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$.
(III) $n_{1}, n_{2}$ are odd: (A2) $n_{1}=3$ :

It is an open problem, whether there exists a linear $\chi$-binding function for $\mathcal{G}^{I}\left(3, n_{2}\right)$. The graph-sequence $G_{r}=C_{n_{2}}\left[K_{r}\right]$, the lexicographic product of the odd cycle $C_{n_{2}}$ and the complete graph $K_{r}$, reveals that we have $f^{*}(\omega) \geq\left(\left(n_{2}+1\right) /\left(n_{2}-1\right)\right) \omega$ for every $\chi$-binding function. We expect that

$$
f^{*}(\omega)=\left(\left(n_{2}+1\right) /\left(n_{2}-1\right)\right) \omega .
$$

Table 1. $\chi$-binding function $f^{*}$ for $\mathcal{G}^{I}\left(n_{1}, n_{2}\right)$.

| $\rightarrow$ <br> $n_{1}, n_{2}$ <br> $\downarrow$ | 3 | 4 | odd $\geq 5$ | even $\geq 6$ |
| :---: | :---: | :---: | :---: | :---: |
| 3 | $f^{*}(\omega)=\omega$ <br> chordal | $\nexists$ linear $f^{*}$ <br> Theorem 7 <br> Conj.[6]: $\exists f^{*}$ | $f^{*}(\omega) \geq\left(\left(n_{2}+1\right) /\left(n_{2}-1\right)\right) \omega$ <br> Conj.: ${ }^{\prime \prime}={ }^{\prime \prime}$ | $f^{*}(\omega)=\omega$ <br> Rusu $[11]$ |
| 4 |  | $f^{*}(\omega)=\omega \leq 2$ <br> chordal <br> bipartite | $f^{*}(\omega)=\omega+1 \leq 3$ <br> Theorem 1 | $f^{*}(\omega)=\omega \leq 2$ <br> $\subset$ bipartite |
| odd <br> $\geq 5$ |  |  | $f^{*}(\omega)=\omega+1 \leq 3$ <br> Markossian,$\ldots[9]$ | $f^{*}(\omega)=\omega+1 \leq 3$ <br> Corollary 6 |
| even <br> $\geq 6$ |  |  |  | $f^{*}(\omega)=\omega \leq 2$ <br> $\subset$ bipartite |

## 2. Proofs

The following well-known easy observation provides a very useful property. If a graph $G$ contains a pair $u, v$ of nonadjacent vertices with $N_{G}(u) \subseteq N_{G}(v)$, then any proper $k$-colouring of $G-u$ can easily be extended to a proper $k$-colouring of $G$.

Therefore we only have to consider those graphs $G$ having the property ( $*$ ):
$(*)$ If $u v \notin E(G)$, then there exist vertices (private neighbours) $p_{u} \in$ $\left(N_{G}(u)-N_{G}(v)\right)$ and $p_{v} \in\left(N_{G}(v)-N_{G}(u)\right)$.
The next lemma provides a useful property of triangle-free and $C_{6}$-free graphs, which will be used in the proof of Theorem 1. This class of graphs forms a superclass of all classes $\mathcal{G}^{I}(4,2 p+1)$ with $p \geq 2$.

Lemma 8. Let $G$ be a triangle-free and $C_{6}$-free graph satisfying property $(*)$. Then for every vertex $x$ of degree $d_{G}(x)=k \geq 3$ with neighbours $x_{1}, x_{2}, \ldots, x_{k}$ there exists a pair $x_{i}, x_{j}$ of neighbours such that $N_{G}\left(x_{i}\right) \cap$ $N_{G}\left(x_{j}\right) \cap N_{G}^{2}(x)=\emptyset$.

Proof. Let $G$ be a triangle-free and $C_{6}$-free graph satisfying property $(*)$. Further suppose to the contrary that there exists a vertex $x$ of degree $d_{G}(x)=k \geq 3$ with neighbours $x_{1}, x_{2}, \ldots, x_{k}$ such that $N_{G}\left(x_{i}\right) \cap N_{G}\left(x_{j}\right)$ $\cap N_{G}^{2}(x) \neq \emptyset$ for all pairs $i, j$ with $1 \leq i<j \leq k$. Choose a vertex $v \in$ $N_{G}^{2}(x)$ such that $\left|N_{G}(x) \cap N_{G}(v)\right|=p$ is maximum. By $(*)$ we know that $p \leq k-1$. We may assume that $N_{G}(x) \cap N_{G}(v)=\left\{x_{1}, \ldots, x_{p}\right\}$. By the assumption there exists a vertex $w \in N_{G}^{2}(x)$ with $w$ likewise adjacent to a vertex $x_{i} \in\left\{x_{1}, \ldots, x_{p}\right\}$ and to a vertex $x_{j} \in\left\{x_{p+1}, \ldots, x_{k}\right\}$. Hence there is a vertex $w \in N_{G}^{2}(x)-\{v\}$ with $N_{G}(v) \cap N_{G}(w) \cap N_{G}(x) \neq \emptyset$ and $N_{G}(w) \cap$ $\left\{x_{p+1}, \ldots, x_{k}\right\} \neq \emptyset$. Choose a vertex $w$ with $N_{G}(w) \cap\left\{x_{p+1}, \ldots, x_{k}\right\} \neq \emptyset$ such that $\left|N_{G}(v) \cap N_{G}(w) \cap N_{G}(x)\right|=q$ is maximum. Hence $1 \leq q \leq p-1$ by the choice of $v$ and $w$. We may assume that $N_{G}(v) \cap N_{G}(w) \cap N_{G}(x)=$ $\left\{x_{1}, \ldots, x_{q}\right\}$ and that $x_{p+1} \in N_{G}(w)$. By the hypothesis there exists a vertex $u \in N_{G}^{2}(x)-\{v, w\}$ such that $x_{p}, x_{p+1} \in N_{G}(u)$. By the choice of $p$ and $q$ there is a vertex $x_{s} \in\left\{x_{1}, \ldots, x_{q}\right\}$ such that $x_{s} \notin N_{G}(u)$. Now $G^{\prime}:=$ $G\left[\left\{x_{s}, x_{p}, x_{p+1}, v, w, u\right\}\right] \cong C_{6}$ or $G^{\prime}$ contains a triangle, a contradiction.

Proof of Theorem 1. We will prove the theorem by induction. We may assume that $G$ is a connected graph and fulfills property (*). By the induction hypothesis there is a $3^{*}$-colouring with root $y$ for every vertex $y$ of any induced subgraph $H$ of $G$ with $|V(H)|<|V(G)|$. Now let $x$ be an arbitrary vertex of $G$. If $\operatorname{dist}_{G}(x, y)<p$ for all vertices $y \in V(G-x)$, then we can easily $3^{*}$-colour $G$ with root $x$. Hence we may assume that $\operatorname{dist}_{G}(x, y) \geq p$ for at least one vertex $y \in V(G-x)$.

Case 1. Assume that $d_{G}(x)=1$ and let $x z \in E(G)$.

Then by induction hypothesis there is a $3^{*}$-colouring with root $z$ of $G-x$, which can be easily extended to a $3^{*}$-colouring with root $x$ of $G$.

Case 2. Assume that $d_{G}(x) \geq 2$.
Also, if $x$ is a cutvertex of $G$, then we can easily $3^{*}$-colour $G$ with root $x$. Hence we may assume that $x$ is not a cutvertex of $G$. Note that $x$ and every neighbour of $x$ has a degree of at least two.

We now consider $N_{G}^{i}(x)$ for $1 \leq i<p$. If there is a vertex $y \in N_{G}^{i}(x)$ such that

$$
N_{G}(y) \cap N_{G}^{i+1}(x) \subseteq\left(\bigcup_{z \in N_{G}^{i}(x)-\{y\}} N_{G}(z)\right) \cap N_{G}^{i+1}(x)
$$

then the levels $N_{G}^{h}(x)$ with $1 \leq h<p$ and $h \neq i$ are the same in $G$ and $G-y$. Thus, we can reduce $G$ to $G-y$ and a $3^{*}$-colouring with root $x$ of $G-y$ can be extended to a $3^{*}$-colouring with root $x$ of $G$ since $G\left[N_{G}^{i-1}(x)\right]$ is independent and monochromatic, $G\left[N_{G}^{i}(x)-\{y\}\right]$ is independent and monochromatic and $G\left[N_{G}^{i+1}(x)\right]$ is 2-coloured. Consequently, we can assume for the remaining part of the proof that

Claim 1. Every vertex $y \in N_{G}^{i}(x)$ has a 'private neighbour' in $N_{G}^{i+1}(x)$ for $1 \leq i<p$.

Subcase 2.1. Assume that $d_{G}(x) \geq 3$.
Let $u^{1}, v^{1}$ and $w^{1}$ be three vertices of $N_{G}^{1}(x)$. Then by Claim 1 there are three 'paths of private neighbours' $u^{1} u^{2} \ldots u^{p}, v^{1} v^{2} \ldots v^{p}$ and $w^{1} w^{2} \ldots w^{p}$. Set $u=u^{p-1}, v=v^{p-1}$ and $w=w^{p-1}$. Let $U^{p}, V^{p}, W^{p}$ be the set of private neighbours of $u, v$ and $w$, respectively.

If $p \geq 3$, then $N_{G}^{p}(x) \cap N_{G}(y) \cap N_{G}(z)=\emptyset$ for all pairs $y, z \in\{u, v, w\}$. Since otherwise there would be an induced $C_{2 p}-$ a contradiction. If $p=2$, then we can choose $u$ and $v$ by Lemma 8 such that $N_{G}^{2}(x) \cap N_{G}(u)$ $\cap N_{G}(v)=\emptyset$.

Now observe that $G\left[U^{p}, V^{p}\right], G\left[U^{p}, W^{p}\right]$ and $G\left[W^{p}, V^{p}\right]$ are $2 K_{2}$-free (and bipartite), since otherwise there would be an induced $C_{6}$.

A useful property of every bipartite and $2 K_{2}$-free graph $H$ is the existence of a labelling of the vertices of each partition set $X=\left\{x_{1}, \ldots, x_{k}\right\}$, such that $N_{H}\left(x_{i}\right) \subset N_{H}\left(x_{j}\right)$ if $i \leq j$.

Then we easily deduce that there are sets $U_{(c)}^{p}, U_{(e)}^{p}, V_{(c)}^{p}$ and $V_{(e)}^{p}$ such that $U^{p}=U_{(c)}^{p} \cup U_{(e)}^{p}, V^{p}=V_{(c)}^{p} \cup V_{(e)}^{p}, G\left[U_{(e)}^{p} \cup V_{(e)}^{p}\right]$ is edgeless and
$G\left[U_{(c)}^{p} \cup V_{(c)}^{p}\right]$ is complete bipartite. Note that the partition of $U^{p}$ and $V^{p}$ into $U_{(c)}^{p}, U_{(e)}^{p}, V_{(c)}^{p}$ and $V_{(e)}^{p}$ is not necessarily unique. But, if $G\left[U^{p}, V^{p}\right]$ is not a complete bipartite graph, it is always possible to choose a partition of $U^{p}$ and $V^{p}$ into $U_{(c)}^{p}, U_{(e)}^{p}, V_{(c)}^{p}$ and $V_{(e)}^{p}$ such that $U_{(e)}^{p} \neq \emptyset$ and $U_{(e)}^{p} \neq \emptyset!$

Subcase 2.1.1. $G\left[U^{p}, V^{p}\right]$ is not a complete bipartite subgraph.
Then there are two non-adjacent vertices $y \in U^{p}$ and $z \in V^{p}$. Now we can choose a partition of $U^{p}$ and $V^{p}$ into $U_{(c)}^{p}, U_{(e)}^{p}, V_{(c)}^{p}$ and $V_{(e)}^{p}$ such that $y \in U_{(e)}^{p}$ and $z \in V_{(e)}^{p}$. Set $S=\left\{v \in V(G) \mid \operatorname{dist}_{G}(x, v) \leq p-1\right\} \cup U_{(c)}^{p} \cup V_{(c)}^{p}$.

Subcase 2.1.1.1. For every vertex $y \in U_{(e)}^{p}$ and $z \in V_{(e)}^{p}$ there exists a path connecting $y$ and $z$ in $G-S$.
Now we choose $y \in U_{(e)}^{p}$ and $z \in V_{(e)}^{p}$ such that $\operatorname{dist}_{G-S}(y, z)=$ $\min \left\{d i s t_{G-S}\left(y^{\prime}, z^{\prime}\right) \mid y^{\prime} \in U_{(e)}^{p}\right.$ and $\left.z^{\prime} \in V_{(e)}^{p}\right\}$. Then obviously there would be an induced cycle of length at least $(2 p+1)+1>2 p+1>4$, a contradiction.

Subcase 2.1.1.2. There exist vertices $y \in U_{(e)}^{p}$ and $z \in V_{(e)}^{p}$ such that there is no path connecting $y$ and $z$ in $G-S$.
Now let $H_{1}$ be a component of $G-S$ and $H_{2}$ be the remaining part of $G-S$. Set $G_{i}=G\left[V\left(H_{i}\right) \cup S\right]$ for $i=1,2$. Then there is a $3^{*}$-colouring with root $x$ for each $G_{i}$ with $i=1,2$. We can choose these $3^{*}$-colourings in such a way that the vertices of $S$ always receive the same colours. Hence we obtain a $3^{*}$-colouring with root $x$ of $G$. In the following, if we will apply this subprocedure, we shortly refer that we apply decomposition.

Subcase 2.1.2. $G\left[U^{p}, V^{p}\right]$ is a complete bipartite subgraph.
If $p \geq 3$, then $G\left[U^{p}, W^{p}\right]$ and $G\left[W^{p}, V^{p}\right]$ are complete as well contradicting that $G$ is triangle-free.

Let $p=2$. If we also have $N_{G}^{2}(x) \cap N_{G}(u) \cap N_{G}(w)=\emptyset$ or $N_{G}^{2}(x) \cap$ $N_{G}(v) \cap N_{G}(w)=\emptyset$, then we either apply decomposition or we deduce that $G\left[U^{p}, W^{p}\right]$ or $G\left[W^{p}, V^{p}\right]$ is complete bipartite. But then there exists a path ayz or azy with $a \in W^{p}, y \in U^{p}$ and $z \in V^{p}$ implying the existence of an induced cycle $C_{6}$ or ayz induces a triangle - a contradiction. Hence $N_{G}^{2}(x) \cap N_{G}(u) \cap N_{G}(w) \neq \emptyset$ and $N_{G}^{2}(x) \cap N_{G}(v) \cap N_{G}(w) \neq \emptyset$. Let $a \in$ $N_{G}^{2}(x) \cap N_{G}(u) \cap N_{G}(w)$ and $b \in N_{G}^{2}(x) \cap N_{G}(v) \cap N_{G}(w)$. We now consider the subgraph $G\left[U^{\prime 2}, V^{\prime 2}\right]$ with $U^{\prime 2}=N_{G}^{2}(x) \cap\left(N_{G}(u)-N_{G}(v)\right)$ and $V^{\prime 2}=$ $N_{G}^{2}(x) \cap\left(N_{G}(v)-N_{G}(u)\right)$. Recall that $N_{G}^{2}(x) \cap\left(N_{G}(u) \cup N_{G}(v)\right)=U^{\prime 2} \cup V^{\prime 2}$ and $G\left[U^{2}, V^{2}\right]$ is a subgraph of $G\left[U^{\prime 2}, V^{\prime 2}\right]$. Note that we obtain analogously that $G\left[U^{\prime 2}, V^{\prime 2}\right]$ is $2 K_{2}$-free (and bipartite). With $a \in U^{\prime 2}, b \in V^{\prime 2},\{a, b\} \subset$ $N_{G}(w)$ and $G$ being triangle-free we deduce that $G\left[U^{\prime 2}, V^{\prime 2}\right]$ is not complete.

But now we analogously can apply decomposition. This settles the case that $d_{G}(x) \geq 3$.

Subcase 2.2. Suppose that $d_{G}(x)=2$.
By the induction hypothesis there is a $3^{*}$-colouring with root $y$ for every vertex $y$ with $d_{G}(y) \neq 2$ and for every $y$ of any induced subgraph $H$ of $G$ with $|V(H)|<|V(G)|$. Note that $N_{G}\left(u^{1}\right) \cap N_{G}\left(v^{1}\right) \cap N_{G}^{2}(x)=\emptyset$ by $(*)$.

Let $p=2$. If $d_{G}\left(u^{1}\right)=d_{G}\left(v^{1}\right)=2$, then because the vertices $u^{2}, u^{1}, x, v^{1}$ and $v^{2}$ are lying on a cycle (! x is no cutvertex of $G!$ ) we obtain adjacency of $v^{2}$ and $u^{2}$. Now it is not very difficult to extend an arbitrary 3 -colouring of $G-\left\{u^{1}, x, v^{1}\right\}$ to a $3^{*}$-colouring with root $x$ of $G$.

Let $d_{G}\left(u^{1}\right) \geq 3$. By the induction hypothesis there is a $3^{*}$-colouring with root $u^{1}$ of $G$. Since $\left\{v^{1}\right\} \cup\left(N_{G}\left(v^{1}\right) \cap N_{G}^{2}(x)\right)$ induces a star we may choose the colours in such a way that (e.g.) $u^{1}$ and $v^{1}$ receive colour $1, x$ and all vertices of $N_{G}\left(u^{1}\right) \cap N_{G}^{2}(x)$ get colour 2 and finally all vertices of $N_{G}\left(v^{1}\right) \cap N_{G}^{2}(x)$ receive colour 3. This gives a $3^{*}$-colouring with root $x$ of $G$.

Let $p \geq 3$. Since $N_{G}^{i}(x)$ is independent for $1 \leq i<p$ and $G$ being $C_{2 q}$-free for $q \geq 3$, we obtain that $N_{G}^{i}\left(u^{1}\right) \cap N_{G}^{i}\left(v^{1}\right) \cap N_{G}^{i+1}(x)=\emptyset$ for $2 \leq i<p$. Hence the set $N_{G}^{p}(x)$ is the disjoint union of the sets $N_{G}^{p-1}\left(u^{1}\right)$ and $N_{G}^{p-1}\left(v^{1}\right)$. Now the $C_{2 q+1}$-freeness of $G(1 \leq q<p)$ forces that $N_{G}^{p-1}\left(u^{1}\right)$ as well as $N_{G}^{p-1}\left(v^{1}\right)$ form independent sets. Moreover, it is not difficult to see that $N_{G}^{p}(x)$ induces a $2 K_{2}$-free bipartite graph. Likewise to the previous case $d_{G}(x) \geq 3$ we can apply decomposition or $N_{G}^{p}(x)$ induces a complete bipartite graph. In the latter case we can proceed analogously to the $p=2$ subcase.

If all vertices of $N^{\prime}:=\left\{v \in V(G) \mid \operatorname{dist}_{G}(x, v) \leq p-1\right\}$ have degree two, then again $N_{G}^{p}(x)$ induces an edge. Now it is not very difficult to extend an arbitrary 3 -colouring of $G-N^{\prime}$ to a $3^{*}$-colouring with root $x$ of $G$.

Now let $u^{i}$ (with $1 \leq i<p$ ) be a vertex of degree at least three having a minimum distance concerning $x$. By the induction hypothesis there is a $3^{*}$ colouring with root $u^{i}$ of $G$. Note that $N_{G}^{p}(x)$ is bicoloured, since $G\left[U^{p}, V^{p}\right]$ is complete bipartite. Now again it is not very difficult to recolour this 3 -colouring in such a way that we obtain a $3^{*}$-colouring with root $x$ of $G$.

Proof of Theorem 2. We will show that every vertex $v$ of $G$ being an endvertex of a longest induced path of $G$ satisfies $d_{G}(v) \leq k+1$. Observe that this implies $\delta(G) \leq k+1$. Let $P:=v_{1} v_{2} \ldots v_{t}$ be a longest induced path of $G$. Suppose $d_{G}\left(v_{t}\right)=s \geq k+2$. Let $N_{G}\left(v_{t}\right)=\left\{v_{t-1}, u_{1}, \ldots, u_{s-1}\right\}$. By the choice of $P$ every vertex $u_{i}$ has a neighbour on $P$. For each $i$ (with
$1 \leq i \leq s-1)$ let $j(i)$ be the largest integer less than or equal to $t-1$ such that $u_{i} v_{j(i)} \in E(G)$. Since $G$ is triangle-free and contains no induced $C_{4}$, no two values of $j(1), \ldots, j(s-1)$ are equal. Hence there are $s-1 \geq k+1$ induced cycles of different lengths $t-j(i)+2$ for $1 \leq i \leq s-1$, a contradiction.

## Proof of Theorem 5.

Case 1. Suppose there is a graph $G \in \mathcal{G}^{I}(2 q, 2 p+1)$ with $q \geq 3$ and $p \geq 2($ and $2 p+1>2 q)$, which satisfies $\delta(G) \geq 3$ and is non-bipartite.

Let $P:=v_{1} v_{2} \ldots v_{t}$ be a longest induced path of $G$. We deduce that $d_{G}\left(v_{t}\right)=3$ because of Theorem 2. Let $N_{G}\left(v_{t}\right)=\left\{v_{t-1}, u_{1}, u_{2}\right\}$. By the choice of $P$ the vertices $u_{1}$ and $u_{2}$ each have at least one neighbour on $P$. For $i=1,2$ let $j(i)$ be the largest integer less than or equal to $t-1$ such that $u_{i} v_{j(i)} \in E(G)$. Hence, $\{j(1), j(2)\}=\{t-(2 p-1), t-(2 q-2)\}$ and say $u_{1} v_{t-(2 p-1)}, u_{2} v_{t-(2 q-2)} \in E(G)$. Furthermore, there exists a maximum $r \geq 1$ such that $r(2 q-2)<2 p-1$ and $u_{2} v_{t-i(2 q-2)} \in E(G)$ for $1 \leq i \leq r$. Now the cycle $v_{t} u_{1} v_{t-(2 p-1)} v_{t-2 p+2} \ldots v_{t-r(2 q-2)} u_{2} v_{t}$ is induced and has odd length. Hence, $(t-r(2 q-2))-(t-(2 p-1))+4=2 p+1$ and by rearranging $2=r(2 q-2)$ contradicting $q \geq 3$.

Case 2. Suppose there is a graph $G \in \mathcal{G}^{I}(2 p+1,2 q)$ with $q \geq 3$ and $p \geq 2$ (and $2 p+1<2 q!$ ), which satisfies $\delta(G) \geq 3$ and is non-bipartite.
Let again $P:=v_{1} v_{2} \ldots v_{t}$ be a longest induced path of $G$. In the remaining proof we will deduce several structural statements concerning this longest induced path $P$ of $G$. It is important to note that these statements also hold for every longest induced path $P^{\prime}$ of $G$. For convenience we call a vertex $v \in V(G)$ an $i$-type vertex, if there exists a longest induced path $P^{\prime}=v_{1}^{\prime} v_{2}^{\prime} \ldots v_{t}^{\prime}$ of $G$ with $v=v_{i}^{\prime}$. Again because of Theorem 2 we obtain $d_{G}\left(v_{t}\right)=3$ or more generally

Claim 1. Every $t$-type vertex has the degree 3 .
Now let $N_{G}\left(v_{t}\right)=\left\{v_{t-1}, u_{1}, u_{2}\right\}$. Again for $i=1,2$ let $j(i)$ be the largest integer less than or equal to $t-1$ such that $u_{i} v_{j(i)} \in E(G)$. Analogously, we have say $j(1)=t-(2 q-2), j(2)=t-(2 p-1)$ and $u_{1} v_{t-(2 q-2)}$, $u_{2} v_{t-(2 p-1)} \in E(G)$. Furthermore, there exists a maximum $r \geq 1$ such that $r(2 p-1)<2 q-2$ and $u_{2} v_{t-i(2 p-1)} \in E(G)$ for $1 \leq i \leq r$. Now the cycle $v_{t} u_{1} v_{t-(2 q-2)} v_{t-2 q+3} \ldots v_{t-r(2 p-1)} u_{2} v_{t}$ is induced and has odd length. Hence, $(t-r(2 p-1))-(t-(2 q-2))+4=2 p+1$ and by rearranging $2 q=(r+1)(2 p-1)$.

We now examine a special case. Suppose that $G$ contains no induced path $P_{2 q}$ containing $2 q$ vertices.

Markossian, Gasparian and Reed [9] showed that all triangle- free and even-hole-free graphs are 2-degenerate. Hence $G$ has to contain an induced even cycle $C=c_{0} c_{1} \ldots c_{2 q-2} c_{2 q-1} c_{0}$. Furthermore, since $\delta(G) \geq 3$ each vertex of $C$ is adjacent to at least one vertex not lying on the cycle $C$, e.g. $\left\{c_{0} d_{0}, c_{1} d_{1}\right\} \subset E(G)$. The $P_{2 q}$-freeness, $\left.G \in \mathcal{G}^{I}(2 p+1,2 q)\right)$ and $2 q=(r+1)$ $(2 p-1)$ forces that $N_{G}\left(d_{i}\right) \cap V(C)=\left\{c_{i}, c_{i+(2 p-1)}, c_{i+2(2 p-1)} \ldots, c_{i+r(2 p-1)}\right\}$ for $i=0,1$. But then the vertex set $\left\{c_{0}, d_{0}, c_{2 p-1}, c_{2 p}, d_{1}, c_{1}\right\}$ induces a 6 cycle enforcing that $q=3$ and $p=2$. Since $G$ is non-bipartite $G$ has to contain an induced 5 -cycle $C^{1}=a_{0} a_{1} a_{2} a_{3} a_{4} a_{0}$. Furthermore, since $\delta(G) \geq$ 3 each vertex of $C^{1}$ is adjacent to at least one vertex not lying on the cycle $C^{1}$, e.g. $\left\{a_{0} b_{0}, a_{1} b_{3}, a_{2} b_{1}, a_{3} b_{4}, a_{4} b_{2}\right\} \subset E(G)$. The $P_{6}$-freeness forces that $C^{2}=b_{0} b_{1} b_{2} b_{3} b_{4} b_{0}$ induces a 5 -cycle. Now the $C_{4}$-freeness of $G$ forces that all vertices of $C^{1}$ have degree 3. Analogously, every vertex of $C^{2}$ has degree 3. Thus, we have $N_{G}\left(V\left(C^{i}\right)\right)-V\left(C^{i}\right)=V\left(C^{3-i}\right)$ for $i=1,2$, i.e., $G \cong P^{*}$. Hence, in the following a longest induced path $P$ contains at least $2 q$ vertices.

We now examine the $(t-1)$-type vertex $v_{t-1}$. Since $\delta(G) \geq 3, P$ is chordless and $G$ is triangle-free, $v_{t-1}$ has a neighbour $w \in V(G)-(V(P) \cup$ $\left.N_{G}\left(v_{t}\right)\right)$.

Subcase 2.1. $w$ has no neighbour on $P$.
Then $P^{\prime}:=v_{1} v_{2} \ldots v_{t-1} w$ is also a longest induced path of $G$ and the $t$-type vertex $w$ has (exactly) two neighbours $w_{1}, w_{2} \in V(G)-(V(P) \cup$ $\left.N_{G}\left(v_{t}\right) \cup\{w\}\right)$. Because $w_{1}$ corresponds to $u_{2}$ and $w_{2}$ to $u_{1}$, we deduce that say $w_{1} v_{t-(2 q-2)} \in E(G)$ and $w_{2} v_{t-(2 p-1)} \in E(G)$. Now $C^{1}:=w_{1} v_{t-(2 q-2)}$ $u_{1} v_{t} v_{t-1} w w_{1}$ forms a cycle of length six. Therefore, the only possible case is again $q=3$ and $p=2$, i.e., $G \in \mathcal{G}^{I}(5,6)$. Now the $C_{3}$ - and $C_{4}$-freeness of $G$ forces that the set $\left\{w, w_{1}, v_{t-1}, v_{t}, u_{2}, v_{t-4}, v_{t-3}\right\}$ induces a $C_{7}$, a contradiction.

Note that the $C_{3}$ - and $C_{4}$-freeness of $G$ always forces that every 7 -cycle is an induced one.

Subcase 2.2. $w$ has a neighbour on $P$.
Then $w v_{t-(2 q-1)} \in E(G)$ or $w v_{t-2 p} \in E(G)$. Now $C^{1}:=w v_{t-(2 q-1)} v_{t-2 q-2}$ $u_{1} v_{t} v_{t-1} w$ or $C^{2}:=w v_{t-2 p} v_{t-(2 p-1)} u_{2} v_{t} v_{t-1} w$ induces a cycle of length six, i.e. $G \in \mathcal{G}^{I}(5,6)$. Therefore, we have either $w v_{t-5} \in E(G)$ or $w v_{t-4} \in E(G)$.

Subcase 2.2.1. $w v_{t-5} \in E(G)$.
Since $G$ is $C_{4}$-free, $u_{2}$ is not adjacent to $w$ and we deduce a contradiction by considering the set $\left\{w, v_{t-5}, v_{t-4}, v_{t-3}, u_{2}, v_{t}, v_{t-1}\right\}$, which induces a $C_{7}$.

Subcase 2.2.2. $w v_{t-4} \in E(G)$.
Claim 2. Every $(t-1)$-type vertex has the degree 3, e.g. $d_{G}\left(v_{t-1}\right)=3$.
Assume $d_{G}\left(v_{t-1}\right)>3$, then there exists a vertex $\bar{w} \in\left(N_{G}\left(v_{t-1}\right)-\right.$ $(\{w\} \cup V(P)))$. Likewise, we deduce that $\bar{w}$ is also adjacent to $v_{t-4}$. But then $\left\{v_{t-1}, w, \bar{w}, v_{t-4}\right\}$ induces a $C_{4}-$ a contradiction. Thus, we have $d_{G}\left(v_{t-1}\right)=3$.

Recall that $u_{1}$ is adjacent to $v_{t-4}$ and $u_{2}$ is adjacent to $v_{t-3}$.
Claim 3. Every $(t-2)$-type vertex has the degree 3 , e.g. $d_{G}\left(v_{t-2}\right)=3$.
Let $x$ be a neighbour of $v_{t-2}$ not lying on $P$. If $x$ is adjacent to $v_{t-5}$ or $v_{t-6}$, then $x v_{t-5} v_{t-4} u_{1} v_{t} v_{t-1} v_{t-2} x$ or $x v_{t-6} v_{t-5} v_{t-4} w v_{t-1} v_{t-2} x$ is a cycle of length 7 , a contradiction. Hence, $x$ has no other neighbours on $P$. The $C_{4}{ }^{-}$ freeness of $G$ forces that $x w \notin E(G)$. Furthermore, we have $x u_{2} \notin E(G)$, since otherwise $x u_{2} v_{t} u_{1} v_{t-4} v_{t-3} v_{t-2} x$ is a cycle of length 7 , a contradiction. Since $d_{G}(x) \geq \delta(G)=3, x$ has a neighbour $y \notin V(P) \cup\left\{w, u_{1}, u_{2}\right\}$. If $y$ is adjacent to $v_{t-4}$ or $v_{t-5}$, then $y v_{t-4} u_{1} v_{t} v_{t-1} v_{t-2} x y$ or $y v_{t-5} v_{t-4} w v_{t-1} v_{t-2} x y$ is a cycle of length 7 , a contradiction. Hence, $P^{\prime \prime \prime}=v_{1} \ldots v_{t-2} x y$ is a longest induced path implying with Claim 1 and Claim 2 that $d_{G}(x)=d_{G}(y)=3$. Observe that $x$ is a $(t-1)$-type and $y$ a $t$-type vertex.

Claim 4. $x$ is adjacent to $u_{1}$.
Suppose to the contrary that $x u_{1} \notin E(G)$. Analogously to $P$, we deduce for $P^{\prime \prime \prime}$ that the $(t-1)$-type vertex $x$ is adjacent to a vertex $y^{*} \notin$ $V(P) \cup\left\{y, w, u_{1}, u_{2}\right\}$. Because $y^{*}$ corresponds to $w$ we also deduce that $y^{*}$ is adjacent to $v_{t-4}$. But then $y^{*} v_{t-4} u_{1} v_{t} v_{t-1} v_{t-2} x y^{*}$ is a cycle of length 7 , a contradiction. Thus we have $x u_{1} \in E(G)$.

Now suppose that $d_{G}\left(v_{t-2}\right)>3$, i.e. there exists a vertex $x^{*} \in N_{G}\left(v_{t-2}\right)$ $-\left\{v_{t-3}, v_{t-1}, x\right\}$. Analogously to Claim 4 we obtain $x^{*} u_{1} \in E(G)$, a contraction since $G$ is $C_{4}$-free. Therefore, we have $d_{G}\left(v_{t-2}\right)=3$.

For convenience we introduce the notation $z$-type vertex with $z \in$ $\left\{w, u_{1}, u_{2}, x, y\right\}$. A vertex $v$ of $G$ is a $z$-type vertex, if there exists a longest induced path $P^{\prime \prime \prime}$, such that the role of $v$ in $P^{\prime \prime \prime}$ corresponds to the role of $z$ for the longest induced path $P=v_{1} v_{2} \ldots v_{t}$.

Claim 5. $w$ is adjacent to $y$.
Because the $(t-1)$-type vertex $x$ of $P^{\prime \prime \prime}=v_{1} v_{2} \ldots v_{t-2} x y$ is adjacent to $u_{1}$, we obtain that $u_{1}$ is a $w$-type vertex. Similarly, we deduce that $v_{t-1}$ is a $x$-type vertex. Recall (Claim 4) that $u_{1}$ is adjacent $x$ and the $(t-4)$-type vertex $v_{t-4}$. Since $w$ is adjacent to the $x$-type vertex $v_{t-1}$ of $P^{\prime \prime \prime}$ and the $(t-4)$-type vertex $v_{t-4}$ of $P^{\prime \prime \prime}$ the $C_{4}$-freeness of $G$ forces that $w$ is the $u_{1}$-type vertex of $P^{\prime \prime \prime}$. Because $y$ is the $t$-type vertex of $P^{\prime \prime \prime}$ this implies Claim 5 .

Claim 6. $y$ is adjacent to $u_{2}$ and therefore $N_{G}(y)=\left\{x, w, u_{2}\right\}$.
Assume to the contrary that $y$ is not adjacent to $u_{2}$. Hence there exists $x^{\prime}$ with $\left\{x^{\prime}\right\}=N_{G}(y)-\{x, w\}$. Note that $x^{\prime} \notin V(P) \cup\left\{u_{1}, u_{2}, x, w\right\}$ and since $P^{\prime \prime \prime}=v_{1} \ldots v_{t-2} x y$ is a longest induced path and $w$ is adjacent to $v_{t-4}$, we obtain that $x^{\prime}$ is adjacent to $v_{t-3}$. But then $x^{\prime} v_{t-3} u_{2} v_{t} v_{t-1} w y x^{\prime}$ is a cycle of length 7 , a contradiction. Thus we have $N_{G}(y)=\left\{x, w, u_{2}\right\}$. Since $u_{2}$ is a $t$-type vertex of the longest induced path $P^{\prime \prime}=v_{1} \ldots v_{t-4} u_{1} x y u_{2}$ we obtain the following Claim 7 by Claim 1 .

Claim 7. $d_{G}\left(u_{2}\right)=3$.
Summarizing all claims we deduce $G^{\prime}:=G\left[\left\{u_{2}, v_{t}, v_{t-1}, v_{t-2}, v_{t-3}, v_{t-4}\right.\right.$, $\left.\left.u_{1}, w, x, y\right\}\right] \cong P^{*}$ and all six non-neighbours of $v_{t-4}$ have degree three. If $G \cong P^{*}$ we are done.

Assume $G \not \not P^{*}$. Note that $C S:=\left\{v_{t-4}, v_{t-3}, u_{1}, w\right\}$ is a cutset of $G$. If $v_{t-4}$ and one of its three neighbours $v_{t-3}, u_{1}$ or $w$ is already a (complete) cutset, then we are done. Hence w.l.o.g. suppose $G_{2}$ is a component of $G-C S$ different to the six cycle $G_{1}:=G\left[\left\{u_{2}, v_{t}, v_{t-1}, v_{t-2}, x, y\right\}\right]$ and say $G^{\prime \prime}:=\left\{u_{1}, w\right\} \cup V\left(G_{2}\right)$ is connected. The $C_{4}$-freeness forces that a shortest path connecting $u_{1}$ and $w$ in $G^{\prime \prime}$ contains at least 2 internal vertices and in $G^{\prime \prime \prime}:=\left\{u_{1}, w\right\} \cup V\left(G_{1}\right)$ there exists a $u_{1}$ and $w$ connecting induced path with 3 internal vertices. Then one can construct easily an induced cycle of length at least 7 , a contradiction. This settles the proof of this theorem.

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