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COLOURING GRAPHS WITH PRESCRIBED INDUCED CYCLE LENGTHS

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Abstract

In this paper we study the chromatic number of graphs with two prescribed induced cycle lengths. It is due to Sumner that triangle-free and P_5-free or triangle-free, P_6-free and C_6-free graphs are 3-colourable. A canonical extension of these graph classes is $\mathcal{G}^I(4,5)$, the class of all graphs whose induced cycle lengths are 4 or 5. Our main result states that all graphs of $\mathcal{G}^I(4,5)$ are 3-colourable. Moreover, we present polynomial time algorithms to 3-colour all triangle-free graphs G of this kind, i.e., we have polynomial time algorithms to 3-colour every $G \in \mathcal{G}^I(n_1,n_2)$ with $n_1, n_2 \geq 4$ (see Table 1). Furthermore, we consider the related problem of finding a χ-binding function for the class $\mathcal{G}^I(n_1,n_2)$. Here we obtain the surprising result that there exists no linear χ-binding function for $\mathcal{G}^I(3,4)$.

Keywords: colouring, chromatic number, induced subgraphs, graph algorithms, χ-binding function.

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1. Introduction and Results

We consider finite undirected simple graphs. For terminology and notation not defined here we refer to [1]. As introduced by Gyárfás [6], a family $\mathcal{G}$ of graphs is called *χ-bound* with *χ-binding function f*, if $\chi(G') \leq f(\omega(G'))$ holds whenever G' is an induced subgraph of $G \in \mathcal{G}$.

Erdös [4] showed that for each pair g, k with $g, k \geq 4$ there exist graphs with girth g and chromatic number k. Hence, triangle-free graphs can have arbitrary large chromatic number. Sumner [13] showed that triangle-free and P_5-free or triangle-free, P_6-free and C_6-free graphs are 3-colourable.

For $t \geq 5$ define $\mathcal{G}_t$ as the class of all triangle-free graphs which are P_t-free and C_i-free for $6 \leq i \leq t$. For $k \geq 1$ and $3 \leq n_1 < n_2 < \cdots < n_k$ let $\mathcal{G}^I(n_1, n_2, \ldots, n_k)$ be the class of all graphs whose induced cycle lengths are equal to one of $n_1, n_2, \ldots, n_k$. Thus

$$\mathcal{G}_5 \subset \mathcal{G}_6 \subset \mathcal{G}_7 \subset \cdots \subset \mathcal{G}^I(4,5)$$

and all graphs G of $\mathcal{G}_5$ and $\mathcal{G}_6$ are 3-colourable by Sumners result. Note that all graphs of $\mathcal{G}_t$ have diameter at most $t-2$ whereas graphs of $\mathcal{G}^I(4,5)$ can have arbitrary diameter.

Our research was motivated by the question whether 3-colourability still holds for a superclass $\mathcal{G}_t$ (of $\mathcal{G}_5$ and $\mathcal{G}_6$) for some $t \geq 7$. Theorem 1 states that all graphs of $\mathcal{G}^I(4,5)$ are 3-colourable. Hence, the answer is affirmative for each $t \geq 7$. Moreover, we can guarantee a 3-colouring with some additional properties. For a fixed integer $p \geq 2$ we call a graph $G \in \mathcal{G}^I(4, 2p+1)$ *3^*-colourable with root v*, if there is a 3-colouring of G such that $G[N_G^p(v)]$ is coloured with two colours, where $N_G^p(v)$ is the set of vertices having distance p from v. Observe that this definition implies the following useful property: If G is 3^*-colourable with root v, then we can choose a 3-colouring such that $G[N_G^i(v)]$ is coloured monochromatic for every $1 \leq i < p$ and $G[N_G^p(v)]$ is coloured with two colours. Furthermore, if this property holds for every vertex of $G \in \mathcal{G}^I(4, 2p+1)$, then we call G *3^*-colourable.* This definition is motivated by the following observation.

If $G_1, G_2 \in \mathcal{G}^I(4, 2p+1)$ and $v_i \in G_i$ for $i = 1, 2$, then the new graph G^* with vertex set $V(G^*) = V(G_1 - v_1) \cup V(G_2 - v_2)$ and edge set $E(G^*) = E(G_1 - v_1) \cup E(G_2 - v_2) \cup \{u_1 u_2 | u_i \in N_{G_i}(v_i)$ for $i = 1, 2\}$ is likewise a member of $\mathcal{G}^I(4, 2p+1)$. The invariance of $\mathcal{G}^I(4, 2p+1)$ concerning this graph operation reasons the equivalence of 3^*- and 3-colourability for the class $\mathcal{G}^I(4, 2p+1)$.

Theorem 1. *Every graph of* $\mathcal{G}^I(4, 2p+1)$ *with* $p \geq 2$ *is* 3^**-colourable.*

The proof of Theorem 1 is based on decomposition and provides a polynomial time algorithm to 3^*-colour a given graph $G \in \mathcal{G}^I(4, 2p+1)$. Note that the class $\mathcal{G}^I(4, 5)$ is a canonical extension of $\mathcal{G}^I(4)$, which are the well-known chordal bipartite graphs (e.g. see [2]). Very recently Brandt [3] examined the maximal (with respect to edge addition) triangle-free members of the class $\mathcal{G}^I(4, 5)$ with emphasis on graph homomorphisms. Brandt also observed that the class $\mathcal{G}^I(4, 5)$ is a natural extension of $\mathcal{G}^I(4)$ — the chordal bipartite graphs — and he introduced for members of $\mathcal{G}^I(4, 5)$ the terminology of *chordal triangle-free graphs.*

Motivated by the first theorem we consider next the classes $\mathcal{G}^I(2q, 2p+1)$ and $\mathcal{G}^I(2p'+1, 2q')$ for $q, q' \geq 3$ and $p, p' \geq 2$. But first we will examine the larger class $\mathcal{G}^I(n_1, n_2, \ldots, n_k)$ with $n_1 \geq 5$. A graph G is r*-degenerate*, if there exists an ordering $(v_1, \ldots, v_n)$ of $V(G)$ such that $d_{G[\{v_i,\ldots,v_n\}]}(v_i) \leq r$ for all $1 \leq i \leq n$.

Theorem 2. *Every graph of* $\mathcal{G}^I(n_1, n_2, \ldots, n_k)$ *with* $k \geq 1$ *and* $n_1 \geq 5$ *is* $(k+1)$*-degenerate. Especially, every vertex* v *of* G *being an endvertex of a longest induced path of* G *satisfies* $d_G(v) \leq k+1$.

Corollary 3. *Every graph of* $\mathcal{G}^I(n_1, n_2, \ldots, n_k)$ *with* $k \geq 1$ *and* $n_1 \geq 5$ *is* $(k+2)$*-colourable.*

The last result reveals an interesting relation to the colouring properties of graphs of the class $\mathcal{G}(n_1, n_2, \ldots, n_k)$, the class of all graphs whose (not necessarily induced) cycle lengths are equal to one of $n_1, n_2, \ldots, n_k$. Now let G be a graph with r different odd and s different even cycle lengths (which need not to be induced). In [10] Mihók and Schiermeyer presented a polynomial time colouring algorithm, called MAXBIP, which recursively constructs maximal bipartite subgraphs. Based on MAXBIP they proved the following theorem, answering thereby a question of B. Bollobás and P. Erdős [5].

Theorem 4 (Mihók and Schiermeyer [10], 1997)**.** $\chi(G) \leq \min\{2r+2, 2s+3\}$.

With $k = r+s$ this also implies $\chi(G) \leq k+2$. The question of B. Bollobás and P. Erdős [5] only concerned $(2r+2)$-colourability of graphs with r different odd cycle lengths (which need not to be induced). This question was first answered affirmative by A. Gyárfás [7]. Additional informations and a related conjecture can be found in the excellent book [8] of T. Jensen and B. Toft on graph colouring problems.

Obviously, Corollary 3 is best possible for $k = 1$. But for $k = 2$ we are able to improve Corollary 3. For the next theorem we need to recall the definition of the famous Petersen graph P^*. This 3-regular, non-bipartite graph P^* of order 10 is a member of the class $\mathcal{G}^I(5,6)$. The Petersen graph P^* consists of two disjoint induced 5-cycles $C^1 = a_0a_1a_2a_3a_4a_0$ and $C^2 = b_0b_1b_2b_3b_4b_0$ and the additional edges $a_0b_0, a_1b_3, a_2b_1, a_3b_4$ and a_4b_2. Obviously P^* is 3-colourable.

Theorem 5. *Every graph G of $\mathcal{G}^I(2q, 2p+1)$ or $\mathcal{G}^I(2p'+1, 2q')$ with $q, q' \geq 3$ and $p, p' \geq 2$ fulfills at least one of the following properties:*

1. *G is bipartite;*
2. *G satisfies $\delta(G) \leq 2$;*
3. *$G \in \mathcal{G}^I(5,6)$ and one of the following properties holds:*
   (a) *$G \cong P^*$;*
   (b) *G contains a clique cutset, i.e., K_1 or K_2 clique cutset.*

Every graph G of $\mathcal{G}^I(2q, 2p+1)$ or $\mathcal{G}^I(2p'+1, 2q')$ with $q, q' \geq 3$ and $p, p' \geq 2$ not isomorphic to P^* fulfills at least one of the three properties because of Theorem 5. Testing whether G is bipartite, has minimal degree two or contains a complete cutset of size at most two can be done very efficiently. Moreover, if $G \in \mathcal{G}^I(5,6)$ is non-bipartite, $\delta(G) \geq 3$ and contains no complete cutset, then $G \cong P^*$, which obviously is 3-colourable. Hence, Theorem 5 provides a polynomial time algorithm to 3-colour a given graph $G \in \mathcal{G}^I(2q, 2p+1)$ or $G \in \mathcal{G}^I(2p'+1, 2q')$ with $q, q' \geq 3$ and $p, p' \geq 2$. This algorithm (recursively) makes use of the fact that the graph (in question) is bipartite, has a vertex of degree at most two, is isomorphic to the Petersen graph or the graph can be decomposed into two smaller graphs according to a complete cutset of size at most two.

Corollary 6. *Every graph G of $\mathcal{G}^I(2q, 2p+1)$ or $\mathcal{G}^I(2p'+1, 2q')$ with $q, q' \geq 3$ and $p, p' \geq 2$ is 3-colourable.*

Now we consider the related problem of finding a (best possible) χ-binding function f^* for $\mathcal{G}^I(n_1, n_2)$ and for completeness also for its subclasses $\mathcal{G}^I(n_1)$.

Recall that a graph is *perfect* if for each induced subgraph H of G the chromatic number $\chi(H)$ equals the corresponding clique number $\omega(H)$. Furthermore the *lexicographic product* $G_1[G_2]$ of two graphs G_1 and G_2 contains the vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and two vertices (a, b) and (c, d) are adjacent in $G_1[G_2]$ if a is adjacent to c in G_1 or $a = c$ and b is adjacent to d in G_2.

For convenience we drop the condition that n_1 is always smaller than n_2 in the definition of $\mathcal{G}^I(n_1, n_2)$.

(I) n_1, n_2 *are even* :
For even n_1 and n_2 all graphs of $\mathcal{G}^I(n_1)$ and $\mathcal{G}^I(n_1, n_2)$ are bipartite and thus perfect with $f^*(\omega) = \omega \leq 2$.

(II) n_1 *is even*, n_2 *is odd* : (A) $n_2 \geq 5$:
By our results (Theorem 1, Corollary 6) every graph of $\mathcal{G}^I(n_2)$ and $\mathcal{G}^I(n_1, n_2)$ is 3-colourable, i.e., with $\omega \leq 2$ we have $f^*(\omega) = \omega + 1 \leq 3$.

(II) n_1 *is even*, n_2 *is odd* : $(B1)$ $n_2 = 3$ and $n_1 \geq 6$:
Recently, Rusu [11] proved that all members of a superclass of $\mathcal{G}^I(3, 2q)$ are perfect for any $q \geq 3$. Hence, we also have $f(\omega) = \omega$. A well-known subclass of $\mathcal{G}^I(3, 2q)$ is $\mathcal{G}^I(3)$ containing the chordal graphs.

(II) n_1 *is even*, n_2 *is odd* : $(B2)$ $n_2 = 3$ and $n_1 = 4$:
In 1987 Gyárfás [6] conjectured (motivated by the Strong Perfect Graph Conjecture) that there exists a χ-binding function for $\mathcal{G}^I(3, 4)$. But this Conjecture is still open. In [6] Gyárfás also suggested to examine whether there exists a linear χ-binding function for hereditary classes of graphs. For $\mathcal{G}^I(3, 4)$ we have constructed the following sequence of graphs (H_i). Starting with $H_1 = \bar{C}_7$, the complement of the 7-cycle, we define $H_{i+1} = \bar{C}_7[H_i]$, the lexicographic product of the graphs $\bar{C}_7$ and H_i. Note that $\omega(H_{i+1}) = 3\omega(H_i)$. Furthermore, in any colouring of H_{i+1} we need for each copy of H_i at least $\chi(H_i)$ different colours. With $\alpha(\bar{C}_7) = 2$ we then observe that every colour of a colouring of H_{i+1} appears in at most two different copies of H_i. Hence, H_i has the order $n(H_i) = 7^i$, the independence number $\alpha(H_i) = 2^i$ and the clique number $\omega(H_i) = 3^i$. Therefore, its chromatic number $\chi(H_i)$ is at least $(7/2)^i$. Thus, the χ-binding function f^* for $\mathcal{G}^I(3, 4)$ satisfies $f^*(\omega) \geq (7/6)^i\omega$ for every integer i. Hence, we obtain the following surprising result:

Theorem 7. *There exists no linear χ-binding function for $\mathcal{G}^I(3, 4)$.*

It is noteworthy to mention that $\mathcal{G}^I(3,4)$ contains all weakly triangulated graphs. Recently, Scott [12] achieved some related results.

(III) n_1, n_2 *are odd* : $(A1)$ $n_1, n_2 \geq 5$:
Markossian, Gasparian and Reed [9] showed that all triangle-free and even-hole-free graphs are 2-degenerate and thus are 3-colourable. Hence, $f^*(\omega) = \omega + 1 \leq 3$ is a χ-binding function for $\mathcal{G}^I(n_1)$ and $\mathcal{G}^I(n_1, n_2)$.

(III) n_1, n_2 *are odd* : $(A2)$ $n_1 = 3$:
It is an open problem, whether there exists a linear χ-binding function for $\mathcal{G}^I(3, n_2)$. The graph-sequence $G_r = C_{n_2}[K_r]$, the lexicographic product of the odd cycle C_{n_2} and the complete graph K_r, reveals that we have $f^*(\omega) \geq ((n_2+1)/(n_2-1))\omega$ for every χ-binding function. We expect that

$$f^*(\omega) = ((n_2+1)/(n_2-1))\omega.$$

Table 1. χ-binding function f^* for $\mathcal{G}^I(n_1, n_2)$.

$\rightarrow$ n_1, n_2 $\downarrow$	3	4	odd ≥ 5	even ≥ 6
3	$f^*(\omega) = \omega$ chordal	$\nexists$ linear f^* Theorem 7 Conj.[6]: $\exists f^*$	$f^*(\omega) \geq ((n_2+1)/(n_2-1))\omega$ Conj.: $'' = ''$	$f^*(\omega) = \omega$ Rusu [11]
4		$f^*(\omega) = \omega \leq 2$ chordal bipartite	$f^*(\omega) = \omega + 1 \leq 3$ Theorem 1	$f^*(\omega) = \omega \leq 2$ $\subset$ bipartite
odd ≥ 5			$f^*(\omega) = \omega + 1 \leq 3$ Markossian,... [9]	$f^*(\omega) = \omega + 1 \leq 3$ Corollary 6
even ≥ 6				$f^*(\omega) = \omega \leq 2$ $\subset$ bipartite

2. Proofs

The following well-known easy observation provides a very useful property. If a graph G contains a pair u, v of nonadjacent vertices with $N_G(u) \subseteq N_G(v)$, then any proper k-colouring of $G - u$ can easily be extended to a proper k-colouring of G.

Therefore we only have to consider those graphs G having the property $(*)$:

$(*)$ If $uv \notin E(G)$, then there exist vertices (*private neighbours*) $p_u \in (N_G(u) - N_G(v))$ and $p_v \in (N_G(v) - N_G(u))$.

The next lemma provides a useful property of triangle-free and C_6-free graphs, which will be used in the proof of Theorem 1. This class of graphs forms a superclass of all classes $\mathcal{G}^I(4, 2p+1)$ with $p \geq 2$.

Lemma 8. *Let G be a triangle-free and C_6-free graph satisfying property $(*)$. Then for every vertex x of degree $d_G(x) = k \geq 3$ with neighbours $x_1, x_2, \ldots, x_k$ there exists a pair x_i, x_j of neighbours such that $N_G(x_i) \cap N_G(x_j) \cap N_G^2(x) = \emptyset$.*

Proof. Let G be a triangle-free and C_6-free graph satisfying property $(*)$. Further suppose to the contrary that there exists a vertex x of degree $d_G(x) = k \geq 3$ with neighbours $x_1, x_2, \ldots, x_k$ such that $N_G(x_i) \cap N_G(x_j) \cap N_G^2(x) \neq \emptyset$ for all pairs i, j with $1 \leq i < j \leq k$. Choose a vertex $v \in N_G^2(x)$ such that $|N_G(x) \cap N_G(v)| = p$ is maximum. By $(*)$ we know that $p \leq k-1$. We may assume that $N_G(x) \cap N_G(v) = \{x_1, \ldots, x_p\}$. By the assumption there exists a vertex $w \in N_G^2(x)$ with w likewise adjacent to a vertex $x_i \in \{x_1, \ldots, x_p\}$ and to a vertex $x_j \in \{x_{p+1}, \ldots, x_k\}$. Hence there is a vertex $w \in N_G^2(x) - \{v\}$ with $N_G(v) \cap N_G(w) \cap N_G(x) \neq \emptyset$ and $N_G(w) \cap \{x_{p+1}, \ldots, x_k\} \neq \emptyset$. Choose a vertex w with $N_G(w) \cap \{x_{p+1}, \ldots, x_k\} \neq \emptyset$ such that $|N_G(v) \cap N_G(w) \cap N_G(x)| = q$ is maximum. Hence $1 \leq q \leq p-1$ by the choice of v and w. We may assume that $N_G(v) \cap N_G(w) \cap N_G(x) = \{x_1, \ldots, x_q\}$ and that $x_{p+1} \in N_G(w)$. By the hypothesis there exists a vertex $u \in N_G^2(x) - \{v, w\}$ such that $x_p, x_{p+1} \in N_G(u)$. By the choice of p and q there is a vertex $x_s \in \{x_1, \ldots, x_q\}$ such that $x_s \notin N_G(u)$. Now $G' := G[\{x_s, x_p, x_{p+1}, v, w, u\}] \cong C_6$ or G' contains a triangle, a contradiction. ■

***Proof of Theorem* 1.** We will prove the theorem by induction. We may assume that G is a connected graph and fulfills property $(*)$. By the induction hypothesis there is a 3^*-colouring with root y for every vertex y of any induced subgraph H of G with $|V(H)| < |V(G)|$. Now let x be an arbitrary vertex of G . If $dist_G(x, y) < p$ for all vertices $y \in V(G - x)$, then we can easily 3^*-colour G with root x. Hence we may assume that $dist_G(x, y) \geq p$ for at least one vertex $y \in V(G - x)$.

Case 1. Assume that $d_G(x) = 1$ and let $xz \in E(G)$.

Then by induction hypothesis there is a 3^*-colouring with root z of $G - x$, which can be easily extended to a 3^*-colouring with root x of G.

Case 2. Assume that $d_G(x) \geq 2$.
Also, if x is a cutvertex of G, then we can easily 3^*-colour G with root x. Hence we may assume that *x is not a cutvertex of G*. Note that x and every neighbour of x has a degree of at least two.

We now consider $N_G^i(x)$ for $1 \leq i < p$. If there is a vertex $y \in N_G^i(x)$ such that

$$N_G(y) \cap N_G^{i+1}(x) \subseteq \left(\bigcup_{z \in N_G^i(x) - \{y\}} N_G(z) \right) \cap N_G^{i+1}(x),$$

then the levels $N_G^h(x)$ with $1 \leq h < p$ and $h \neq i$ are the same in G and $G-y$. Thus, we can reduce G to $G-y$ and a 3^*-colouring with root x of $G-y$ can be extended to a 3^*-colouring with root x of G since $G[N_G^{i-1}(x)]$ is independent and monochromatic, $G[N_G^i(x)-\{y\}]$ is independent and monochromatic and $G[N_G^{i+1}(x)]$ is 2-coloured. Consequently, we can assume for the remaining part of the proof that

Claim 1. Every vertex $y \in N_G^i(x)$ has a 'private neighbour' in $N_G^{i+1}(x)$ for $1 \leq i < p$.

Subcase 2.1. Assume that $d_G(x) \geq 3$.
Let u^1, v^1 and w^1 be three vertices of $N_G^1(x)$. Then by Claim 1 there are three 'paths of private neighbours' $u^1u^2 \dots u^p$, $v^1v^2 \dots v^p$ and $w^1w^2 \dots w^p$. Set $u = u^{p-1}$, $v = v^{p-1}$ and $w = w^{p-1}$. Let U^p, V^p, W^p be the set of private neighbours of u, v and w, respectively.

If $p \geq 3$, then $N_G^p(x) \cap N_G(y) \cap N_G(z) = \emptyset$ for all pairs $y, z \in \{u, v, w\}$. Since otherwise there would be an induced C_{2p} – a contradiction. If $p = 2$, then we can choose u and v by Lemma 8 such that $N_G^2(x) \cap N_G(u) \cap N_G(v) = \emptyset$.

Now observe that $G[U^p, V^p]$, $G[U^p, W^p]$ and $G[W^p, V^p]$ are $2K_2$-free (and bipartite), since otherwise there would be an induced C_6.

A useful property of every bipartite and $2K_2$-free graph H is the existence of a labelling of the vertices of each partition set $X = \{x_1, \dots, x_k\}$, such that $N_H(x_i) \subset N_H(x_j)$ if $i \leq j$.

Then we easily deduce that there are sets $U_{(c)}^p$, $U_{(e)}^p$, $V_{(c)}^p$ and $V_{(e)}^p$ such that $U^p = U_{(c)}^p \cup U_{(e)}^p$, $V^p = V_{(c)}^p \cup V_{(e)}^p$, $G[U_{(e)}^p \cup V_{(e)}^p]$ is edgeless and

$G[U^p_{(c)} \cup V^p_{(c)}]$ is complete bipartite. Note that the partition of U^p and V^p into $U^p_{(c)}$, $U^p_{(e)}$, $V^p_{(c)}$ and $V^p_{(e)}$ is not necessarily unique. But, if $G[U^p, V^p]$ is not a complete bipartite graph, it is always possible to choose a partition of U^p and V^p into $U^p_{(c)}$, $U^p_{(e)}$, $V^p_{(c)}$ and $V^p_{(e)}$ such that $U^p_{(e)} \neq \emptyset$ and $U^p_{(e)} \neq \emptyset$!

Subcase 2.1.1. $G[U^p, V^p]$ is not a complete bipartite subgraph.
Then there are two non-adjacent vertices $y \in U^p$ and $z \in V^p$. Now we can choose a partition of U^p and V^p into $U^p_{(c)}$, $U^p_{(e)}$, $V^p_{(c)}$ and $V^p_{(e)}$ such that $y \in U^p_{(e)}$ and $z \in V^p_{(e)}$. Set $S = \{v \in V(G)|dist_G(x, v) \leq p-1\} \cup U^p_{(c)} \cup V^p_{(c)}$.

Subcase 2.1.1.1. For every vertex $y \in U^p_{(e)}$ and $z \in V^p_{(e)}$ there exists a path connecting y and z in $G - S$.
Now we choose $y \in U^p_{(e)}$ and $z \in V^p_{(e)}$ such that $dist_{G-S}(y, z) = \min\{dist_{G-S}(y', z')\ |y' \in U^p_{(e)}$ and $z' \in V^p_{(e)}\}$. Then obviously there would be an induced cycle of length at least $(2p+1)+1 > 2p+1 > 4$, a contradiction.

Subcase 2.1.1.2. There exist vertices $y \in U^p_{(e)}$ and $z \in V^p_{(e)}$ such that there is no path connecting y and z in $G - S$.
Now let H_1 be a component of $G - S$ and H_2 be the remaining part of $G - S$. Set $G_i = G[V(H_i) \cup S]$ for $i = 1, 2$. Then there is a 3^*-colouring with root x for each G_i with $i = 1, 2$. We can choose these 3^*-colourings in such a way that the vertices of S always receive the same colours. Hence we obtain a 3^*-colouring with root x of G. In the following, if we will apply this subprocedure, we shortly refer that we *apply decomposition*.

Subcase 2.1.2. $G[U^p, V^p]$ is a complete bipartite subgraph.
If $p \geq 3$, then $G[U^p, W^p]$ and $G[W^p, V^p]$ are complete as well contradicting that G is triangle-free.

Let $p = 2$. If we also have $N^2_G(x) \cap N_G(u) \cap N_G(w) = \emptyset$ or $N^2_G(x) \cap N_G(v) \cap N_G(w) = \emptyset$, then we either apply decomposition or we deduce that $G[U^p, W^p]$ or $G[W^p, V^p]$ is complete bipartite. But then there exists a path ayz or azy with $a \in W^p, y \in U^p$ and $z \in V^p$ implying the existence of an induced cycle C_6 or ayz induces a triangle – a contradiction. Hence $N^2_G(x) \cap N_G(u) \cap N_G(w) \neq \emptyset$ and $N^2_G(x) \cap N_G(v) \cap N_G(w) \neq \emptyset$. Let $a \in N^2_G(x) \cap N_G(u) \cap N_G(w)$ and $b \in N^2_G(x) \cap N_G(v) \cap N_G(w)$. We now consider the subgraph $G[U'^2, V'^2]$ with $U'^2 = N^2_G(x) \cap (N_G(u) - N_G(v))$ and $V'^2 = N^2_G(x) \cap (N_G(v) - N_G(u))$. Recall that $N^2_G(x) \cap (N_G(u) \cup N_G(v)) = U'^2 \cup V'^2$ and $G[U^2, V^2]$ is a subgraph of $G[U'^2, V'^2]$. Note that we obtain analogously that $G[U'^2, V'^2]$ is $2K_2$-free (and bipartite). With $a \in U'^2$, $b \in V'^2$, $\{a, b\} \subset N_G(w)$ and G being triangle-free we deduce that $G[U'^2, V'^2]$ is not complete.

But now we analogously can apply decomposition. This settles the case that $d_G(x) \geq 3$.

Subcase 2.2. Suppose that $d_G(x) = 2$.
By the induction hypothesis there is a 3^*-colouring with root y for every vertex y with $d_G(y) \neq 2$ and for every y of any induced subgraph H of G with $|V(H)| < |V(G)|$. Note that $N_G(u^1) \cap N_G(v^1) \cap N_G^2(x) = \emptyset$ by $(*)$.

Let $p = 2$. If $d_G(u^1) = d_G(v^1) = 2$, then because the vertices u^2, u^1, x, v^1 and v^2 are lying on a cycle (! x is no cutvertex of G !) we obtain adjacency of v^2 and u^2. Now it is not very difficult to extend an arbitrary 3-colouring of $G - \{u^1, x, v^1\}$ to a 3^*-colouring with root x of G.

Let $d_G(u^1) \geq 3$. By the induction hypothesis there is a 3^*-colouring with root u^1 of G. Since $\{v^1\} \cup (N_G(v^1) \cap N_G^2(x))$ induces a star we may choose the colours in such a way that (e.g.) u^1 and v^1 receive colour 1, x and all vertices of $N_G(u^1) \cap N_G^2(x)$ get colour 2 and finally all vertices of $N_G(v^1) \cap N_G^2(x)$ receive colour 3. This gives a 3^*-colouring with root x of G.

Let $p \geq 3$. Since $N_G^i(x)$ is independent for $1 \leq i < p$ and G being C_{2q}-free for $q \geq 3$, we obtain that $N_G^i(u^1) \cap N_G^i(v^1) \cap N_G^{i+1}(x) = \emptyset$ for $2 \leq i < p$. Hence the set $N_G^p(x)$ is the disjoint union of the sets $N_G^{p-1}(u^1)$ and $N_G^{p-1}(v^1)$. Now the C_{2q+1}-freeness of G $(1 \leq q < p)$ forces that $N_G^{p-1}(u^1)$ as well as $N_G^{p-1}(v^1)$ form independent sets. Moreover, it is not difficult to see that $N_G^p(x)$ induces a $2K_2$-free bipartite graph. Likewise to the previous case $d_G(x) \geq 3$ we can apply decomposition or $N_G^p(x)$ induces a complete bipartite graph. In the latter case we can proceed analogously to the $p = 2$ subcase.

If all vertices of $N' := \{v \in V(G) | dist_G(x, v) \leq p - 1\}$ have degree two, then again $N_G^p(x)$ induces an edge. Now it is not very difficult to extend an arbitrary 3-colouring of $G - N'$ to a 3^*-colouring with root x of G.

Now let u^i (with $1 \leq i < p$) be a vertex of degree at least three having a minimum distance concerning x. By the induction hypothesis there is a 3^*-colouring with root u^i of G. Note that $N_G^p(x)$ is bicoloured, since $G[U^p, V^p]$ is complete bipartite. Now again it is not very difficult to recolour this 3-colouring in such a way that we obtain a 3^*-colouring with root x of G. ■

***Proof of Theorem* 2.** We will show that every vertex v of G being an endvertex of a longest induced path of G satisfies $d_G(v) \leq k + 1$. Observe that this implies $\delta(G) \leq k + 1$. Let $P := v_1 v_2 \ldots v_t$ be a longest induced path of G. Suppose $d_G(v_t) = s \geq k + 2$. Let $N_G(v_t) = \{v_{t-1}, u_1, \ldots, u_{s-1}\}$. By the choice of P every vertex u_i has a neighbour on P. For each i (with

$1 \leq i \leq s-1$) let $j(i)$ be the largest integer less than or equal to $t-1$ such that $u_i v_{j(i)} \in E(G)$. Since G is triangle-free and contains no induced C_4, no two values of $j(1), \ldots, j(s-1)$ are equal. Hence there are $s-1 \geq k+1$ induced cycles of different lengths $t-j(i)+2$ for $1 \leq i \leq s-1$, a contradiction. ■

***Proof of Theorem* 5.**

Case 1. Suppose there is a graph $G \in \mathcal{G}^I(2q, 2p+1)$ with $q \geq 3$ and $p \geq 2$ (and $2p+1 > 2q$), which satisfies $\delta(G) \geq 3$ and is non-bipartite.

Let $P := v_1 v_2 \ldots v_t$ be a longest induced path of G. We deduce that $d_G(v_t) = 3$ because of Theorem 2. Let $N_G(v_t) = \{v_{t-1}, u_1, u_2\}$. By the choice of P the vertices u_1 and u_2 each have at least one neighbour on P. For $i = 1, 2$ let $j(i)$ be the largest integer less than or equal to $t-1$ such that $u_i v_{j(i)} \in E(G)$. Hence, $\{j(1), j(2)\} = \{t-(2p-1), t-(2q-2)\}$ and say $u_1 v_{t-(2p-1)}, u_2 v_{t-(2q-2)} \in E(G)$. Furthermore, there exists a maximum $r \geq 1$ such that $r(2q-2) < 2p-1$ and $u_2 v_{t-i(2q-2)} \in E(G)$ for $1 \leq i \leq r$. Now the cycle $v_t u_1 v_{t-(2p-1)} v_{t-2p+2} \ldots v_{t-r(2q-2)} u_2 v_t$ is induced and has odd length. Hence, $(t-r(2q-2))-(t-(2p-1))+4 = 2p+1$ and by rearranging $2 = r(2q-2)$ contradicting $q \geq 3$.

Case 2. Suppose there is a graph $G \in \mathcal{G}^I(2p+1, 2q)$ with $q \geq 3$ and $p \geq 2$ (and $2p+1 < 2q$!), which satisfies $\delta(G) \geq 3$ and is non-bipartite.
Let again $P := v_1 v_2 \ldots v_t$ be a longest induced path of G. In the remaining proof we will deduce several structural statements concerning this longest induced path P of G. It is important to note that these statements also hold for every longest induced path P' of G. For convenience we call a vertex $v \in V(G)$ an *i-type vertex*, if there exists a longest induced path $P' = v_1' v_2' ... v_t'$ of G with $v = v_i'$. Again because of Theorem 2 we obtain $d_G(v_t) = 3$ or more generally

Claim 1. Every t-type vertex has the degree 3.
Now let $N_G(v_t) = \{v_{t-1}, u_1, u_2\}$. Again for $i = 1, 2$ let $j(i)$ be the largest integer less than or equal to $t-1$ such that $u_i v_{j(i)} \in E(G)$. Analogously, we have say $j(1) = t-(2q-2)$, $j(2) = t-(2p-1)$ and $u_1 v_{t-(2q-2)}$, $u_2 v_{t-(2p-1)} \in E(G)$. Furthermore, there exists a maximum $r \geq 1$ such that $r(2p-1) < 2q-2$ and $u_2 v_{t-i(2p-1)} \in E(G)$ for $1 \leq i \leq r$. Now the cycle $v_t u_1 v_{t-(2q-2)} v_{t-2q+3} \ldots v_{t-r(2p-1)} u_2 v_t$ is induced and has odd length. Hence, $(t-r(2p-1))-(t-(2q-2))+4 = 2p+1$ and by rearranging $2q = (r+1)(2p-1)$.

We now examine a special case. Suppose that G contains no induced path P_{2q} containing $2q$ vertices.

Markossian, Gasparian and Reed [9] showed that all triangle- free and even-hole-free graphs are 2-degenerate. Hence G has to contain an induced even cycle $C = c_0c_1 \dots c_{2q-2}c_{2q-1}c_0$. Furthermore, since $\delta(G) \geq 3$ each vertex of C is adjacent to at least one vertex not lying on the cycle C, e.g. $\{c_0d_0, c_1d_1\} \subset E(G)$. The P_{2q}-freeness, $G \in \mathcal{G}^I(2p+1, 2q))$ and $2q = (r+1)$ $(2p-1)$ forces that $N_G(d_i) \cap V(C) = \{c_i, c_{i+(2p-1)}, c_{i+2(2p-1)} \dots, c_{i+r(2p-1)}\}$ for $i = 0, 1$. But then the vertex set $\{c_0, d_0, c_{2p-1}, c_{2p}, d_1, c_1\}$ induces a 6-cycle enforcing that $q = 3$ and $p = 2$. Since G is non-bipartite G has to contain an induced 5-cycle $C^1 = a_0a_1a_2a_3a_4a_0$. Furthermore, since $\delta(G) \geq 3$ each vertex of C^1 is adjacent to at least one vertex not lying on the cycle C^1, e.g. $\{a_0b_0, a_1b_3, a_2b_1, a_3b_4, a_4b_2\} \subset E(G)$. The P_6-freeness forces that $C^2 = b_0b_1b_2b_3b_4b_0$ induces a 5-cycle. Now the C_4-freeness of G forces that all vertices of C^1 have degree 3. Analogously, every vertex of C^2 has degree 3. Thus, we have $N_G(V(C^i)) - V(C^i) = V(C^{3-i})$ for $i = 1, 2$, i.e., $G \cong P^*$. Hence, in the following a longest induced path P contains at least $2q$ vertices.

We now examine the $(t-1)$-type vertex v_{t-1}. Since $\delta(G) \geq 3$, P is chordless and G is triangle-free, v_{t-1} has a neighbour $w \in V(G) - (V(P) \cup N_G(v_t))$.

Subcase 2.1. w has no neighbour on P.
Then $P' := v_1v_2 \dots v_{t-1}w$ is also a longest induced path of G and the t-type vertex w has (exactly) two neighbours $w_1, w_2 \in V(G) - (V(P) \cup N_G(v_t) \cup \{w\})$. Because w_1 corresponds to u_2 and w_2 to u_1, we deduce that say $w_1v_{t-(2q-2)} \in E(G)$ and $w_2v_{t-(2p-1)} \in E(G)$. Now $C^1 := w_1v_{t-(2q-2)}$ $u_1v_tv_{t-1}ww_1$ forms a cycle of length six. Therefore, the only possible case is again $q = 3$ and $p = 2$, i.e., $G \in \mathcal{G}^I(5, 6)$. Now the C_3- and C_4-freeness of G forces that the set $\{w, w_1, v_{t-1}, v_t, u_2, v_{t-4}, v_{t-3}\}$ induces a C_7, a contradiction.

Note that the C_3- and C_4-freeness of G always forces that every 7-cycle is an induced one.

Subcase 2.2. w has a neighbour on P.
Then $wv_{t-(2q-1)} \in E(G)$ or $wv_{t-2p} \in E(G)$. Now $C^1 := wv_{t-(2q-1)}v_{t-2q-2}$ $u_1v_tv_{t-1}w$ or $C^2 := wv_{t-2p}v_{t-(2p-1)}u_2v_tv_{t-1}w$ induces a cycle of length six, i.e. $G \in \mathcal{G}^I(5, 6)$. Therefore, we have either $wv_{t-5} \in E(G)$ or $wv_{t-4} \in E(G)$.

Subcase 2.2.1. $wv_{t-5} \in E(G)$.
Since G is C_4-free, u_2 is not adjacent to w and we deduce a contradiction by considering the set $\{w, v_{t-5}, v_{t-4}, v_{t-3}, u_2, v_t, v_{t-1}\}$, which induces a C_7.

Subcase 2.2.2. $wv_{t-4} \in E(G)$.

Claim 2. Every $(t-1)$-type vertex has the degree 3, e.g. $d_G(v_{t-1}) = 3$.

Assume $d_G(v_{t-1}) > 3$, then there exists a vertex $\bar{w} \in (N_G(v_{t-1}) - (\{w\} \cup V(P)))$. Likewise, we deduce that $\bar{w}$ is also adjacent to v_{t-4}. But then $\{v_{t-1}, w, \bar{w}, v_{t-4}\}$ induces a C_4 – a contradiction. Thus, we have $d_G(v_{t-1}) = 3$.

Recall that u_1 is adjacent to v_{t-4} and u_2 is adjacent to v_{t-3}.

Claim 3. Every $(t-2)$-type vertex has the degree 3, e.g. $d_G(v_{t-2}) = 3$.
Let x be a neighbour of v_{t-2} not lying on P. If x is adjacent to v_{t-5} or v_{t-6}, then $xv_{t-5}v_{t-4}u_1v_tv_{t-1}v_{t-2}x$ or $xv_{t-6}v_{t-5}v_{t-4}wv_{t-1}v_{t-2}x$ is a cycle of length 7, a contradiction. Hence, x has no other neighbours on P. The C_4-freeness of G forces that $xw \notin E(G)$. Furthermore, we have $xu_2 \notin E(G)$, since otherwise $xu_2v_tu_1v_{t-4}v_{t-3}v_{t-2}x$ is a cycle of length 7, a contradiction. Since $d_G(x) \geq \delta(G) = 3$, x has a neighbour $y \notin V(P) \cup \{w, u_1, u_2\}$. If y is adjacent to v_{t-4} or v_{t-5}, then $yv_{t-4}u_1v_tv_{t-1}v_{t-2}xy$ or $yv_{t-5}v_{t-4}wv_{t-1}v_{t-2}xy$ is a cycle of length 7, a contradiction. Hence, $P^{'''} = v_1 \ldots v_{t-2}xy$ is a longest induced path implying with Claim 1 and Claim 2 that $d_G(x) = d_G(y) = 3$. Observe that x is a $(t-1)$-type and y a t-type vertex.

Claim 4. x is adjacent to u_1.
Suppose to the contrary that $xu_1 \notin E(G)$. Analogously to P, we deduce for $P^{'''}$ that the $(t-1)$-type vertex x is adjacent to a vertex $y^* \notin V(P) \cup \{y, w, u_1, u_2\}$. Because y^* corresponds to w we also deduce that y^* is adjacent to v_{t-4}. But then $y^*v_{t-4}u_1v_tv_{t-1}v_{t-2}xy^*$ is a cycle of length 7, a contradiction. Thus we have $xu_1 \in E(G)$.

Now suppose that $d_G(v_{t-2}) > 3$, i.e. there exists a vertex $x^* \in N_G(v_{t-2}) - \{v_{t-3}, v_{t-1}, x\}$. Analogously to Claim 4 we obtain $x^*u_1 \in E(G)$, a contraction since G is C_4-free. Therefore, we have $d_G(v_{t-2}) = 3$.

For convenience we introduce the notation *z-type vertex* with $z \in \{w, u_1, u_2, x, y\}$. A vertex v of G is a z-type vertex, if there exists a longest induced path $P^{'''}$, such that the role of v in $P^{'''}$ corresponds to the role of z for the longest induced path $P = v_1v_2...v_t$.

Claim 5. w is adjacent to y.
Because the $(t-1)$-type vertex x of $P^{'''} = v_1 v_2 ... v_{t-2} xy$ is adjacent to u_1, we obtain that u_1 is a w-type vertex. Similarly, we deduce that v_{t-1} is a x-type vertex. Recall (Claim 4) that u_1 is adjacent x and the $(t-4)$-type vertex v_{t-4}. Since w is adjacent to the x-type vertex v_{t-1} of $P^{'''}$ and the $(t-4)$-type vertex v_{t-4} of $P^{'''}$ the C_4-freeness of G forces that w is the u_1-type vertex of $P^{'''}$. Because y is the t-type vertex of $P^{'''}$ this implies Claim 5.

Claim 6. y is adjacent to u_2 and therefore $N_G(y) = \{x, w, u_2\}$.
Assume to the contrary that y is not adjacent to u_2. Hence there exists x' with $\{x'\} = N_G(y) - \{x, w\}$. Note that $x' \notin V(P) \cup \{u_1, u_2, x, w\}$ and since $P^{'''} = v_1 ... v_{t-2} xy$ is a longest induced path and w is adjacent to v_{t-4}, we obtain that x' is adjacent to v_{t-3}. But then $x' v_{t-3} u_2 v_t v_{t-1} w y x'$ is a cycle of length 7, a contradiction. Thus we have $N_G(y) = \{x, w, u_2\}$. Since u_2 is a t-type vertex of the longest induced path $P'' = v_1 ... v_{t-4} u_1 x y u_2$ we obtain the following Claim 7 by Claim 1.

Claim 7. $d_G(u_2) = 3$.
Summarizing all claims we deduce $G' := G[\{u_2, v_t, v_{t-1}, v_{t-2}, v_{t-3}, v_{t-4}, u_1, w, x, y\}] \cong P^*$ and all six non-neighbours of v_{t-4} have degree three. If $G \cong P^*$ we are done.

Assume $G \ncong P^*$. Note that $CS := \{v_{t-4}, v_{t-3}, u_1, w\}$ is a cutset of G. If v_{t-4} and one of its three neighbours v_{t-3}, u_1 or w is already a (complete) cutset, then we are done. Hence w.l.o.g. suppose G_2 is a component of $G - CS$ different to the six cycle $G_1 := G[\{u_2, v_t, v_{t-1}, v_{t-2}, x, y\}]$ and say $G'' := \{u_1, w\} \cup V(G_2)$ is connected. The C_4-freeness forces that a shortest path connecting u_1 and w in G'' contains at least 2 internal vertices and in $G''' := \{u_1, w\} \cup V(G_1)$ there exists a u_1 and w connecting induced path with 3 internal vertices. Then one can construct easily an induced cycle of length at least 7, a contradiction. This settles the proof of this theorem. ■

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