# REMARKS ON PARTIALLY SQUARE GRAPHS, HAMILTONICITY AND CIRCUMFERENCE 

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#### Abstract

Given a graph $G$, its partially square graph $G^{*}$ is a graph obtained by adding an edge $(u, v)$ for each pair $u, v$ of vertices of $G$ at distance 2 whenever the vertices $u$ and $v$ have a common neighbor $x$ satisfying the condition $N_{G}(x) \subseteq N_{G}[u] \cup N_{G}[v]$, where $N_{G}[x]=N_{G}(x) \cup\{x\}$. In the case where $G$ is a claw-free graph, $G^{*}$ is equal to $G^{2}$. We define $\sigma_{t}^{\circ}=\min \left\{\sum_{x \in S} d_{G}(x): S\right.$ is an independent set in $G^{*}$ and $\left.|S|=t\right\}$. We give for hamiltonicity and circumference new sufficient conditions depending on $\sigma^{\circ}$ and we improve some known results.


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## 1. Introduction

We shall use standard graph theory notation. A finite, undirected graph $G$ consists of a vertex set $V$ and an edge set $E$. We denote the open neighborhood and closed neighborhood of a vertex $u$ of $G$ by $N(u)=\{x \in V$ : $(x, u) \in E\}$ and $N[u]=N(u) \cup\{u\}$, respectively. Finally we denote by $d(u)$ the degree of $u$. Ainouche [1] defined, for each pair $a, b$ of vertices at distance 2 in $G$, a parameter $J(a, b)=\{u \in N(a) \cap N(b): N[u] \subseteq N[a] \cup N[b]\}$. He introduces the concept of partially square graph $G^{*}$ of a given graph $G$. Given a graph $G$, its partially square graph $G^{*}$ is the graph obtained by adding an edge $(u, v)$ for each pair $u, v$ of vertices of $G$ at distance 2 whenever
$J(u, v)$ is not empty, so $G^{*}=(V, E \cup\{(u, v): \operatorname{dist}(u, v)=2, J(u, v) \neq \emptyset\})$. In particular this condition is satisfied if at least a common neighbor of $u$ and $v$ does not center a claw (an induced $K_{1,3}$ ).

Obviously $E(G) \subseteq E\left(G^{*}\right) \subseteq E\left(G^{2}\right)$. On one side we have $G^{*}=G^{2}$ if for each pair $u, v$ of vertices of $G$ at distance $2, J(u, v) \neq \emptyset$. On the other side $G^{*}$ can be equal to $G$ if $G=K_{p, q}$ with $p, q \geq 3$.

Ainouche and Kouider [2] used the square partially graph to improve some known results, in particular they proved the following result.

Theorem 1. Let $G$ be a $k$-connected graph $(k \geq 2)$ and $G^{*}$ its partially square graph. If $\alpha\left(G^{*}\right) \leq k$, then $G$ is hamiltonian.

In this paper we discuss some best known results on a longest cycle and hamiltonicity in a given graph $G$, where the sufficient condition depends on the degree sum of an independent set of vertices. Among these results, we consider first, the following result of Bermond [3].

Theorem 2. If $G$ is a 2-connected graph such that the degree sum of any independent set of two vertices is greater than d then $G$ either is hamiltonian or contains a cycle of length at least d.

Bondy [4] proved that

Theorem 3. If $G$ is $k$-connected $(k \geq 2)$ of order $n$ such that the degree sum of any independent set of $k+1$ vertices is strictly greater than $(k+1) \frac{(n-1)}{2}$, then $G$ is hamiltonian.

Finally, Fournier and Fraisse [5] generalize Bondy's theorem as follows.
Theorem 4. If $G$ is $k$-connected $(k \geq 2)$ of order $n$ such that the degree sum of any independent set of $k+1$ vertices is at least $m$, then $G$ contains a cycle of length at least $\min (\lceil 2 m /(k+1)\rceil, n)$.

We denote the minimal degree sum of independent sets of order $t(t=$ $1,2, \ldots$ ) in $G$ by
$\sigma_{t}=\min \left\{\sum_{x \in S} d_{G}(x): S\right.$ is an independent set in $G$ and $\left.|S|=t\right\}$.
Moreover, we define a kind of minimal $G$-degree sum of independent sets of order $t$ in $G^{*}$ as follows

$$
\sigma_{t}^{\circ}=\min \left\{\sum_{x \in S} d_{G}(x): S \text { is an independent set in } G^{*} \text { and }|S|=t\right\}
$$

Observe that if $S$ is an independent set in $G^{*}$ then $S$ is an independent set in $G$, but the opposite is false, as we can show it in Figure 1. The set $\{1,3,5\}$ is not independent in $G^{*}$. Then $\sigma_{t}^{\circ} \geq \sigma_{t}$. We suppose by convention that if $\alpha\left(G^{*}\right)<t$ then $\sigma_{t}^{\circ}$ is infinite.

a) $G$

b) $G^{*}$

Figure 1
As $\sigma_{t}^{\circ} \geq \sigma_{t}$, then the following theorems, which we prove in this note, are better than Theorems 2 and 4 , respectively.

Theorem 5. Let $G$ be 2-connected graph. Then either $G$ is hamiltonian or contains a cycle with length at least $\sigma_{2}^{\circ}$.

Theorem 6. Let $G$ be a $k$-connected graph $(k \geq 2)$. Then either $G$ is hamiltonian or it contains a cycle of length at least $\frac{2 \sigma_{k+1}^{\circ}}{k+1}$.

We deduce the following extension of Theorem 3 to $\sigma^{\circ}$ :
Corollary 7. Let $G$ be a $k$-connected graph $(k \geq 2)$ of order $n$. If $\sigma_{k+1}^{\circ}>$ $(k+1) \frac{(n-1)}{2}$, then $G$ is hamiltonian.

The $k$-connected graph $G$ (with $k \geq 2$ and $k$ is even) of Figure 2 is given as follows. There exist $k$ independent vertices adjacent to each vertex of $(k+2)$ copies of the complete graph $K_{p}$. The copies of $K_{p}$ are regrouped by pairs. The vertices of each pair are adjacent to a vertex. Since the connectivity of $G$ is equal to $k$, then $2 p$ is at least equal to $k$. As $k$ is even, we may construct a hamiltonian cycle in $G$. For $p \geq k$, we have $\sigma_{k+1}=(k+1)(p+k)$. The bound in Corollary 7 is $(k+1) \frac{(n-1)}{2}=(k+1) \frac{(2 p(k+2)+3 k)}{4}$. Since $\sigma_{k+1} \leq(k+1) \frac{(n-1)}{2}$, Theorem 3 does not allow to deduce that $G$ is hamiltonian. But if we consider $G^{*}$, as the independent set of $G$ which gives a minimum degree sum is not obviously an independent set in $G^{*}$, we obtain $\sigma_{k+1}^{\circ}=p \frac{(k+2)^{2}}{2}$
(an independent set which engenders $\sigma_{k+1}^{\circ}$ is given by $\frac{(k+2)}{2}$ vertices of degree $2 p$ each one and $\left((k+1)-\frac{(k+2)}{2}\right)$ vertices of degree $(k+2) p$ each one). For $p \geq$ $\frac{3}{2} k$, we deduce that $\sigma_{k+1}^{\circ}$ is greater than $(k+1) \frac{(n-1)}{2}=(k+1) \frac{(2 p(k+2)+3 k)}{4}$. So from Corollary 7, $G$ is hamiltonian. Moreover, note that $n=\frac{(2 p+3)(k+2)}{2}-2$. Then for $2 \leq \frac{k}{2} \leq p \leq \frac{3}{2} k-4$, we remark that $\min \left\{n, \frac{2 \sigma_{k+1}}{k+1}\right\}=\frac{2 \sigma_{k+1}}{k+1}$ and $\min \left\{n, \frac{2 \sigma_{k+1}^{\circ}}{k+1}\right\}=\frac{2 \sigma_{k+1}^{\circ}}{k+1}$. As $\frac{2 \sigma_{k+1}^{\circ}}{k+1}>\frac{2 \sigma_{k+1}}{k+1}$, we can deduce that the bound given by Theorem 6 is more close to $n$ (because the longest cycle has length $n$ in this case) than the one given by Theorem 4.


Figure 2

## 2. Terminologies

Let $C$ be a longest cycle of a $k$-connected and non-hamiltonian graph $G$ and the orientation of $C$ is fixed. For $u \in V(C), u^{+}$(resp. $u^{-}$) represents its successor (resp. predecessor) on $C$. If $u, v \in V(C)$ then $(u, C, v)$ represents the path given by the consecutive vertices on $C$ ordered from $u$ to $v$ (including $u$ and $v$ ) following the orientation chosen of $C$. The same vertices visited in the opposite orientation give the path $(v, \bar{C}, u)$. Let $R=G \backslash C$. Let $d_{1}$ be a vertex of $C$ such that the number of its neighbors belonging to $R, d_{R}\left(d_{1}\right) \neq 0$. Let $P_{0}$ be a longest path starting from $d_{1}$ on $C$, such that $V\left(P_{0}\right) \backslash\left\{d_{1}\right\} \subseteq R$. Let $x_{0}$ be an extremity of $P_{0}$ in $R$ and $H$ be a connected component of $x_{0}$ in $R$. Let $N_{C}(H)$ be the set of vertices of $C$ which have at least a neighbor in $H$. Then $d_{1} \in N_{C}(H)$. Note $N_{C}(H)=\left\{d_{1}, . ., d_{m}\right\}$, where the indices are taken modulo $m$. Because $C$ is a longest cycle, we have $|V(C)|>\left|N_{C}(H)\right|=m$ and therefore, $m \geq k$. We suppose that following the orientation of $C$, we meet $d_{1}, . ., d_{m}$, respectively. Since every
path $\left(d_{i}, C, d_{i+1}\right)$ contains at least one internal vertex, $C_{i}=\left(d_{i}^{+}, C, d_{i+1}^{-}\right)$ is a, possibly trivial, path $(i=1, \ldots, m)$. Each pair of vertices $d_{i}, d_{j}$ of $\left\{d_{1}, d_{2}, \ldots, d_{m}\right\}$ is joined by a path of length at least two, in which all internal vertices (of this path) are in $H$. We denote this (not oriented) path by $\left(d_{i}, H, d_{j}\right)$.

Given a path $P=\left(a_{1}, a_{2}, \ldots, a_{q}\right), q \geq 2$ and a vertex $u \notin V(P)$. We say that $u$ is $P$-insertible if there exists an $i, 1 \leq i<q$, such that the vertices $a_{i}$ and $a_{i+1}$ are both adjacent to $u$. The edge $\left(a_{i}, a_{i+1}\right)$ is called an insertion edge for $u$. In particular, a vertex $u \in V\left(C_{i}\right)$ is called insertible if it is $\left(d_{i+1}, C, d_{i}\right)$-insertible.

Given four vertices $a, b, u, v$ of $C$, we say that the edges $(u, a),(v, b)$ are crossing (if they exist) if the four vertices arrive on $C$ in the order $a, v, u, b$.

## 3. Definition of an Independent Set

Let us recall the following lemma (see [1]) on properties of insertible vertices, where $m, C_{i}, d_{i}(i=1, \ldots, m)$ are used as defined in Section 2.

Lemma 8. Let $G$ be a $k$-connected and non-hamiltonian graph (with $k \geq 2$ ), $C$ be a longest cycle of $G$ and $H$ be a connected component of $R=G \backslash C$. Then
(a) for each $i \in\{1,2, \ldots, m\}, C_{i}$ contains a non-insertible vertex.

Let $x_{i}$ be the first non-insertible vertex on $C_{i}(i=1,2, \ldots, m)$ and $x_{0} \in$ $V(H)$. Set $W_{0}=V(H)$ and $W_{i}=V\left(\left(d_{i}^{+}, C, x_{i}\right)\right)$ for each $1 \leq i \leq m$. For each $1 \leq i<j \leq m$, choose $w_{i} \in W_{i}$ and $w_{j} \in W_{j}$. Then
(b) $\left(w_{i}, w_{j}\right) \notin E(G)$.
(c) There does not exist a vertex $z \in V\left(\left(w_{i}, C, w_{j}\right)\right)$ such that $\left(w_{i}, z^{+}\right)$, $\left(w_{j}, z\right) \in E$ (i.e., the edges $\left(w_{i}, z^{+}\right)$and $\left(w_{j}, z\right)$ are not crossing).

For a longest cycle $C$ of a $k$-connected ( $k \geq 2$ ) non-hamiltonian graph $G$ with a fixed orientation of $C$, let now $P_{0}, x_{0}, H, m, d_{i}, C_{i}(i=1, \ldots, m)$ be as defined in Section 2, and $W_{0}, x_{i}, W_{i}(i=1, \ldots, m)$ as defined in Lemma 8. Furtheremore, let $X=\left\{x_{0}, x_{1}, \ldots, x_{m}\right\}$. For each $s \in\{1,2, \ldots, m\}$, let $A_{s}$ be the set of vertices $u$ belonging to ( $x_{s}^{+}, C, d_{s+1}^{-}$) which verify the two following properties:
(i) $d_{R}(u) \neq 0$ and
(ii) $\left(x_{s}, u^{-}\right),\left(x_{s}, u\right),\left(x_{s}, u^{+}\right) \in E$ if $x_{s} \neq u^{-}$, and $\left(x_{s}, u\right),\left(x_{s}, u^{+}\right) \in E$, otherwise.


Figure 3
Let

$$
\begin{aligned}
& I=\left\{s \in\{1,2, \ldots, m\}: d_{R}\left(x_{s}\right)=0 \text { and } A_{s}=\emptyset\right\}, \\
& J_{1}=\left\{s \in\{1,2, \ldots, m\}: d_{R}\left(x_{s}\right)=0 \text { and } A_{s} \neq \emptyset\right\} \text { and } \\
& J_{2}=\left\{s \in\{1,2, \ldots, m\}: d_{R}\left(x_{s}\right) \neq 0\right\} .
\end{aligned}
$$

By the definitions of $I, J_{1}$ and $J_{2}$, we can deduce that $I, J_{1}$ and $J_{2}$ form a partition of the set $\{1,2, \ldots, m\}$ (i.e., $I \cap J_{1}=I \cap J_{2}=J_{1} \cap J_{2}=\emptyset$ and $I \cup J_{1} \cup J_{2}=\{1,2, \ldots, m\}$ ).
In the case where $s \in J_{1}$, we denote by $u_{s}$ the first vertex on $\left(x_{s}^{+}, C, d_{s+1}^{-}\right)$ which is in $A_{s}$. Let $b_{s}=u_{s}$ if $s \in J_{1}$ and $b_{s}=x_{s}$ if $s \in I \cup J_{2}$. For $s \in J_{1} \cup J_{2}$, let $P_{s}$ be a longest path with an extremity $b_{s}$ (on the cycle) such that $V\left(P_{s}\right) \backslash\left\{b_{s}\right\} \subset R$. Let $y_{s}$ be the extremity of $P_{s}$ on $R$ (Figure 3).

Let $S=\left\{x_{0}\right\} \cup\left\{x_{i}: i \in I\right\} \cup\left\{y_{s}: s \in J_{1} \cup J_{2}\right\}$. In particular, if $J_{1} \cup J_{2}=\emptyset$, then $S=X=\left\{x_{0}, x_{1}, \ldots, x_{m}\right\}$. It is proved in [2] that $X$ is an independent set in $G^{*}$.

Let $s \in I \cup J_{1} \cup J_{2}$. Put $W_{s}^{\prime}=\left(d_{s}^{+}, C, x_{s}, P_{s}, y_{s}\right)$ if $s \in J_{2}, W_{s}^{\prime}=$ $\left(d_{s}^{+}, C, x_{s}, b_{s}, P_{s}, y_{s}\right)$ if $s \in J_{1}$ and $W_{s}^{\prime}=\left(d_{s}^{+}, C, x_{s}\right)$ if $s \in I$ (see Figure 3, the $W_{s}^{\prime}$ are given in bold). We deduce that if $s \in J_{1} \cup J_{2}$, then $V\left(W_{s}^{\prime}\right)=$ $W_{s} \cup V\left(P_{s}\right)$.

## 4. Lemmas

For all statements of this section we make the following:

Supposition. Let $G$ be a $k$-connected, non-hamiltonian graph with $k \geq 2$ and $C$ a longest cycle of $G$.

Furthermore, we use all denotations introduced in Sections 2 and 3. We shall prove a series of lemmas on some properties of the elements of $S$ in $G$. In the following lemma we prove a property of the vertices of $b_{j}^{-}, b_{j}$, and $b_{j}^{+}$, for $1 \leq j \leq m$ which will be always used in the further proofs.

Lemma 9. For each pair of indices $\{i, j\}$, with $1 \leq i \neq j \leq m, N_{C}\left(b_{j}\right) \cap$ $W_{i}=\emptyset$; in particular, neither $\left(b_{j}^{-}, b_{j}\right)$ nor $\left(b_{j}, b_{j}^{+}\right)$are insertion edges for the vertices of $W_{i}$.

Proof. Suppose first that $j \in I \cup J_{2}$. As $i \neq j$, then by Lemma 8(b), the vertex $b_{j}$ (which is $x_{j}$ in this case) cannot have neighbors in $W_{i}$. In the case where $j \in J_{1}$, we know that $b_{j}=u_{j}$. Let $w$ be a neighbor of $u_{j}$ in $W_{i}$, with $i \neq j$. Observe that the edges $\left(w, u_{j}\right)$ and $\left(x_{j}, u_{j}^{+}\right)$are crossing. We obtain then a contradiction with Lemma 8(c) applied to $w \in W_{i}$ and $x_{j} \in W_{j}$.

We deduce that for each pair of indices $\{i, j\}$, with $1 \leq i \neq j \leq m$, neither $\left(b_{j}^{-}, b_{j}\right)$ nor $\left(b_{j}, b_{j}^{+}\right)$can be an insertion edge for vertices of $W_{i}$.
From now on, we denote by $T=\left(a \ldots x\left[c \ldots s_{j} \ldots d\right] y \ldots b\right)$ a segment of $C$, where the sub-segment $c \ldots s_{j} \ldots d$ (which is in brackets in $T$ ) is considered only if $j \in J_{1}$. In other words, $T=\left(a, \ldots, x, c, \ldots, s_{j}, \ldots, d, y, \ldots, b\right)$ if $j \in J_{1}$ and $T=(a, \ldots, x, y, \ldots, b)$, otherwise.
Now, we prove other properties of $S$ in $G$. In particular, we prove that it is an independent set in $G$ and $G^{*}$.

Lemma 10. The following statements are true:
(a) let $w_{0}$ be a vertex of $W_{0}$ and $w_{i}$ be a vertex of $W_{i} \cup\left\{b_{i}\right\}$, for each $1 \leq i \leq m$. For each pair of indices $\{i, j\}$, with $0 \leq i \neq j \leq m$, the vertices $w_{i}, w_{j}$ are not joined by a path all internal vertices of which are in $R$. In particular, $S$ is an independent set of $m+1$ elements in $G$.
(b) for each $j \in J_{1} \cup J_{2}$, we have $N_{C}\left(y_{j}\right) \subset\left(V(C) \backslash\left[\cup_{i=1}^{m}\left(W_{i} \cup\left\{b_{i}\right\}\right) \cup\right.\right.$ $\left.\left.\left\{d_{j}\right\}\right]\right) \cup\left\{b_{j}\right\}$.
(c) $S$ is an independent set in $G^{*}$.

Proof. (a) The proof is by contradiction. We may suppose $i<j$. Denote by $\left(w_{i}, L, w_{j}\right)$ a path which joins $w_{i}$ and $w_{j}$ and which is assumed to have all its internal vertices in $R$.

Suppose first that $i=0$. Then $1 \leq j \leq m$. Put $P=\left(w_{0}, L,\left[b_{j}, \bar{C}\right], x_{j}\right.$, $\left.\left[b_{j}^{+}\right], C, d_{j+1}\right)$ if $w_{j}=b_{j}$ and $P=\left(w_{0}, L, w_{j}, C, d_{j+1}\right)$ if $w_{j} \in W_{j}$. By definition, all insertible vertices of $W_{j}$ admit their insertion edges on $Q=$ $\left(d_{j+1}, C, d_{j}\right)$. We can insert in $Q$ the vertices of $\left(d_{j}^{+}, C, x_{j}^{-}\right)$if $w_{j}=b_{j}$ and the vertices of $\left(d_{j}^{+}, C, w_{j}^{-}\right)$if $w_{j} \in W_{j}$. Then the new path obtained from $Q$ in this way and combined with a subpath of the walk $\left(d_{j}, H, w_{0}, P\right)$ joining $d_{j}$ and $d_{j+1}$ gives a cycle longer than $C$, which is a contradiction. Consequently, $i \neq 0$. Since there does not exist a path between a vertex of $W_{0}$ and $w_{j}$, then $V(L) \cap V(H)=\emptyset$. In case $w_{i} \in W_{i}$ and $w_{j} \in W_{j}$ the cycle $\left(d_{i}, H, d_{j}, \bar{C}, w_{i}, L, w_{j}, C, d_{i}\right)$ contradicts the maximality of $C$.

Suppose now that $w_{i}=b_{i}$ or $w_{j}=b_{j}$. For each $s \in\{i, j\}$, if $w_{s}=b_{s}$ then we construct a path $Q_{1}$ from the path $Q=\left(d_{i}, \bar{C}, w_{j}, L, w_{i}, C, d_{j}\right)$, by replacing the path $\left(w_{s}, C, d_{s+1}\right)$ by the path $\left(w_{s},\left[b_{s}, \bar{C}\right], x_{s},\left[b_{s}^{+}\right], C, d_{s+1}\right)$. By the definition of the insertion, the edge $\left(b_{j}, b_{j}^{+}\right)$is not an insertion edge for the vertices of $W_{j}$ and the edge $\left(b_{i}, b_{i}^{+}\right)$is not an insertion edge for the vertices of $W_{i}$. Moreover, by Lemma $9,\left(b_{j}, b_{j}^{+}\right)$is not an insertion edge for the vertices of $W_{i}$ and the edge $\left(b_{i}, b_{i}^{+}\right)$is not an insertion edge for the vertices of $W_{j}$, and by Lemma $8(\mathrm{~b})$ there is no edge between $W_{i}$ and $W_{j}$. We deduce that all vertices of $C$ which do not belong to $Q_{1}$ are vertices contained in $\left(d_{i}^{+}, C, x_{i}^{-}\right)$or in $\left(d_{j}^{+}, C, x_{j}^{-}\right)$; therefore, they are insertible with insertion edges belonging to $Q_{1}$. We can insert these insertible vertices in $Q_{1}$, in order to obtain a path $Q_{1}^{\prime}$. The combination of $Q_{1}^{\prime}$ and $\left(d_{i}, H, d_{j}\right)$ forms a cycle longer than $C$, a contradiction.

We deduce that $W_{i}^{\prime} \cap W_{j}^{\prime}=\emptyset, W_{0} \cap W_{j}^{\prime}=\emptyset$ and that any two vertices belonging respectively to $W_{i}^{\prime}$ (or $W_{0}$ ) and $W_{j}^{\prime}$ are not adjacent, for each $1 \leq i \neq j \leq m$. Consequently, $S$ is an independent set of $m+1$ elements in $G$.
(b) Suppose there exists a neighbor $u$ of $y_{j}$ on $\left[\cup_{i=1}^{m}\left(W_{i} \cup\left\{b_{i}\right\}\right) \cup\left\{d_{j}\right\}\right] \backslash$ $\left\{b_{j}\right\}$. By (a), for each $1 \leq i \neq j \leq m$, we have $u \notin N_{C}\left(y_{j}\right) \cap W_{i} \cup\left\{b_{i}\right\}$.

Remark that if $j \in J_{2}$, then $b_{j} \in W_{j}$ (in this case $x_{j}=b_{j}$ ) and $b_{j}$ can be neighbor of $y_{j}$. If $j \in J_{1}$, then $u \neq x_{j}$ because of $d_{R}\left(x_{j}\right)=0$. Thus $u \in V\left(\left(d_{j}, C, x_{j}^{-}\right)\right)$. Let $Q=\left(u, \bar{C},\left[b_{j}^{+}\right], x_{j},\left[C, b_{j}\right]\right)$. By the definition of the insertion, the insertion edges of the vertices of $\left(d_{j}^{+}, C, x_{j}^{-}\right)$belong to $E\left(\left(d_{j+1}, C, d_{j}\right)\right)$. So they belong to $E(Q)$. Then we can construct $Q^{\prime}$ from $Q$ by inserting the vertices of $V\left(\left(u^{+}, C, x_{j}^{-}\right)\right)$(if there are any, i.e., if $u \neq x_{j}^{-}$). Consequently, the paths $Q^{\prime}$ and ( $u, y_{j}, P_{j}, b_{j}$ ) give a cycle longer than $C$, which is a contradiction.
(c) The proof is by contradiction. Since $S$ is an independent set in $G$, we suppose that there exists a pair of distinct vertices $a_{i}, a_{j} \in S$ such that $\left(a_{i}, a_{j}\right) \in E\left(G^{*}\right) \backslash E(G)$. So there exists $v \in J\left(a_{i}, a_{j}\right)$ (such that $\left.\left(a_{i}, v\right),\left(v, a_{j}\right) \in E(G)\right)$. We suppose first that $v \in V(R)$. Without loss of generality, we assume that $i<j$ and $j \in I \cup J_{1} \cup J_{2}$. If $j \in I$ then $a_{j}=x_{j}$. This contradicts the fact that $d_{R}\left(x_{j}\right)=0$. By (a), $j \notin J_{1} \cup J_{2}$. Therefore $v \in V(C)$. In this case $v, v^{+} \in N_{C}\left(a_{i}\right) \cup N_{C}\left(a_{j}\right)$. So either $a_{i}\left(\right.$ or $\left.a_{j}\right)$ is $C$-insertible or $\left(a_{i}, v^{+}\right),\left(v, a_{j}\right)$ are crossing, which is a contradiction.

Hence, by (a) and (c) for each pair of distinct vertices $a_{i}, a_{j} \in S$, $\left(a_{i}, a_{j}\right) \notin E\left(G^{*}\right)$. We deduce that $S$ is an independent set in $G^{*}$.

Corollary 11. For each pair of distinct vertices $a_{i}$ and $a_{j}$ of $S \backslash\left\{x_{0}\right\}$, we have $d_{C}\left(a_{i}\right)+d_{C}\left(a_{j}\right) \leq|C|$.

As usual we write $|Q|$ instead of $|V(Q)|$ for a cycle or a path $Q$.
Proof. We begin by the following claim.
Claim 12. Let $P=\left(u_{1}, u_{2}, \ldots, u_{r}\right)$ be a segment of $C$ (in the same orientation as $C)$ with $r \geq 1$. Let $a_{i}, a_{j} \in S \backslash\left\{x_{0}\right\}$, with $a_{i}, a_{j} \notin V(P)$ and $i \neq j$; it can be assumed - if necessary by commuting the denotations of the indices $i, j$ - that $P$ is a subpath of $\left(b_{i}, C, b_{j}\right)$. Then
(a) $d_{P}\left(a_{i}\right)+d_{P}\left(a_{j}\right) \leq r+1$.
(b) In particular, if $\left(u_{1}, a_{j}\right) \notin E$ and $\left(u_{2}, a_{i}\right) \notin E$, then $d_{P}\left(a_{i}\right)+d_{P}\left(a_{j}\right) \leq r$.

Proof of Claim 12. (a) Define $N_{P}^{+}\left(a_{j}\right)=\left\{u_{i+1}:\left(u_{i}, a_{j}\right) \in E\right\}$, where $u_{r+1}:=u_{r}^{+}$. Thus $N_{P}^{+}\left(a_{j}\right) \subseteq\left\{u_{2}, u_{3}, \ldots, u_{r}, u_{r+1}\right\}$ and $N_{P}\left(a_{i}\right) \subseteq$ $\left\{u_{1}, u_{2}, \ldots, u_{r}\right\}$. One can see that the property (c) of Lemma 8 remains true if we replace $w_{i}, w_{j}$ and $\left(w_{i}, C, w_{j}\right)$ by $a_{i}, a_{j}$ and $\left(b_{i}, C, b_{j}\right)$, respectively. So there does not exist $z$ on $P$ such that the edges $\left(a_{i}, z^{+}\right)$and $\left(z, a_{j}\right)$ are crossing. Then $N_{P}^{+}\left(a_{j}\right) \cap N_{P}\left(a_{i}\right)=\emptyset$. We deduce that $d_{P}\left(a_{j}\right)+d_{P}\left(a_{i}\right)=$ $\left|N_{P}^{+}\left(a_{j}\right) \cap N_{P}\left(a_{i}\right)\right|+\left|N_{P}^{+}\left(a_{j}\right) \cup N_{P}\left(a_{i}\right)\right| \leq r+1$.
(b) If $\left(u_{1}, a_{j}\right) \notin E$ and $\left(u_{2}, a_{i}\right) \notin E$, then $u_{2} \notin N_{P}^{+}\left(a_{j}\right) \cup N_{P}\left(a_{i}\right)$, and hence $d_{P}\left(a_{j}\right)+d_{P}\left(a_{i}\right) \leq r$.

Return now to the proof of Corollary 11.
Put $U_{s}=\left(d_{s}^{+}, C, d_{s+1}\right)$, for each $1 \leq s \leq m$. We prove that for each $1 \leq s \leq m, d_{U_{s}}\left(a_{i}\right)+d_{U_{s}}\left(a_{j}\right) \leq\left|U_{s}\right|$.

Case 1. $i$ and $j$ belong to $I \cup J_{1}$.

Put $L_{s}=\left(x_{s}^{+}, C, d_{s+1}\right)$. By Claim 12(a), $d_{L_{s}}\left(a_{i}\right)+d_{L_{s}}\left(a_{j}\right) \leq\left|L_{s}\right|+1$, for each $s, 1 \leq s \leq m$. If $s \notin\{i, j\}$, by Lemma 10(a), $d_{W_{s}}\left(a_{i}\right)=d_{W_{s}}\left(a_{j}\right)=0$. Then $d_{U_{s}}\left(a_{i}\right)+d_{U_{s}}\left(a_{j}\right) \leq\left|L_{s}\right|+1 \leq\left|U_{s}\right|$ since $\left|U_{s}\right|=\left|L_{s}\right|+\left|W_{s}\right| \geq\left|L_{s}\right|+1$. If $s \in\{i, j\}$, we have $d_{W_{s}}\left(a_{s}\right)=0$ if $s \in J_{1}$ and $d_{W_{s}}\left(a_{s}\right) \leq\left|W_{s}\right|-1$ if $s \in I$. As $d_{W_{s}}\left(a_{s^{\prime}}\right)=0$, with $s^{\prime} \in\{i, j\}$ and $s^{\prime} \neq s$, then $d_{W_{s}}\left(a_{i}\right)+d_{W_{s}}\left(a_{j}\right) \leq\left|W_{s}\right|-1$. Thus $d_{U_{s}}\left(a_{i}\right)+d_{U_{s}}\left(a_{j}\right) \leq\left(\left|L_{s}\right|+1\right)+\left(\left|W_{s}\right|-1\right)=\left|U_{s}\right|$.

Case 2. $i$ or $j$ belongs to $J_{2}$.
If $s \notin\{i, j\}$ or $s \in\{i, j\} \cap\left(I \cup J_{1}\right)$, the arguments are similar than those of the above case. If $s \in\{i, j\} \cap J_{2}$, put $L_{s}=\left(x_{s}, C, d_{s+1}\right)$. Without loss of generality, put $s=i$ and $s \in J_{2}$. By Lemma 10 (a), we have $\left(x_{s}, a_{j}\right) \notin E$. As $s \in J_{2}$ and $C$ is maximal, then $\left(x_{s}^{+}, a_{s}\right) \notin E$. Remark that by Claim $12(\mathrm{~b})$, these two last hypotheses allow to deduce that $d_{L_{s}}\left(a_{s}\right)+d_{L_{s}}\left(a_{j}\right) \leq\left|L_{s}\right|$. Moreover, by Lemma $10(\mathrm{~b}), d_{W_{s} \backslash\left\{x_{s}\right\}}\left(a_{s}\right)+d_{W_{s} \backslash\left\{x_{s}\right\}}\left(a_{j}\right)=0$. As we have always, $\left|U_{s}\right| \geq\left|L_{s}\right|$, then $d_{U_{s}}\left(a_{s}\right)+d_{U_{s}}\left(a_{j}\right) \leq\left|U_{s}\right|$.
Consequently, for each $1 \leq s \leq m, d_{U_{s}}\left(a_{i}\right)+d_{U_{s}}\left(a_{j}\right) \leq\left|U_{s}\right|$. As $|C|=$ $\sum_{s=1}^{s=m}\left|U_{s}\right|$, then $d_{C}\left(a_{i}\right)+d_{C}\left(a_{j}\right) \leq|C|$.

Lemma 13. If $i \in I$ then $d_{G}\left(x_{i}\right)=d_{C}\left(x_{i}\right)$ and $d_{G}\left(x_{0}\right)+d_{C}\left(x_{i}\right) \leq|C|$.
Proof. The proof is by contradiction and it is similar to the one of Lemma 5 given in [5], except that instead of considering $d_{i}^{+}$we consider $x_{i}$. It is clear that in the proof we take into account the insertible vertices in all the constructions of longest cycles (as we have done it in the above lemmas), this contradicts the maximality of $C$.

Finally, we recall the following lemma of Fournier and Fraisse (see [5]) which is useful for the proofs of theorems.

Lemma 14. Let $P$ be a path of maximum length between all paths which have a given extremity, $a$, on $C$ and all the other vertices are not on $C$. Let $x$ the second extremity of $P$. Then we have $d_{G}(x) \leq \frac{|C|}{2}$.

## 5. Proofs of Theorems

We define the following variants of $I, J_{1}$ and $J_{2}$ (which had been defined in Section 3):

$$
\begin{aligned}
& I(k)=\{1,2, \ldots, k\} \cap I \\
& J_{1}(k)=\{1,2, \ldots, k\} \cap J_{1} \text { and }
\end{aligned}
$$

$$
J_{2}(k)=\{1,2, \ldots, k\} \cap J_{2} .
$$

It is clear that $I(k) \cup J_{1}(k) \cup J_{2}(k)=\{1,2, \ldots, k\}$. In order to prove Theorems 5 and 6 , we suppose that the graph $G$ is not hamiltonian.

Proof of Theorem 5. If there exists at least an index $i \in\{1,2\}$, such that $i \in J_{1} \cup J_{2}$, then by Lemma $14, d_{G}\left(y_{i}\right) \leq \frac{|C|}{2}$ and $d_{G}\left(x_{0}\right) \leq \frac{|C|}{2}$. Thus using Lemma 10 (c), $\sigma_{2}^{\circ} \leq d_{G}\left(y_{i}\right)+d_{G}\left(x_{0}\right) \leq|C|$. Consequently, $|C| \geq \sigma_{2}^{\circ}$. Otherwise, for each $i \in\{1,2\}$, we have $i \in I$. By Corollary 11, $d_{C}\left(x_{1}\right)+$ $d_{C}\left(x_{2}\right) \leq|C|$ and since $\left(x_{1}, x_{2}\right) \notin E\left(G^{*}\right)$ and $d_{R}\left(x_{1}\right)=d_{R}\left(x_{2}\right)=0$ then $\sigma_{2}^{\circ} \leq d_{G}\left(x_{1}\right)+d_{G}\left(x_{2}\right) \leq|C|$. Consequently, $|C| \geq \sigma_{2}^{\circ}$.

Proof of Theorem 6. We may suppose that $|I(k)|=p \leq k$ and $I(k)=$ $\left\{i_{1}, i_{2}, \ldots, i_{p}\right\}$. Recall that $d_{R}\left(x_{i}\right)=0$, for $i \in I$. If $p=0$ then by Lemmas 10 and $14, \sigma_{k+1}^{\circ} \leq \sum_{j \in J_{1}(k) \cup J_{2}(k) \cup\{0\}} d_{G}\left(y_{j}\right) \leq \frac{(k+1)}{2}|C|$. Consequently, $|C| \geq$ $2 \frac{\sigma_{k+1}^{\circ}}{k+1}$. If $p=1$, then by Lemma $13, d_{G}\left(x_{0}\right)+d_{C}\left(x_{i_{1}}\right) \leq|C|$ and by Lemma $14, d_{G}\left(y_{j}\right) \leq \frac{|C|}{2}$, for each $j \in J_{1}(k) \cup J_{2}(k)$. As $d_{R}\left(x_{i_{1}}\right)=0$, we get then $\sigma_{k+1}^{\circ} \leq d_{G}\left(x_{0}\right)+d_{G}\left(x_{i_{1}}\right)+\sum_{j \in J_{1}(k) \cup J_{2}(k)} d_{G}\left(y_{j}\right) \leq \frac{(k+1)}{2}|C|$. So, $|C| \geq 2 \frac{\sigma_{k+1}^{o}}{k+1}$.
Finally, if $p \geq 2$ then by Corollary 11,

$$
\begin{aligned}
& d_{C}\left(x_{i_{1}}\right)+d_{C}\left(x_{i_{2}}\right) \leq|C|, \\
& d_{C}\left(x_{i_{2}}\right)+d_{C}\left(x_{i_{3}}\right) \leq|C|, \\
& \vdots \\
& d_{C}\left(x_{i_{p-1}}\right)+d_{C}\left(x_{i_{p}}\right) \leq|C|, \\
& d_{C}\left(x_{i_{p}}\right)+d_{C}\left(x_{i_{1}}\right) \leq|C| .
\end{aligned}
$$

Thus $\sum_{i \in I(k)} d_{C}\left(x_{i}\right) \leq p \frac{|C|}{2}$. By Lemma 14, $\sum_{j \in J_{1}(k) \cup J_{2}(k) \cup\{0\}} d_{G}\left(y_{j}\right) \leq$ $(k-p+1) \frac{|C|}{2}$. So $\sigma_{k+1}^{\circ} \leq \sum_{i \in I(k)} d_{C}\left(x_{i}\right)+\sum_{j \in J_{1}(k) \cup J_{2}(k) \cup\{0\}} d_{G}\left(y_{j}\right) \leq$ $p \frac{|C|}{2}+(k-p+1) \frac{|C|}{2} \leq(k+1) \frac{|C|}{2}$. Then $|C| \geq 2 \frac{\sigma_{k+1}^{\circ}}{k+1}$.

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