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DOMINATION SUBDIVISION NUMBERS

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Abstract

A set S of vertices of a graph $G = (V, E)$ is a *dominating set* if every vertex of $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and the *domination subdivision number* $sd_\gamma(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the domination number. Arumugam conjectured that $1 \le sd_\gamma(G) \le 3$ for any graph G. We give a counterexample to this conjecture. On the other hand, we show that $sd_\gamma(G) \le \gamma(G) + 1$ for any graph G without isolated vertices, and give constant upper bounds on $sd_\gamma(G)$ for several families of graphs.

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1. Introduction

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood of* v, denoted by $N(v)$, is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. Given a set $S \subseteq V$ of vertices and a vertex $u \in S$, the *private neighbor set of* u, with respect to S, is the set $pn[u, S] = N[u] - N[S - \{u\}]$. We say that every vertex $v \in pn[u, S]$ is a *private neighbor* of u (with respect to S). Such a vertex v is adjacent to u but is not adjacent to any other vertex of S. Note that if a vertex $u \in S$ is not adjacent to any other vertex of S, then it is an isolated vertex in the subgraph $G[S]$ induced by S. In this case, $u \in pn[u, S]$, and we say that u is its own private neighbor. A set S is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a $\gamma(G)$*-set*. For a more thorough treatment of domination parameters and for terminology not presented here, see [5, 6].

An edge $uv \in E(G)$ is *subdivided* by deleting the edge uv, and adding a new vertex x and two new edges ux and xv. The vertex x is called a *subdivision vertex*. Arumugam [1] defined the *domination subdivision number* of a graph G, which we denote $sd_\gamma(G)$, to equal the minimum number of edges that must be subdivided (where no edge in G can be subdivided more than once, that is, no edge incident to a subdivision vertex can be subdivided) in order to create a graph whose domination number is greater than the domination number of G. We assume here that every graph is of order $n \geq 3$, since the domination number of the graph K_2 does not change when its only edge is subdivided.

Although it may not be immediately obvious that $sd_\gamma(G)$ is defined for all connected graphs of order $n \geq 3$, we will show this shortly.

Arumugam showed the following result for trees.

Theorem 1 [1]**.** *For any tree T of order $n \geq 3$, $1 \leq sd_\gamma(T) \leq 3$.*

Moreover, he made the following interesting conjecture.

Conjecture 2 [1]**.** *For any graph G of order $n \geq 3$, $1 \leq sd_\gamma(G) \leq 3$.*

Our purpose is threefold, namely, to settle Conjecture 2; to give an upper bound on the domination subdivision number of a graph in terms of its domination number; and to give constant upper bounds on the domination subdivision numbers for several families of graphs. In Section 2 we show that $sd_\gamma(G)$ is defined for all connected graphs G of order $n \geq 3$, and show that $sd_\gamma(G) \leq \gamma(G) + 1$ for all graphs G without isolated vertices. In Section 3, we give a counterexample to Conjecture 2 by showing that $sd_\gamma(G) = 4$ for a particular family of graphs. On the other hand, since the only counterexample that we have found has $sd_\gamma(G) = 4$, we think there may still be a constant upper bound on $sd_\gamma(G)$ for all graphs G. In support of this, in Section 4 we establish constant upper bounds on $sd_\gamma(G)$ for several classes of graphs.

2. Bounds

Haynes, Hedetniemi, and Hedetniemi [4] gave the following upper bound for $sd_\gamma(G)$ for arbitrary graphs.

Theorem 3. *For any connected graph G and edge uv, where $deg(u) \geq 2$ and $deg(v) \geq 2$,*

$$sd_\gamma(G) \leq deg(u) + deg(v) - 1.$$

Using Theorem 3 one can show that $sd_\gamma(G)$ is defined for every connected graph G of order $n \geq 3$. Every such graph G either has an edge uv, where $deg(u) \geq 2$ and $deg(v) \geq 2$, or it does not. If G has such an edge uv, then the proof of Theorem 3 shows that the domination number of G must increase if every edge incident to either u or v is subdivided. If G does not have such an edge, then for every edge uv, either $deg(u) = 1$ or $deg(v) = 1$. But this implies that G is a star $K_{1,n}$. But for $G = K_{1,n}$, since $n \geq 3$, it is easy to see that the domination number is increased by subdividing any edge, that is, $sd_\gamma(G) = 1$. Therefore, $sd_\gamma(G)$ is defined for every connected graph of order $n \geq 3$.

Although the upper bound in Theorem 3 for the subdivision number of an arbitrary graph is not a constant, it can be used to obtain a constant upper bound for the domination subdivision number of all graphs in some classes of graphs, such as the following.

Corollary 4. *For any $r \times s$ grid graph $G_{r,s}$,*

$$1 \leq sd_\gamma(G_{r,s}) \leq 4.$$

Corollary 4 follows from the simple observation that if either $r = 1$ or $s = 1$, then $sd_\gamma(G_{r,s}) \leq 3$ by Theorem 1, and otherwise $G_{r,s}$ must contain a corner vertex of degree two which is adjacent to a vertex of degree three.

Corollary 5. *For any k-regular graph G, where $k \geq 2$,*

$$1 \leq sd_\gamma(G) \leq 2k - 1.$$

Corollary 6. *For any cubic graph G,*

$$1 \leq sd_\gamma(G) \leq 5.$$

Next we prove a couple of useful lemmas. A vertex which is adjacent to only one other vertex is called a *leaf*, and its neighbor is called a *support vertex.* A vertex which is adjacent to two or more leaves is called a *strong support vertex.*

Lemma 7. *If G has a strong support vertex, then $sd_\gamma(G) = 1$.*

Proof. Let w be adjacent to leaves u and v. Subdividing either edge wu or wv will increase the domination number. Thus, $sd_\gamma(G) = 1$. ■

Lemma 8. *If G has adjacent support vertices, then $sd_\gamma(G) \leq 3$.*

Proof. Let w and x be adjacent support vertices, and let u and y be leaves adjacent to w and x, respectively. Subdividing edges wu, wx, and xy will increase the domination number. Thus, $sd_\gamma(G) \leq 3$. ■

We next show that $sd_\gamma(G) = 1$ for any graph G having $\gamma(G) = 1$.

Proposition 9. *If G is a graph of order $n \geq 3$ and $\gamma(G) = 1$, then $sd_\gamma(G) = 1$.*

Proof. If you subdivide any edge in a graph of order n whose domination number equals one, the resulting graph cannot have domination number equal to one. ■

We are able to determine an upper bound on $sd_\gamma(G)$ in terms of $\gamma(G)$ for graphs G with no isolated vertices. For this purpose we need the following result which establishes a connection between the matching number of a graph and its domination subdivision number. A *matching* in a graph G is a set M of edges having the property that no two edges in M have a vertex in common. The maximum cardinality of a matching in G is called the *matching number* of G and is denoted $\beta_1(G)$.

Theorem 10. *If G is a graph with $\gamma(G) = k$ and $\beta_1(G) \geq k+1$, then $sd_\gamma(G) \leq k+1$.*

Proof. Let G be a graph with $\gamma(G) = k$ and assume that $\beta_1(G) \geq k+1$. Let the edges of a matching of order $k+1$ be given by $M = \{u_1v_1, u_2v_2, \ldots, u_{k+1}v_{k+1}\}$, and let G' be the graph which results from subdividing each edge in M, by adding subdivision vertices $a_1, a_2, \ldots, a_{k+1}$, where a_i subdivides the edge u_iv_i for $1 \leq i \leq k+1$. Now any dominating set of G' must contain at least one vertex from each triple $\{u_i, a_i, v_i\}$. Since these $k+1$ triples are pairwise disjoint, it follows that $\gamma(G') \geq k+1$. Therefore, $sd_\gamma(G) \leq k+1$. ∎

Corollary 11. *If $\gamma(G) < \beta_1(G)$, then $sd_\gamma(G) \leq \gamma(G)+1$.*

Note that for any graph G without isolates, $\gamma(G) \leq \beta_1(G)$ (see [5]). Thus to prove that $sd_\gamma(G) \leq \gamma(G)+1$ for all connected graphs G, we need to consider the graphs G for which $\gamma(G) = \beta_1(G)$. We first consider the graphs which have a perfect matching, i.e., the graphs G for which $\gamma(G) = \beta_1(G) = n/2$. These graphs were characterized independently by Payan and Xuong [8] and by Fink, Jacobson, Kinch, and Roberts [2]. The *corona* $H \circ K_1$ is the graph formed from a copy of H by adding a new vertex v' and a pendant edge vv' for each vertex $v \in V(H)$.

Theorem 12 [2, 8]**.** *If G is a connected graph having $\gamma(G) = n/2$, then either G is isomorphic to the cycle C_4 or $G = H \circ K_1$ is the corona of some connected graph H.*

Our next corollary follows directly from the facts that $sd_\gamma(C_4) = 3$ and $sd_\gamma(H \circ K_1) \leq 3$, for any nontrivial connected graph H (by Lemma 8).

Corollary 13. *If G is a graph with $\gamma(G) = \beta_1(G) = n/2$, then $sd_\gamma(G) \leq 3$.*

The only remaining graphs that we need to consider are those having $\gamma(G) = \beta_1(G) < n/2$, that is, G has equal domination and matching numbers, but does not have a perfect matching. These graphs were characterized independently by Randerath and Volkmann [9] and Hare and McCuaig [3]. We will use the following result from [3].

Let $\mathcal{L}$ be the set of leaves of G; $\mathcal{N}$ be the set of support vertices of G; and define $\mathcal{I} = \{x \in V(G) - (\mathcal{N} \cup \mathcal{L}) : N(x) \subseteq \mathcal{N}\}$. Note that $\mathcal{I}$ is an independent set of vertices.

Let $\mathcal{G}$ be the class of graphs G without isolated vertices having the following properties.

1. If H_2 is the collection of the bipartite connected components of $G-(\mathcal{N}\cup\mathcal{L})$, then the vertices of H_2 can be partitioned into two independent sets A and B such that:
For any two distinct vertices a_1 and a_2 in A with a common neighbor b in B, there exists a vertex $b_1 \in B-\{b\}$ such that $N_G(b_1) = \{a_1, a_2\}$. Furthermore, the only vertices of H_2 which have neighbors in $\mathcal{N}$ are vertices in B.

2. Every nonbipartite component H of $G-(\mathcal{N}\cup\mathcal{L})$ is one of the graphs shown in Figure 1, where each of the dashed edges may or may not be an edge of H. Furthermore, only the starred vertices can have neighbors in $\mathcal{N}$.

Theorem 14 (Hare and McCuaig [3])**.** *A graph G with no isolated vertices and no perfect matching has $\gamma(G) = \beta_1(G)$ if and only if G is in $\mathcal{G}$.*

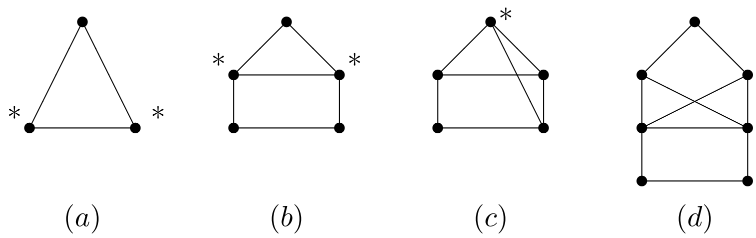

Figure 1. Nonbipartite components of $G-(\mathcal{N}\cup\mathcal{L})$. Dashed line represents an optional edge, and an asterisk indicates that a vertex may have a neighbor in $\mathcal{N}$.

Theorem 15. *If G is a connected graph G of order $n \geq 3$ having $\gamma(G) = \beta_1(G)$, then $sd_\gamma(G) \leq 3$.*

Proof. From Proposition 9 we know that if $\gamma(G) = 1$, then $sd_\gamma(G) = 1$; and from Corollary 13 we know that if $\gamma(G) = \beta_1(G)$ and G has a perfect matching, then $sd_\gamma(G) \leq 3$. Thus, the only case remaining is a graph G, where $\gamma(G) = \beta_1(G)$ and G does not have a perfect matching.

From Theorem 14, we know that $G \in \mathcal{G}$. Assume that G contains a nonbipartite component in $G-(\mathcal{N}\cup\mathcal{L})$. Theorem 14 asserts that the only allowable nonbipartite components are illustrated in Figure 1. Since components (b) and (d) have adjacent vertices of degree two (in G), we know from Theorem 3 that $sd_\gamma(G) \leq 3$ if $G-(\mathcal{N}\cup\mathcal{L})$ has either of these components. Component (c) is a C_5 with two chords. One can verify that there exist three edges in this graph whose subdivisions will result in a component requiring at least three vertices to dominate it in G, and hence the resulting graph will have a domination number larger than $\gamma(G)$. Thus, if this component is present in $G - (\mathcal{N} \cup \mathcal{L})$, then $sd_\gamma(G) \leq 3$. Finally, component (a) is a K_3 where two of its vertices may have neighbors in $\mathcal{N}$. Since G is connected, at least one of these vertices, say x, is adjacent to a vertex in $\mathcal{N}$. It is easy to see that subdividing all three edges of this K_3 causes the domination number to increase. Hence, $sd_\gamma(G) \leq 3$ in this case.

Therefore, we may assume that G does not contain any nonbipartite components in $G - (\mathcal{N} \cup \mathcal{L})$. If $\mathcal{N}$ is not an independent set, then Lemma 8 implies that $sd_\gamma(G) \leq 3$.

Hence, if G is not bipartite, then $sd_\gamma(G) \leq 3$.

Thus, we may assume that G is bipartite. Let the sets A and B be defined as in Property 1 of the graphs in $\mathcal{G}$. Suppose $\mathcal{I} \neq \emptyset$.

Let $uv, vw \in E(G)$ where $u \in \mathcal{I}$, $v \in \mathcal{N}$, and $w \in \mathcal{L}$. Then the subdivision of uv, and vw will cause the domination number to increase. Thus, $sd_\gamma(G) \leq 2$.

Hence, we may assume that $\mathcal{I} = \emptyset$.

Suppose there exists a vertex $a \in A$ such that $N(a) \cap N(A - a) = \emptyset$. Then the vertices in $N(a)$ are only adjacent with vertices in $\mathcal{N}$. Since a is not a leaf, a must be adjacent to at least two vertices in B. Let $b_1, b_2 \in B$, $v \in \mathcal{N}$, and $w \in \mathcal{L}$ such that $\{ab_1, ab_2, vb_1, vw\} \subseteq E(G)$. Then the subdivision of ab_2, vb_1, and vw will cause the domination number to increase. Therefore, $sd_\gamma(G) \leq 3$.

Hence, we may assume that for every $a \in A$, $N(a) \cap N(A - \{a\}) \neq \emptyset$.

We have that $\mathcal{N}$ is independent and that $\mathcal{I} = \emptyset$. Since G is a connected graph of order $n \geq 3$, it follows that $V - (\mathcal{N} \cup \mathcal{L}) \neq \emptyset$. But since $\mathcal{I} = \emptyset$, it follows that $B \neq \emptyset$ and $A \neq \emptyset$.

Let $a_1 \in A$. Since $N(a) - N(A - \{a_1\}) \neq \emptyset$, there exists $a_2 \in A$ such that $N(a_1) \cap N(a_2) \neq \emptyset$. Moreover, given the structure of the graphs in $\mathcal{G}$, if a_1 and a_2 have a common neighbor $b \in B$, then there exists a vertex $b_1 \in B$ such that $N(b_1) = \{a_1, a_2\}$. Using the same argument and b_1 as the common neighbor we deduce that there is a vertex $b_2 \neq b_1$ in B such that $N(b_2) = \{a_1, a_2\}$. Thus in G, the vertices a_1, a_2, b_1, and b_2 induce a C_4. Subdividing three edges of the C_4 yields a C_7 which requires at least three vertices to dominate; hence, the domination number increases. Hence, $sd_\gamma(G) \leq 3$. Thus, if G is bipartite (and $\gamma(G) = \beta_1(G)$), then $sd_\gamma(G) \leq 3$.

In conclusion, if $\gamma(G) = \beta_1(G)$ and G does not have a perfect matching, then $sd_\gamma(G) \leq 3$. ■

Our main result of this section follows directly from Proposition 9, Corollary 11, Corollary 13, and Theorem 15.

Theorem 16. *If G is a connected graph of order $n \geq 3$, then*

$$sd_\gamma(G) \leq \gamma(G) + 1.$$

3. The Counterexample

In this section we settle Conjecture 2 with a counterexample. Let G_t denote the Cartesian product $K_t \times K_t$. We can think of G_t as having t disjoint copies of K_t in "rows" and t disjoint copies of K_t in "columns". In other words we can think of the vertices of the $K_t \times K_t$ as a matrix where vertex $v_{i,j}$ is in the ith row (copy of K_t) and the jth column (copy of K_t). For ease of discussion, we will be using the words row and column to mean a "copy of K_t".

Theorem 17. *For any positive integer $t \geq 4$, $sd_\gamma(K_t \times K_t) = 4$.*

Proof. Let $G_t = K_t \times K_t$. We first show that $\gamma(G_t) = t$. It is immediate that $\gamma(G_t) \leq t$, since $\{v_{1,1}, v_{2,1}, \ldots, v_{t,1}\}$ is a dominating set. Assume that $\gamma(G_t) \leq t - 1$. Then, without loss of generality, we may assume that row 1 has no vertex in any $\gamma(G_t)$-set S. But then the vertices in S dominate at most $t - 1$ vertices of row 1. Thus, there is at least one undominated vertex in row 1, a contradiction. Therefore, $\gamma(G_t) = t$.

To aid in our arguments, we observe the following facts about γ-sets S in the graph G_t for $t \geq 4$.

1. Every dominating set contains a permutation of the row indices or a permutation of the column indices (or both). These permutations must exist since $\{v_{1,1}, v_{2,2}, \ldots, v_{t,t}\}$ must all be dominated and a vertex is only dominated by a vertex in the same row or column.

2. For any $k \times k$ square B, $|V(B) \cap S| \leq k$. This result builds on the first observation, for if $|V(B) \cap S| \geq k + 1$, then at least one of those k rows has two or more vertices in it and at least one of the columns has two or more vertices in it. Hence, no permutation can exist.

Next we show that $sd_\gamma(G_t) \geq 4$ by showing that if any set of three or fewer edges of G_t are subdivided, then the resulting graph can still be dominated by t vertices. Notice that if a set S dominates the graph resulting from subdividing three arbitrary edges, then S dominates the graph resulting from subdividing any pair of these edges. Hence, we consider the subdivision of three arbitrary edges of G_t to form G'_t. Without loss of generality, the only distinct possibilities are the following:

Case 1. All three edges are from the same row (respectively, column) i. Then $S = \{v_{i,j} : 1 \leq j \leq t\}$ dominates G'_t.

Case 2. Two edges are from the same row (respectively, column) and one edge is from a different row or column.

(a) The two edges from the same row (respectively, column) are adjacent. Without loss of generality, we may assume that these edges are $v_{1,1}v_{1,2}$ and $v_{1,2}v_{1,3}$. There are eight distinct possibilities, up to isomorphism, for the third edge:

(1) $v_{1,2}v_{2,2}$. Then $S = \{v_{1,2}, v_{2,1}, v_{i,i} : 3 \leq i \leq t\}$ dominates G'_t.
(2) $v_{1,1}v_{2,1}$. Then $S = \{v_{1,1}, v_{1,3}, v_{2,2}, v_{i,i} : 4 \leq i \leq t\}$ dominates G'_t.
(3) $v_{2,1}v_{2,2}$. Then $S = \{v_{1,2}, v_{2,1}, v_{i,i} : 3 \leq i \leq t\}$ dominates G'_t.
(4) $v_{2,1}v_{3,1}$. Then $S = \{v_{1,2}, v_{2,1}, v_{i,i} : 3 \leq i \leq t\}$ dominates G'_t.
(5) $v_{2,3}v_{2,4}$. Then $S = \{v_{1,2}, v_{3,2}, v_{2,3}, v_{i,i} : 4 \leq i \leq t\}$ dominates G'_t.
(6) $v_{2,4}v_{2,5}$ if $t \geq 5$. Then $S = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,4}, v_{i,i} : 5 \leq i \leq t\}$ dominates G'_t.
(7) $v_{2,4}v_{3,4}$. Then $S = \{v_{1,2}, v_{2,1}, v_{3,4}, v_{4,3}, v_{i,i} : 5 \leq i \leq t\}$ dominates G'_t.
(8) $v_{1,4}v_{2,4}$. Then $S = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,4}, v_{1,i} : 5 \leq i \leq t\}$.

(b) The two edges in the same row (respectively, column) are not adjacent. Without loss of generality, we may assume that these edges are $v_{1,1}v_{1,2}$ and $v_{1,3}v_{1,4}$. There are eight distinct possibilities for the third edge:

(1) $v_{1,1}v_{2,1}$. Then $S = \{v_{1,1}, v_{1,3}, v_{2,2}, v_{i,i} : 4 \leq i \leq t\}$ dominates G'_t.
(2) $v_{2,1}v_{2,2}$. Then $S = \{v_{1,2}, v_{2,1}, v_{1,3}, v_{i,i} : 4 \leq i \leq t\}$ dominates G'_t.
(3) $v_{2,2}v_{2,3}$. Then $S = \{v_{1,1}, v_{1,3}, v_{2,2}, v_{i,i} : 4 \leq i \leq t\}$ dominates G'_t.
(4) $v_{2,1}v_{3,1}$. Then $S = \{v_{1,2}, v_{2,1}, v_{3,3}, v_{1,4}, v_{i,i} : 5 \leq i \leq t\}$ dominates G'_t.
(5) $v_{2,4}v_{2,5}$ if $t \geq 5$. Then $S = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,4}, v_{i,i} : 5 \leq i \leq t\}$ dominates G'_t.

(6) $v_{2,5}v_{3,5}$ if $t \geq 5$. Then $S = \{v_{1,1}, v_{1,3}, v_{3,2}, v_{2,5}, v_{2,4}, v_{i,i} : \ 6 \leq i \leq t\}$ dominates G'_t.

(7) $v_{2,5}v_{2,6}$ if $t \geq 5$. Then $S = \{v_{2,5}, v_{1,i} : \ 1 \leq i \neq 5 \leq t\}$ dominates G'_t.

(8) $v_{1,5}v_{2,5}$ if $t \geq 5$. Then $S = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{2,4}, v_{1,i} : \ 5 \leq i \leq t\}$.

Case 3. The three edges are from distinct copies of K_t, that is, no two edges are from the same row (respectively, column).

(a) The three edges are in distinct rows (respectively, columns) of G_t. If these edges are incident to vertices in at least three different columns (respectively, rows), then it is straightforward to find a dominating set that contains t vertices that are a permutation of the columns (respectively, rows). The only other possibility is that the edges are, without loss of generality, $v_{1,1}v_{1,2}, v_{2,1}v_{2,2}$, and $v_{3,1}v_{3,2}$. In this case, $S = \{v_{1,1}, v_{2,2}, v_{3,1}, v_{i,i} : 4 \leq i \leq t\}$ dominates G'_t.

(b) Two of the edges are in distinct rows (respectively, columns) and one is from a column (respectively, row) of G_t. In this case, one can select an endvertex from each of the subdivided edges in such a way to dominate three columns (respectively, rows) with the exception of one vertex, say $v_{i,j}$. Thus, any vertex from row i in one of the $t-3$ remaining columns can dominate $v_{i,j}$. At this point, we have four vertices dominating four columns, that is, $t-4$ vertices can be selected to dominate the remaining $t-4$ columns. Hence, G'_t can be dominated with t vertices.

Hence, in every case, $\gamma(G'_t) = \gamma(G_t) = t$, so $sd_\gamma(G_t) \geq 4$. All that remains is to show that $sd_\gamma(G_t) \leq 4$. Form G'_t from G_t by subdividing four edges in a 3×3 block of vertices as illustrated by Figure 2. Then any dominating set of G'_t must contain four vertices from the 3×3 block. Therefore, there are only $t-4$ vertices available to dominate $t-3$ rows and $t-3$ columns which cannot be done. Thus, by a subdivision of four edges, the resulting graph has a domination number greater than t, so $sd_\gamma(G_t) \leq 4$ for $t \geq 3$. ■

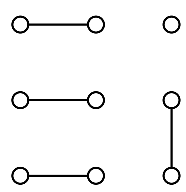

Figure 2. Four edges whose subdivision increases the domination number.

4. Constant Upper Bounds on the Domination Subdivision Number for Specific Families of Graphs

The next result can be used to establish constant upper bounds for $sd_\gamma(G)$ for several classes of graphs.

Define a vertex u to be *triangular* if every vertex v adjacent to u is contained in a triangle with u. Stated equivalently, a vertex is triangular if the induced subgraph $G[N(u)]$ contains no isolated vertices. Notice, by definition, if a vertex u is triangular, then $deg(u) \geq 2$. We say that a graph G is *triangular* if it contains at least one triangular vertex, and is *completely triangular* if every vertex in G is triangular.

Theorem 18. *If a graph G contains a triangular vertex u, then $sd_\gamma(G) \leq deg(u) + 1$.*

Proof. Let $u \in V$ and assume that u is a triangular vertex, that is, $deg(u) \geq 2$ and $G[N(u)]$ has no isolated vertices. Let G_u be the graph which results from subdividing every edge incident with u in G, and one additional edge between two vertices v, w in $G[N(u)]$. We know that such an edge vw exists, since u is triangular. Let a be the subdivision vertex between v and w in G_u.

Now either $\gamma(G_u) > \gamma(G)$, in which case $sd_\gamma(G) \leq deg(u) + 1$, or $\gamma(G_u) = \gamma(G)$. Assume that $\gamma(G_u) = \gamma(G)$, and let S be any $\gamma(G_u)$-set.

Case 1. $u \in S$. In this case at least one of $\{v, a, w\}$ must also be in S to dominate a. But if either $v \in S$ or $w \in S$ (or both), then $S - \{u\}$ is also a dominating set of G, contradicting the minimality of $\gamma(G)$. If $v, w \notin S$ and $a \in S$, it follows that $S - \{a, u\} \cup \{v\}$ is a dominating set of G of cardinality $\gamma(G) - 1$, again a contradiction.

Case 2. $u \notin S$, $b \in S$, where b is the subdivision vertex between u and v. Again one of $\{v, a, w\}$ must be in S. If $v \in S$ or $w \in S$, then $S - \{b\}$ is a dominating set of G; and if $a \in S$, then $S - \{a, b\} \cup \{v\}$ is a dominating set of G. In all cases, we contradict the minimality of $\gamma(G)$.

Case 3. $u \notin S$, $b \notin S$, and $c \in S$, where c is a subdivision vertex on an edge ux, where $x \neq v$, and $x \neq w$. We can assume, without loss of generality, that S contains only one subdivision vertex adjacent to u, since if there are any other subdivision vertices in S which are adjacent to u, they can be exchanged with the vertices adjacent to u in G to which they are adjacent in G_u. This means that S contains exactly one subdivision vertex, say c, adjacent to u.

This also means that every vertex, other than possibly x, which is adjacent to u in G must be in S, in order to dominate all of the subdivision vertices (other than c) adjacent to u in G_u. Notice that we can assume that $x \notin S$, since if $x \in S$, then $S - \{c\}$ is a dominating set of G.

Since $G[N(u)]$ has no isolated vertices, by assumption, x must be adjacent to at least one vertex, say y, where y is also adjacent to u in G. We have already established that $y \in S$. It follows therefore that $pn[c, S] = \{c, u\}$, and therefore $S - \{c\}$ is a dominating set of G, a contradiction. ■

Corollary 19. *For every completely triangular graph G, $sd_\gamma(G) \leq \delta(G)+1$.*

A vertex v in a graph G is called *simplicial* if the induced subgraph $G[N[v]]$ is a complete graph. Clearly every simplicial vertex of degree at least two is triangular.

Corollary 20. *If a graph G contains a simplicial vertex u of degree at least two, then $sd_\gamma(G) \leq deg(u) + 1$.*

A *k-tree* is any graph which can be obtained from a complete graph on $k+1$ vertices, by repeatedly adding a new vertex and joining it to every vertex in a complete subgraph of the existing graph of order k. It is easy to see that every k-tree is completely triangular.

Corollary 21. *For every k-tree G, $k \geq 2$, $sd_\gamma(G) \leq k + 1$.*

A graph G is called *chordal* if every cycle of G of length greater than three has a *chord*, that is, an edge between two nonconsecutive vertices of the cycle. Every k-tree is a chordal graph. In fact, it is easy to see that every 2-connected chordal graph is completely triangular.

Corollary 22. *For every 2-connected chordal graph G, $sd_\gamma(G) \leq \delta(G) + 1$.*

A *maximal outerplanar graph* is a 2-tree which is obtained from a copy of K_3 by repeatedly adding a new vertex and joining it to two adjacent vertices on the exterior face of the existing graph. It is easy to see that every maximal outerplanar graph G contains at least two vertices of degree two, which is the minimum degree of any vertex in G, and that each such vertex is a simplicial vertex. Notice that every maximal outerplanar graph is completely triangular.

Corollary 23. *For every maximal outerplanar graph G,*

$$sd_\gamma(G) \leq \delta(G) + 1 = 3.$$

The same upper bound for $sd_\gamma(G)$ in fact holds for any graph having a vertex of degree two which is contained in a triangle since such a vertex is necessarily triangular.

Corollary 24. *For any graph G having a vertex of degree two which forms a triangle with two other vertices,*

$$1 \leq sd_\gamma(G) \leq 3.$$

It is well known that every maximal planar graph contains at least one vertex of degree at most five. One can also observe that every maximal planar graph is completely triangular.

Corollary 25. *For every maximal planar graph G, $sd_\gamma(G) \leq \delta(G) + 1 \leq 6$.*

These observations suggest the following:

Conjecture 26. *For every graph G with $\delta(G) \geq 2$, $sd_\gamma(G) \leq \delta(G) + 1$.*

Corollary 20 can be improved if we know more about the structure of the simplicial vertices in a graph. A *clique* is any maximal complete subgraph of a graph G.

Theorem 27. *If G is a graph having a clique containing exactly two simplicial vertices and at least two non-simplicial vertices, then $1 \leq sd_\gamma(G) \leq 2$.*

Proof. It is straightforward to see that if you subdivide one edge between the two adjacent simplicial vertices in such a clique, and then subdivide any one edge between a simplicial vertex and a non-simplicial vertex in the clique, then the resulting graph will have a domination number greater than the domination number of the original graph. ∎

Given three or more simplicial vertices in a clique, the domination subdivision number is even smaller.

Theorem 28. *If G is a graph having three or more pairwise-adjacent simplicial vertices, then $sd_\gamma(G) = 1$.*

Proof. Let u, v, w be three pairwise adjacent simplicial vertices in a graph G. We will assume that these vertices are all adjacent to at least one non-simplicial vertex, else G is a complete graph, and $sd_\gamma(G) = sd_\gamma(K_n) = 1$.

Let C be the set of non-simplicial vertices adjacent to u, v, w, and let D be the set of simplicial vertices in the clique containing u, v, w and the vertices in C. Let G_a be the graph obtained from G by subdividing the edge uv, and let vertex a be the subdivision vertex on this subdivided edge. We will show that $\gamma(G_a) > \gamma(G)$.

First, we will show that no $\gamma(G)$-set S is a dominating set of G_a.

Case 1. $S \cap C \neq \emptyset$. This implies that $S \cap D = \emptyset$, else S is not a minimal dominating set of G. But this, in turn, implies that S does not dominate the vertex a.

Case 2. $S \cap C = \emptyset$. This implies that $S \cap D \neq \emptyset$, which in turn implies that $|S \cap D| = 1$. If $u \in S$, then S does not dominate v. If $v \in S$, then S does not dominate u. If neither u nor v is in S, then S does not dominate a.

Second, we will show that no set S_a containing a and of cardinality $\gamma(G)$ is a dominating set of G_a. Let S_a be a set of cardinality $\gamma(G)$ containing a.

Case 1. $S_a \cap C \neq \emptyset$: in this case $S_a - \{a\}$ is a dominating set of G, contradicting the minimality of $\gamma(G)$.

Case 2. $S_a \cap C = \emptyset$ but $S_a \cap D \neq \emptyset$: again, in this case, $S_a - \{a\}$ dominates G, a contradiction.

Case 3. $S_a \cap C = \emptyset$ and $S_a \cap D = \emptyset$; in this case S_a does not dominate vertex w. ∎

5. Open Problems

Although we have been able to establish constant upper bounds for the domination subdivision numbers of several infinite families of graphs, we do not know in general if these bounds are sharp. We conclude by presenting a list of open problems suggested by this paper.

1. Is the following revised Arumugam conjecture true? For any graph G, $1 \leq sd_\gamma(G) \leq 4$.

2. Is the bound in Corollary 4 sharp, that is, does there exist a grid graph G for which $sd_\gamma(G) = 4$?
3. Is the bound in Corollary 6 sharp, that is, does there exist a cubic graph G for which $sd_\gamma(G) = 5$?
4. Characterize the class of graphs for which $sd_\gamma(G) = \gamma(G) + 1$.
5. Is the bound in Corollary 23 sharp, that is, does there exist a maximal outerplanar graph G for which $sd_\gamma(G) = 3$?
6. Is the bound in Corollary 25 sharp, that is, does there exist a maximal planar graph G for which $sd_\gamma(G) = 6$?
7. Is the following conjecture true? For every graph G with $\delta(G) \geq 2$, $sd_\gamma(G) \leq \delta(G) + 1$.
8. Characterize the class of graphs for which $sd_\gamma(G) = 1$.
9. Is $sd_\gamma(K_t \times K_t \times K_t) = 5$?

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