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REMARKS ON PARTIALLY SQUARE GRAPHS, HAMILTONICITY AND CIRCUMFERENCE

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Abstract

Given a graph G, its *partially square* graph G^* is a graph obtained by adding an edge (u,v) for each pair u, v of vertices of G at distance 2 whenever the vertices u and v have a common neighbor x satisfying the condition $N_G(x) \subseteq N_G[u] \cup N_G[v]$, where $N_G[x] = N_G(x) \cup \{x\}$. In the case where G is a claw-free graph, G^* is equal to G^2. We define $\sigma_t^\circ = \min\{\sum_{x \in S} d_G(x) : S$ is an independent set in G^* and $|S| = t\}$. We give for hamiltonicity and circumference new sufficient conditions depending on σ° and we improve some known results.

Keywords: partially square graph, claw-free graph, independent set, hamiltonicity and circumference.

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1. Introduction

We shall use standard graph theory notation. A finite, undirected graph G consists of a vertex set V and an edge set E. We denote the open neighborhood and closed neighborhood of a vertex u of G by $N(u) = \{x \in V : (x,u) \in E\}$ and $N[u] = N(u) \cup \{u\}$, respectively. Finally we denote by $d(u)$ the degree of u. Ainouche [1] defined, for each pair a, b of vertices at distance 2 in G, a parameter $J(a,b) = \{u \in N(a) \cap N(b) : N[u] \subseteq N[a] \cup N[b]\}$. He introduces the concept of partially square graph G^* of a given graph G. Given a graph G, its *partially square* graph G^* is the graph obtained by adding an edge (u,v) for each pair u, v of vertices of G at distance 2 whenever

$J(u,v)$ is not empty, so $G^* = (V, E \cup \{(u,v) : \mathrm{dist}(u,v) = 2, J(u,v) \neq \emptyset\})$. In particular this condition is satisfied if at least a common neighbor of u and v does not center a claw (an induced $K_{1,3}$).

Obviously $E(G) \subseteq E(G^*) \subseteq E(G^2)$. On one side we have $G^* = G^2$ if for each pair u, v of vertices of G at distance 2, $J(u,v) \neq \emptyset$. On the other side G^* can be equal to G if $G = K_{p,q}$ with $p, q \geq 3$.

Ainouche and Kouider [2] used the square partially graph to improve some known results, in particular they proved the following result.

Theorem 1. *Let G be a k-connected graph ($k \geq 2$) and G^* its partially square graph. If $\alpha(G^*) \leq k$, then G is hamiltonian.*

In this paper we discuss some best known results on a longest cycle and hamiltonicity in a given graph G, where the sufficient condition depends on the degree sum of an independent set of vertices. Among these results, we consider first, the following result of Bermond [3].

Theorem 2. *If G is a 2-connected graph such that the degree sum of any independent set of two vertices is greater than d then G either is hamiltonian or contains a cycle of length at least d.*

Bondy [4] proved that

Theorem 3. *If G is k-connected ($k \geq 2$) of order n such that the degree sum of any independent set of $k+1$ vertices is strictly greater than $(k+1)\frac{(n-1)}{2}$, then G is hamiltonian.*

Finally, Fournier and Fraisse [5] generalize Bondy's theorem as follows.

Theorem 4. *If G is k-connected ($k \geq 2$) of order n such that the degree sum of any independent set of $k+1$ vertices is at least m, then G contains a cycle of length at least $\min(\lceil 2m/(k+1) \rceil, n)$.*

We denote the minimal degree sum of independent sets of order t ($t = 1, 2, \ldots$) in G by

$\sigma_t = \min\{\sum_{x \in S} d_G(x) :\ S$ is an independent set in G and $|S| = t\}$.

Moreover, we define a kind of minimal G-degree sum of independent sets of order t in G^* as follows

$\sigma_t^\circ = \min\{\sum_{x \in S} d_G(x) :\ S$ is an independent set in G^* and $|S| = t\}$.

Observe that if S is an independent set in G^* then S is an independent set in G, but the opposite is false, as we can show it in Figure 1. The set $\{1,3,5\}$ is not independent in G^*. Then $\sigma_t^\circ \geq \sigma_t$. We suppose by convention that if $\alpha(G^*) < t$ then σ_t° is infinite.

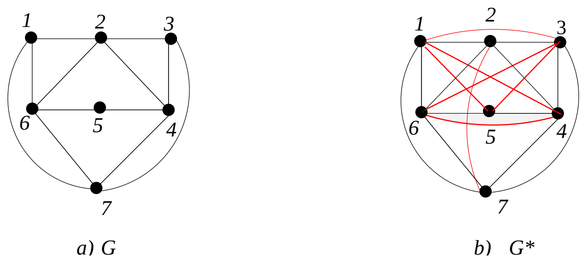

Figure 1

As $\sigma_t^\circ \geq \sigma_t$, then the following theorems, which we prove in this note, are better than Theorems 2 and 4, respectively.

Theorem 5. *Let G be 2-connected graph. Then either G is hamiltonian or contains a cycle with length at least σ_2°.*

Theorem 6. *Let G be a k-connected graph ($k \geq 2$). Then either G is hamiltonian or it contains a cycle of length at least $\frac{2\sigma_{k+1}^\circ}{k+1}$.*

We deduce the following extension of Theorem 3 to σ°:

Corollary 7. *Let G be a k-connected graph ($k \geq 2$) of order n. If $\sigma_{k+1}^\circ > (k+1)\frac{(n-1)}{2}$, then G is hamiltonian.*

The k-connected graph G (with $k \geq 2$ and k is even) of Figure 2 is given as follows. There exist k independent vertices adjacent to each vertex of $(k+2)$ copies of the complete graph K_p. The copies of K_p are regrouped by pairs. The vertices of each pair are adjacent to a vertex. Since the connectivity of G is equal to k, then $2p$ is at least equal to k. As k is even, we may construct a hamiltonian cycle in G. For $p \geq k$, we have $\sigma_{k+1} = (k+1)(p+k)$. The bound in Corollary 7 is $(k+1)\frac{(n-1)}{2} = (k+1)\frac{(2p(k+2)+3k)}{4}$. Since $\sigma_{k+1} \leq (k+1)\frac{(n-1)}{2}$, Theorem 3 does not allow to deduce that G is hamiltonian. But if we consider G^*, as the independent set of G which gives a minimum degree sum is not obviously an independent set in G^*, we obtain $\sigma_{k+1}^\circ = p\frac{(k+2)^2}{2}$

(an independent set which engenders σ°_{k+1} is given by $\frac{(k+2)}{2}$ vertices of degree $2p$ each one and $((k+1)-\frac{(k+2)}{2})$ vertices of degree $(k+2)p$ each one). For $p \geq \frac{3}{2}k$, we deduce that σ°_{k+1} is greater than $(k+1)\frac{(n-1)}{2} = (k+1)\frac{(2p(k+2)+3k)}{4}$. So from Corollary 7, G is hamiltonian. Moreover, note that $n = \frac{(2p+3)(k+2)}{2} - 2$. Then for $2 \leq \frac{k}{2} \leq p \leq \frac{3}{2}k - 4$, we remark that $\min\{n, \frac{2\sigma_{k+1}}{k+1}\} = \frac{2\sigma_{k+1}}{k+1}$ and $\min\{n, \frac{2\sigma^\circ_{k+1}}{k+1}\} = \frac{2\sigma^\circ_{k+1}}{k+1}$. As $\frac{2\sigma^\circ_{k+1}}{k+1} > \frac{2\sigma_{k+1}}{k+1}$, we can deduce that the bound given by Theorem 6 is more close to n (because the longest cycle has length n in this case) than the one given by Theorem 4.

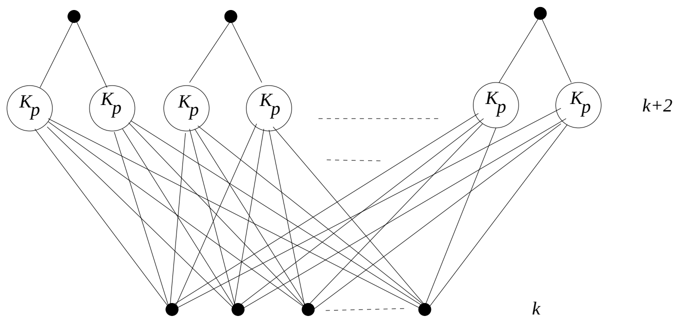

Figure 2

2. Terminologies

Let C be a longest cycle of a k-connected and non-hamiltonian graph G and the orientation of C is fixed. For $u \in V(C), u^+$(resp. u^-) represents its successor (resp. predecessor) on C. If $u, v \in V(C)$ then (u, C, v) represents the path given by the consecutive vertices on C ordered from u to v (including u and v) following the orientation chosen of C. The same vertices visited in the opposite orientation give the path $(v, \overline{C}, u)$. Let $R = G \setminus C$. Let d_1 be a vertex of C such that the number of its neighbors belonging to R, $d_R(d_1) \neq 0$. Let P_0 be a longest path starting from d_1 on C, such that $V(P_0) \setminus \{d_1\} \subseteq R$. Let x_0 be an extremity of P_0 in R and H be a connected component of x_0 in R. Let $N_C(H)$ be the set of vertices of C which have at least a neighbor in H. Then $d_1 \in N_C(H)$. Note $N_C(H) = \{d_1, .., d_m\}$, where the indices are taken modulo m. Because C is a longest cycle, we have $|V(C)| > |N_C(H)| = m$ and therefore, $m \geq k$. We suppose that following the orientation of C, we meet $d_1, .., d_m$, respectively. Since every path (d_i, C, d_{i+1}) contains at least one internal vertex, $C_i = (d_i^+, C, d_{i+1}^-)$ is a, possibly trivial, path $(i = 1, \dots, m)$. Each pair of vertices d_i, d_j of $\{d_1, d_2, ..., d_m\}$ is joined by a path of length at least two, in which all internal vertices (of this path) are in H. We denote this (not oriented) path by (d_i, H, d_j).

Given a path $P = (a_1, a_2, \dots, a_q)$, $q \geq 2$ and a vertex $u \notin V(P)$. We say that u is *P-insertible* if there exists an i, $1 \leq i < q$, such that the vertices a_i and a_{i+1} are both adjacent to u. The edge (a_i, a_{i+1}) is called an *insertion edge* for u. In particular, a vertex $u \in V(C_i)$ is called *insertible* if it is (d_{i+1}, C, d_i)*-insertible.*

Given four vertices a, b, u, v of C, we say that the edges $(u, a), (v, b)$ are *crossing* (if they exist) if the four vertices arrive on C in the order a, v, u, b.

3. Definition of an Independent Set

Let us recall the following lemma (see [1]) on properties of insertible vertices, where m, C_i, d_i $(i = 1, \dots, m)$ are used as defined in Section 2.

Lemma 8. *Let G be a k-connected and non-hamiltonian graph (with $k \geq 2$), C be a longest cycle of G and H be a connected component of $R = G \setminus C$. Then*

(a) *for each $i \in \{1, 2, \dots, m\}$, C_i contains a non-insertible vertex.*

Let x_i be the first non-insertible vertex on C_i $(i = 1, 2, \dots, m)$ and $x_0 \in V(H)$. Set $W_0 = V(H)$ and $W_i = V((d_i^+, C, x_i))$ for each $1 \leq i \leq m$. For each $1 \leq i < j \leq m$, choose $w_i \in W_i$ and $w_j \in W_j$. Then

(b) $(w_i, w_j) \notin E(G)$.

(c) *There does not exist a vertex $z \in V((w_i, C, w_j))$ such that (w_i, z^+), $(w_j, z) \in E$ (i.e., the edges (w_i, z^+) and (w_j, z) are not crossing).*

For a longest cycle C of a k-connected $(k \geq 2)$ non-hamiltonian graph G with a fixed orientation of C, let now P_0, x_0, H, m, d_i, C_i $(i = 1, \dots, m)$ be as defined in Section 2, and W_0, x_i, W_i $(i = 1, \dots, m)$ as defined in Lemma 8. Furtheremore, let $X = \{x_0, x_1, \dots, x_m\}$. For each $s \in \{1, 2, \dots, m\}$, let A_s be the set of vertices u belonging to (x_s^+, C, d_{s+1}^-) which verify the two following properties:

(i) $d_R(u) \neq 0$ and

(ii) $(x_s, u^-), (x_s, u), (x_s, u^+) \in E$ if $x_s \neq u^-$, and $(x_s, u), (x_s, u^+) \in E$, otherwise.

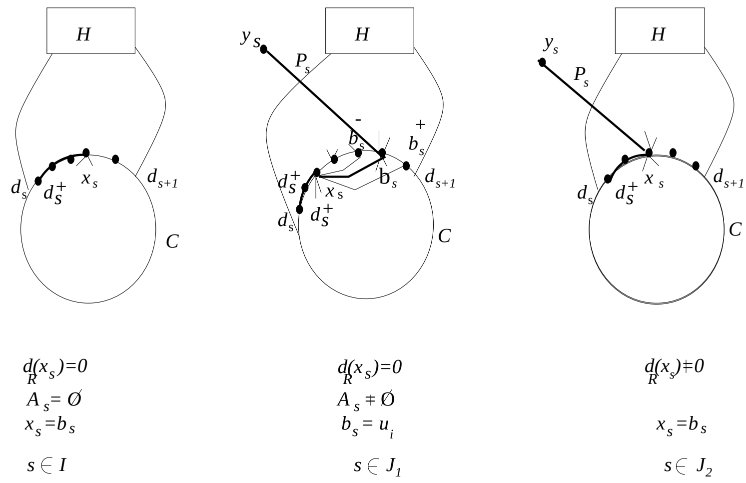

Figure 3

Let

$I = \{s \in \{1, 2, \ldots, m\} : d_R(x_s) = 0 \text{ and } A_s = \emptyset\}$,

$J_1 = \{s \in \{1, 2, \ldots, m\} : d_R(x_s) = 0 \text{ and } A_s \neq \emptyset\}$ and

$J_2 = \{s \in \{1, 2, \ldots, m\} : d_R(x_s) \neq 0\}$.

By the definitions of I, J_1 and J_2, we can deduce that I, J_1 and J_2 form a partition of the set $\{1, 2, \ldots, m\}$ (i.e., $I \cap J_1 = I \cap J_2 = J_1 \cap J_2 = \emptyset$ and $I \cup J_1 \cup J_2 = \{1, 2, \ldots, m\}$).

In the case where $s \in J_1$, we denote by u_s the first vertex on (x_s^+, C, d_{s+1}^-) which is in A_s. Let $b_s = u_s$ if $s \in J_1$ and $b_s = x_s$ if $s \in I \cup J_2$. For $s \in J_1 \cup J_2$, let P_s be a longest path with an extremity b_s (on the cycle) such that $V(P_s) \setminus \{b_s\} \subset R$. Let y_s be the extremity of P_s on R (Figure 3).

Let $S = \{x_0\} \cup \{x_i : i \in I\} \cup \{y_s : s \in J_1 \cup J_2\}$. In particular, if $J_1 \cup J_2 = \emptyset$, then $S = X = \{x_0, x_1, \ldots, x_m\}$. It is proved in [2] that X is an independent set in G^*.

Let $s \in I \cup J_1 \cup J_2$. Put $W'_s = (d_s^+, C, x_s, P_s, y_s)$ if $s \in J_2$, $W'_s = (d_s^+, C, x_s, b_s, P_s, y_s)$ if $s \in J_1$ and $W'_s = (d_s^+, C, x_s)$ if $s \in I$ (see Figure 3, the W'_s are given in bold). We deduce that if $s \in J_1 \cup J_2$, then $V(W'_s) = W_s \cup V(P_s)$.

4. Lemmas

For all statements of this section we make the following:

Supposition. *Let G be a k-connected, non-hamiltonian graph with $k \geq 2$ and C a longest cycle of G.*

Furthermore, we use all denotations introduced in Sections 2 and 3. We shall prove a series of lemmas on some properties of the elements of S in G. In the following lemma we prove a property of the vertices of b_j^-, b_j, and b_j^+, for $1 \leq j \leq m$ which will be always used in the further proofs.

Lemma 9. *For each pair of indices $\{i, j\}$, with $1 \leq i \neq j \leq m$, $N_C(b_j) \cap W_i = \emptyset$; in particular, neither (b_j^-, b_j) nor (b_j, b_j^+) are insertion edges for the vertices of W_i.*

Proof. Suppose first that $j \in I \cup J_2$. As $i \neq j$, then by Lemma 8(b), the vertex b_j (which is x_j in this case) cannot have neighbors in W_i. In the case where $j \in J_1$, we know that $b_j = u_j$. Let w be a neighbor of u_j in W_i, with $i \neq j$. Observe that the edges (w, u_j) and (x_j, u_j^+) are crossing. We obtain then a contradiction with Lemma 8(c) applied to $w \in W_i$ and $x_j \in W_j$.

We deduce that for each pair of indices $\{i, j\}$, with $1 \leq i \neq j \leq m$, neither (b_j^-, b_j) nor (b_j, b_j^+) can be an insertion edge for vertices of W_i. ■

From now on, we denote by $T = (a \ldots x[c \ldots s_j \ldots d]y \ldots b)$ a segment of C, where the sub-segment $c \ldots s_j \ldots d$ (which is in brackets in T) is considered only if $j \in J_1$. In other words, $T = (a, \ldots, x, c, \ldots, s_j, \ldots, d, y, \ldots, b)$ if $j \in J_1$ and $T = (a, \ldots, x, y, \ldots, b)$, otherwise.

Now, we prove other properties of S in G. In particular, we prove that it is an independent set in G and G^*.

Lemma 10. *The following statements are true:*

(a) *let w_0 be a vertex of W_0 and w_i be a vertex of $W_i \cup \{b_i\}$, for each $1 \leq i \leq m$. For each pair of indices $\{i, j\}$, with $0 \leq i \neq j \leq m$, the vertices w_i, w_j are not joined by a path all internal vertices of which are in R. In particular, S is an independent set of $m+1$ elements in G.*

(b) *for each $j \in J_1 \cup J_2$, we have $N_C(y_j) \subset (V(C) \setminus [\cup_{i=1}^{m}(W_i \cup \{b_i\}) \cup \{d_j\}]) \cup \{b_j\}$.*

(c) *S is an independent set in G^*.*

Proof. (a) The proof is by contradiction. We may suppose $i < j$. Denote by (w_i, L, w_j) a path which joins w_i and w_j and which is assumed to have all its internal vertices in R.

Suppose first that $i = 0$. Then $1 \le j \le m$. Put $P = (w_0, L, [b_j, \overline{C}], x_j, [b_j^+], C, d_{j+1})$ if $w_j = b_j$ and $P = (w_0, L, w_j, C, d_{j+1})$ if $w_j \in W_j$. By definition, all insertible vertices of W_j admit their insertion edges on $Q = (d_{j+1}, C, d_j)$. We can insert in Q the vertices of (d_j^+, C, x_j^-) if $w_j = b_j$ and the vertices of (d_j^+, C, w_j^-) if $w_j \in W_j$. Then the new path obtained from Q in this way and combined with a subpath of the walk (d_j, H, w_0, P) joining d_j and d_{j+1} gives a cycle longer than C, which is a contradiction. Consequently, $i \ne 0$. Since there does not exist a path between a vertex of W_0 and w_j, then $V(L) \cap V(H) = \emptyset$. In case $w_i \in W_i$ and $w_j \in W_j$ the cycle $(d_i, H, d_j, \overline{C}, w_i, L, w_j, C, d_i)$ contradicts the maximality of C.

Suppose now that $w_i = b_i$ or $w_j = b_j$. For each $s \in \{i, j\}$, if $w_s = b_s$ then we construct a path Q_1 from the path $Q = (d_i, \overline{C}, w_j, L, w_i, C, d_j)$, by replacing the path (w_s, C, d_{s+1}) by the path $(w_s, [b_s, \overline{C}], x_s, [b_s^+], C, d_{s+1})$. By the definition of the insertion, the edge (b_j, b_j^+) is not an insertion edge for the vertices of W_j and the edge (b_i, b_i^+) is not an insertion edge for the vertices of W_i. Moreover, by Lemma 9, (b_j, b_j^+) is not an insertion edge for the vertices of W_i and the edge (b_i, b_i^+) is not an insertion edge for the vertices of W_j, and by Lemma 8(b) there is no edge between W_i and W_j. We deduce that all vertices of C which do not belong to Q_1 are vertices contained in (d_i^+, C, x_i^-) or in (d_j^+, C, x_j^-); therefore, they are insertible with insertion edges belonging to Q_1. We can insert these insertible vertices in Q_1, in order to obtain a path Q_1'. The combination of Q_1' and (d_i, H, d_j) forms a cycle longer than C, a contradiction.

We deduce that $W_i' \cap W_j' = \emptyset$, $W_0 \cap W_j' = \emptyset$ and that any two vertices belonging respectively to W_i' (or W_0) and W_j' are not adjacent, for each $1 \le i \ne j \le m$. Consequently, S is an independent set of $m + 1$ elements in G.

(b) Suppose there exists a neighbor u of y_j on $[\cup_{i=1}^m (W_i \cup \{b_i\}) \cup \{d_j\}] \setminus \{b_j\}$. By (a), for each $1 \le i \ne j \le m$, we have $u \notin N_C(y_j) \cap W_i \cup \{b_i\}$.

Remark that if $j \in J_2$, then $b_j \in W_j$ (in this case $x_j = b_j$) and b_j can be neighbor of y_j. If $j \in J_1$, then $u \ne x_j$ because of $d_R(x_j) = 0$. Thus $u \in V((d_j, C, x_j^-))$. Let $Q = (u, \overline{C}, [b_j^+], x_j, [C, b_j])$. By the definition of the insertion, the insertion edges of the vertices of (d_j^+, C, x_j^-) belong to $E((d_{j+1}, C, d_j))$. So they belong to $E(Q)$. Then we can construct Q' from Q by inserting the vertices of $V((u^+, C, x_j^-))$ (if there are any, i.e., if $u \ne x_j^-$). Consequently, the paths Q' and (u, y_j, P_j, b_j) give a cycle longer than C, which is a contradiction.

(c) The proof is by contradiction. Since S is an independent set in G, we suppose that there exists a pair of distinct vertices $a_i, a_j \in S$ such that $(a_i, a_j) \in E(G^*) \setminus E(G)$. So there exists $v \in J(a_i, a_j)$ (such that $(a_i, v), (v, a_j) \in E(G)$). We suppose first that $v \in V(R)$. Without loss of generality, we assume that $i < j$ and $j \in I \cup J_1 \cup J_2$. If $j \in I$ then $a_j = x_j$. This contradicts the fact that $d_R(x_j) = 0$. By (a), $j \notin J_1 \cup J_2$. Therefore $v \in V(C)$. In this case $v, v^+ \in N_C(a_i) \cup N_C(a_j)$. So either a_i (or a_j) is C-insertible or (a_i, v^+), (v, a_j) are crossing, which is a contradiction.

Hence, by (a) and (c) for each pair of distinct vertices $a_i, a_j \in S$, $(a_i, a_j) \notin E(G^*)$. We deduce that S is an independent set in G^*. ■

Corollary 11. *For each pair of distinct vertices a_i and a_j of $S \setminus \{x_0\}$, we have $d_C(a_i) + d_C(a_j) \leq |C|$.*

As usual we write $|Q|$ instead of $|V(Q)|$ for a cycle or a path Q.

Proof. We begin by the following claim.

Claim 12. *Let $P = (u_1, u_2, \ldots, u_r)$ be a segment of C (in the same orientation as C) with $r \geq 1$. Let $a_i, a_j \in S \setminus \{x_0\}$, with $a_i, a_j \notin V(P)$ and $i \neq j$; it can be assumed — if necessary by commuting the denotations of the indices i, j — that P is a subpath of (b_i, C, b_j). Then*

(a) $d_P(a_i) + d_P(a_j) \leq r + 1$.

(b) *In particular, if $(u_1, a_j) \notin E$ and $(u_2, a_i) \notin E$, then $d_P(a_i) + d_P(a_j) \leq r$.*

***Proof of Claim* 12.** (a) Define $N_P^+(a_j) = \{u_{i+1} : (u_i, a_j) \in E\}$, where $u_{r+1} := u_r^+$. Thus $N_P^+(a_j) \subseteq \{u_2, u_3, \ldots, u_r, u_{r+1}\}$ and $N_P(a_i) \subseteq \{u_1, u_2, \ldots, u_r\}$. One can see that the property (c) of Lemma 8 remains true if we replace w_i, w_j and (w_i, C, w_j) by a_i, a_j and (b_i, C, b_j), respectively. So there does not exist z on P such that the edges (a_i, z^+) and (z, a_j) are crossing. Then $N_P^+(a_j) \cap N_P(a_i) = \emptyset$. We deduce that $d_P(a_j) + d_P(a_i) = |N_P^+(a_j) \cap N_P(a_i)| + |N_P^+(a_j) \cup N_P(a_i)| \leq r + 1$.

(b) If $(u_1, a_j) \notin E$ and $(u_2, a_i) \notin E$, then $u_2 \notin N_P^+(a_j) \cup N_P(a_i)$, and hence $d_P(a_j) + d_P(a_i) \leq r$. ■

Return now to the proof of Corollary 11.

Put $U_s = (d_s^+, C, d_{s+1})$, for each $1 \leq s \leq m$. We prove that for each $1 \leq s \leq m$, $d_{U_s}(a_i) + d_{U_s}(a_j) \leq |U_s|$.

Case 1. i and j belong to $I \cup J_1$.

Put $L_s = (x_s^+, C, d_{s+1})$. By Claim 12(a), $d_{L_s}(a_i) + d_{L_s}(a_j) \leq |L_s| + 1$, for each s , $1 \leq s \leq m$. If $s \notin \{i, j\}$, by Lemma 10(a), $d_{W_s}(a_i) = d_{W_s}(a_j) = 0$. Then $d_{U_s}(a_i)+d_{U_s}(a_j) \leq |L_s|+1 \leq |U_s|$ since $|U_s| = |L_s|+|W_s| \geq |L_s|+1$. If $s \in \{i, j\}$, we have $d_{W_s}(a_s) = 0$ if $s \in J_1$ and $d_{W_s}(a_s) \leq |W_s|-1$ if $s \in I$. As $d_{W_s}(a_{s'}) = 0$, with $s' \in \{i, j\}$ and $s' \neq s$, then $d_{W_s}(a_i)+d_{W_s}(a_j) \leq |W_s|-1$. Thus $d_{U_s}(a_i) + d_{U_s}(a_j) \leq (|L_s| + 1) + (|W_s| - 1) = |U_s|$.

Case 2. i or j belongs to J_2.
If $s \notin \{i, j\}$ or $s \in \{i, j\} \cap (I \cup J_1)$, the arguments are similar than those of the above case. If $s \in \{i, j\} \cap J_2$, put $L_s = (x_s, C, d_{s+1})$. Without loss of generality, put $s = i$ and $s \in J_2$. By Lemma 10(a), we have $(x_s, a_j) \notin E$. As $s \in J_2$ and C is maximal, then $(x_s^+, a_s) \notin E$. Remark that by Claim 12(b), these two last hypotheses allow to deduce that $d_{L_s}(a_s) + d_{L_s}(a_j) \leq |L_s|$. Moreover, by Lemma 10(b), $d_{W_s \setminus \{x_s\}}(a_s) + d_{W_s \setminus \{x_s\}}(a_j) = 0$. As we have always, $|U_s| \geq |L_s|$, then $d_{U_s}(a_s) + d_{U_s}(a_j) \leq |U_s|$.

Consequently, for each $1 \leq s \leq m$, $d_{U_s}(a_i) + d_{U_s}(a_j) \leq |U_s|$. As $|C| = \sum_{s=1}^{s=m} |U_s|$, then $d_C(a_i) + d_C(a_j) \leq |C|$. ■

Lemma 13. *If $i \in I$ then $d_G(x_i) = d_C(x_i)$ and $d_G(x_0) + d_C(x_i) \leq |C|$.*

Proof. The proof is by contradiction and it is similar to the one of Lemma 5 given in [5], except that instead of considering d_i^+ we consider x_i. It is clear that in the proof we take into account the insertible vertices in all the constructions of longest cycles (as we have done it in the above lemmas), this contradicts the maximality of C. ■

Finally, we recall the following lemma of Fournier and Fraisse (see [5]) which is useful for the proofs of theorems.

Lemma 14. *Let P be a path of maximum length between all paths which have a given extremity, a, on C and all the other vertices are not on C. Let x the second extremity of P. Then we have $d_G(x) \leq \frac{|C|}{2}$.*

5. Proofs of Theorems

We define the following variants of I, J_1 and J_2 (which had been defined in Section 3):

$I(k) = \{1, 2, \ldots, k\} \cap I$,
$J_1(k) = \{1, 2, \ldots, k\} \cap J_1$ and

$J_2(k) = \{1, 2, \ldots, k\} \cap J_2$.

It is clear that $I(k) \cup J_1(k) \cup J_2(k) = \{1, 2, \ldots, k\}$. In order to prove Theorems 5 and 6, we suppose that the graph G is not hamiltonian.

***Proof of Theorem* 5.** If there exists at least an index $i \in \{1, 2\}$, such that $i \in J_1 \cup J_2$, then by Lemma 14, $d_G(y_i) \le \frac{|C|}{2}$ and $d_G(x_0) \le \frac{|C|}{2}$. Thus using Lemma 10(c), $\sigma_2^\circ \le d_G(y_i) + d_G(x_0) \le |C|$. Consequently, $|C| \ge \sigma_2^\circ$. Otherwise, for each $i \in \{1, 2\}$, we have $i \in I$. By Corollary 11, $d_C(x_1) + d_C(x_2) \le |C|$ and since $(x_1, x_2) \notin E(G^*)$ and $d_R(x_1) = d_R(x_2) = 0$ then $\sigma_2^\circ \le d_G(x_1) + d_G(x_2) \le |C|$. Consequently, $|C| \ge \sigma_2^\circ$. ■

***Proof of Theorem* 6.** We may suppose that $|I(k)| = p \le k$ and $I(k) = \{i_1, i_2, \ldots, i_p\}$. Recall that $d_R(x_i) = 0$, for $i \in I$. If $p = 0$ then by Lemmas 10 and 14, $\sigma_{k+1}^\circ \le \sum_{j \in J_1(k) \cup J_2(k) \cup \{0\}} d_G(y_j) \le \frac{(k+1)}{2}|C|$. Consequently, $|C| \ge 2\frac{\sigma_{k+1}^\circ}{k+1}$. If $p = 1$, then by Lemma 13, $d_G(x_0) + d_C(x_{i_1}) \le |C|$ and by Lemma 14, $d_G(y_j) \le \frac{|C|}{2}$, for each $j \in J_1(k) \cup J_2(k)$. As $d_R(x_{i_1}) = 0$, we get then $\sigma_{k+1}^\circ \le d_G(x_0) + d_G(x_{i_1}) + \sum_{j \in J_1(k) \cup J_2(k)} d_G(y_j) \le \frac{(k+1)}{2}|C|$. So, $|C| \ge 2\frac{\sigma_{k+1}^\circ}{k+1}$. Finally, if $p \ge 2$ then by Corollary 11,

$$
\begin{aligned}
&d_C(x_{i_1}) + d_C(x_{i_2}) \le |C|,\\
&d_C(x_{i_2}) + d_C(x_{i_3}) \le |C|,\\
&\vdots\\
&d_C(x_{i_{p-1}}) + d_C(x_{i_p}) \le |C|,\\
&d_C(x_{i_p}) + d_C(x_{i_1}) \le |C|.
\end{aligned}
$$

Thus $\sum_{i \in I(k)} d_C(x_i) \le p\frac{|C|}{2}$. By Lemma 14, $\sum_{j \in J_1(k) \cup J_2(k) \cup \{0\}} d_G(y_j) \le (k - p + 1)\frac{|C|}{2}$. So $\sigma_{k+1}^\circ \le \sum_{i \in I(k)} d_C(x_i) + \sum_{j \in J_1(k) \cup J_2(k) \cup \{0\}} d_G(y_j) \le p\frac{|C|}{2} + (k - p + 1)\frac{|C|}{2} \le (k + 1)\frac{|C|}{2}$. Then $|C| \ge 2\frac{\sigma_{k+1}^\circ}{k+1}$. ■

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