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ON VARIETIES OF ORGRAPHS

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Abstract

In this paper we investigate varieties of orgraphs (that is, oriented graphs) as classes of orgraphs closed under isomorphic images, suborgraph identifications and induced suborgraphs, and we study the lattice of varieties of tournament-free orgraphs.

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1. Introduction

In mathematics we often study classes of structures of the same type closed under some constructions. In particular in universal algebra we consider classes of algebras of the same type closed under direct products, subalgebras and homomorphic images [4]. In theory of posets, some authors investigated classes of posets closed under direct products and retracts [7]. The mentioned classes of algebras and posets are called varieties of algebras and varieties of posets, respectively. Analogously, there is a literature on varieties (properties) of graphs closed under isomorphic images and moreover, closed under induced subgraphs [11], closed under induced subgraphs and identifications [8], closed under induced subgraphs and contractions [9], closed under generalized hereditary operators [3], [10], etc. An interesting survey paper on additive and hereditary properties of graphs is [2].

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In this paper we pay attention to varieties of oriented graphs (called *orgraphs* [1]). Throughout this paper by orgraph we mean a directed graph $\mathcal{G}(V,E)$ without loops with the following property:

for every two distinct vertices $u,v \in V$ at most one of edges uv and vu is an arc from E.

We briefly write uv instead of $[u,v]$ for vertices $u,v \in V$. The cardinality of a set A we will denote by $|A|$.

An orgraph $\mathcal{G}(V,E)$ is called *a tournament* if for each pair of vertices $u,v \in V$ either $uv \in E$ or $vu \in E$ (see [5]).

Let $\mathcal{G}(V,E)$ be an orgraph. Whenever uv is an arc the vertex u is called an *adjacent vertex to* v and v is called an *adjacent vertex from* u. An *outdegree* (an *indegree*) of a vertex $v \in V$ in the orgraph $\mathcal{G}(V,E)$ is the number of vertices adjacent from v (to v). If the outdegree of a vertex v is i and the indegree of v is j we will say that v is of type $v_{(j)}^{(i)}$ and write simply v_j^i, when no confusion can arise.

A suborgraph $\mathcal{P}(V_1,E_1)$ of an orgraph $\mathcal{G}(V,E)$ is called a *weak path* (of length n) if the next three conditions are satisfied:

(i) $V_1 = \{v_0,v_1,\dots,v_n\}$, where the vertices $v_0,v_1,\dots,v_n \in V$ are all distinct;

(ii) either $v_iv_{i+1} \in E_1$ or $v_{i+1}v_i \in E_1$, for each $i = 0,1,\dots,n-1$;

(iii) if $\{i,j\} \neq \{k,k+1\}$, for each $k \in \{0,1,\dots,n-1\}$, then $v_iv_j \notin E_1$.

We often refer to a weak path by the natural sequence of its vertices, e.g. $\mathcal{P} = v_0v_1\dots v_n$, and we call $\mathcal{P}$ a weak path between v_0 and v_n. By a *path* from u to w we mean a weak path $v_0v_1\dots v_n$ for which $u = v_0$, $w = v_n$ and v_iv_{i+1} is an arc for each $i = 0,1,\dots n-1$.

Let $\mathcal{P} = v_0v_1\dots v_n$ be a weak path in an orgraph $\mathcal{G}(V,E)$ and let $n \geq 2$. If $v_nv_0 \in E$ or $v_0v_n \in E$ then the corresponding suborgraph $\mathcal{C} = \mathcal{P} + v_nv_0$ or $\mathcal{C} = \mathcal{P} + v_0v_n$ is called a *weak circle* of $\mathcal{G}$. As with weak paths, we often denote a weak circle by its (cyclic) sequence of vertices; the above weak circle might be written as $v_0v_1\dots v_nv_0$. A weak circle $v_0v_1\dots v_nv_0$ is called a *circle* if $v_0v_1, v_1v_2, \dots, v_nv_0$ are its arcs.

We will briefly write w-path and w-circle instead of weak path and weak circle, respectively. Analogously to the notions of w-path and path, and w-circle and circle, the notions of a w-*tree* and *tree* (with a fixed root) can be introduced.

An orgraph $\mathcal{G}(V, E)$ is called *weakly connected* if there exists a weak path between u and v for every pair of vertices $u, v \in V$.

Let $\mathcal{C} = v_0v_1 \dots v_nv_0$ be a w-circle. Suppose that the vertex v_0 is (in the orgraph $\mathcal{C}$) of type v_0^2. Denote successively by $\mathcal{P}_1$, $\mathcal{P}_2, \dots, \mathcal{P}_k$ paths in the orgraph $\mathcal{C}$ from vertices v_i of type v_0^2 to vertices v_j of type v_2^0, for which $i < j$ and $v_1 \in V(P_1)$ and analogously by $\mathcal{N}_1$, $\mathcal{N}_2, \dots, \mathcal{N}_k$ opposite paths from vertices v_i of type v_0^2 to vertices v_j of type v_2^0 for which $i > j$ or $i = 0$ (and so $v_n \in V(\mathcal{N}_k)$). In this case we will say that the w-circle $\mathcal{C}$ is of type $\mathcal{C}_{(p_1,n_1,p_2,n_2,\dots,p_k,n_k)}$, where p_i, n_i are the lengths of the paths $\mathcal{P}_i$, $\mathcal{N}_i$, respectively, for each $i \in \{1, \dots, k\}$. If the w-circle $\mathcal{C}$ does not contain a vertex of type v_0^2 then $\mathcal{C}$ is of the type $\mathcal{C}_{(l,0)}$, where l is the length of the circle $\mathcal{C}$.

The circle $\mathcal{C}_{(4,0)}$ and the w-circles $\mathcal{C}_{(3,1)}$, $\mathcal{C}_{(2,2)}$ and $\mathcal{C}_{(1,1,1,1)}$ are depicted in Figure 1a-d.

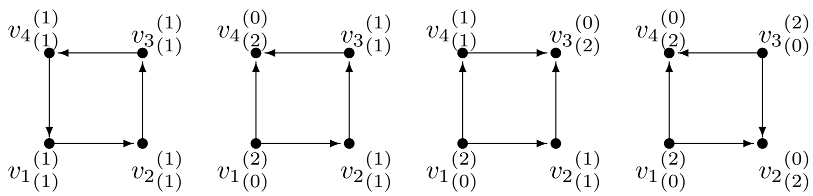

Figure 1a-d

Note that the notation of w-circles is ambiguous. For instance, the w-circle of the type $\mathcal{C}_{(3,1,2,2,1,4)}$ is also of the type $\mathcal{C}_{(2,2,1,4,3,1)}$, $\mathcal{C}_{(1,4,3,1,2,2)}$ (we have moved the members in two positions), $\mathcal{C}_{(4,1,2,2,1,3)}$ (here we have taken the inverse order), $\mathcal{C}_{(2,2,1,3,4,1)}$, $\mathcal{C}_{(1,3,4,1,2,2)}$ (we have again moved the members in two positions). However, we often identify a w-circle with its type.

Definition 1.1. Let $\mathcal{C}_{(p_1,n_1,p_2,n_2,\dots,p_k,n_k)}$ be a w-circle. The number

$$|p_1 + \dots + p_k - n_1 \dots - n_k|$$

is said to be the *characteristic* of the w-circle $\mathcal{C}_{(p_1,n_1,p_2,n_2,\dots,p_k,n_k)}$. We will denote it by $ch(\mathcal{C})$.

The operator of suborgraph identification is a modification of the subgraph identification from [8].

Definition 1.2. Let $\mathcal{G}_1(V_1, E_1)$, $\mathcal{G}_2(V_2, E_2)$ be disjoint weakly connected orgraphs, let $\mathcal{G}'_1(V'_1, E'_1)$, $\mathcal{G}'_2(V'_2, E'_2)$ be weakly connected induced suborgraphs of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively and let $f : \mathcal{G}'_1 \to \mathcal{G}'_2$ be an isomorphism. The *suborgraph identification of* $\mathcal{G}_1$ *with* $\mathcal{G}_2$ *under* f is the orgraph $\mathcal{G}(V, E) = \mathcal{G}_1 \cup^f \mathcal{G}_2$, where
$V = V_1 \cup (V_2 - V'_2)$,
$E = \{uv;\ u, v \in V \text{ and } uv \in E_1 \cup E_2 \text{ or } f(u)v \in E_2 \text{ or } uf(v) \in E_2\}$.

Where no confusion can arise we will call it simply *the suborgraph identification under the suborgraph* $\mathcal{G}'_1$ instead of the suborgraph identification under the isomorphism f; or we briefly call it gluing in the suborgraph $\mathcal{G}'_1$.

As we focus on varieties of orgraphs in the case when orgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are not disjoint we may take instead of the orgraph $\mathcal{G}_2$ an orgraph $\mathcal{G}_3$ isomorphic with $\mathcal{G}_2$ and disjoint with $\mathcal{G}_1$ (for details see [8]).

The fact that $f : \mathcal{G}'_1 \to \mathcal{G}'_2$ is an isomorphism of a weakly connected induced suborgraph $\mathcal{G}'_1 \subseteq \mathcal{G}_1$ onto a weakly connected induced suborgraph $\mathcal{G}'_2 \subseteq \mathcal{G}_2$ will be denoted by $f : \mathcal{G}_1 \rightarrowtail \mathcal{G}_2$.

It is obvious that $\mathcal{G}_1 \cup^f \mathcal{G}_2 \cong \mathcal{G}_2 \cup^{f^{-1}} \mathcal{G}_1$.

Let $\mathbb{K}$ be a family of weakly connected orgraphs. Denote

$$\gamma(\mathbb{K}) = \{\mathcal{G}_1 \cup^f \mathcal{G}_2;\ \mathcal{G}_1, \mathcal{G}_2 \in \mathbb{K},\ f : \mathcal{G}_1 \rightarrowtail \mathcal{G}_2\}$$

and

$$\Gamma(\mathbb{K}) = \gamma(\mathbb{K}) \cup \gamma^2(\mathbb{K}) \cup \cdots = \cup_{n=1}^{\infty} \gamma^n(\mathbb{K}),$$

where $\gamma^k(\mathbb{K}) = \gamma(\gamma^{k-1}(\mathbb{K}))$, for every $k > 1$.

Since $\mathcal{G} \cup^{id} \mathcal{G} = \mathcal{G}$, where id is the identity on the orgraph $\mathcal{G}$, we get

$$\mathbb{K} \subseteq \gamma(\mathbb{K}) \subseteq \gamma^2(\mathbb{K}) \subseteq \ldots.$$

Further, let $S(\mathbb{K})$ and $I(\mathbb{K})$ denote the set of all weakly connected induced suborgraphs of orgraphs in $\mathbb{K}$ and the set of all isomorphic images of orgraphs in $\mathbb{K}$, respectively.

To put things in the right context within set theory, we will assume that vertex sets of all orgraphs are subsets of a fixed countable infinite set W.

Definition 1.3. A set $\mathbb{K}$ of orgraphs closed under isomorphic images, induced weakly connected suborgraphs and suborgraph identifications will be called *a variety*, i.e., $\mathbb{K}$ is a variety if

$$I(\mathbb{K}) \subseteq \mathbb{K}\,,\ S(\mathbb{K}) \subseteq \mathbb{K} \text{ and } \Gamma(\mathbb{K}) \subseteq \mathbb{K}\,.$$

Obviously, I, S, Γ are closure operators on the system of all sets of weakly connected orgraphs. By [5, Theorem 5.2] we obtain the next statement.

Theorem 1.1. *The set of all varieties of orgraphs with set inclusion as the partial ordering is a complete lattice.*

The next statement comes from [8].

Lemma 1.2. *Let $\mathcal{G}$ be a weakly connected orgraph. The orgraph $\mathcal{G}$ contains a family $\mathbb{T}$ of tournaments (as induced suborgraphs) and a family $\mathbb{C}$ of w-circles such that $\mathcal{G} \in I\Gamma(\mathbb{T} \cup \mathbb{C})$.*

2. The Lattice of Varieties

Henceforth by an orgraph we will mean a weakly connected orgraph . The lattice of varieties of orgraphs will be denoted by $\mathbf{L}(I, S, \Gamma)$ or briefly by $\mathbf{L}$. In this part we describe the lower part of the lattice $\mathbf{L}(I, S, \Gamma)$ which consists of all varieties containing no nontrivial tournament.

Obviously, the least element of the lattice $\mathbf{L}$ is the variety $\mathbf{0}$ of all one-vertex orgraphs. The smallest variety containing a set K of orgraphs will be denoted by $\mathbf{V}(K)$ or $\langle K \rangle$ and we will say that it is generated by K. If $K = \{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$, we denote it simply by $\mathbf{V}(\mathcal{G}_1, \ldots, \mathcal{G}_n)$ or $\langle \mathcal{G}_1, \ldots, \mathcal{G}_n \rangle$.

Denote by $\mathbf{D}$ the lattice of all nonnegative integers with the relation of divisibility as the partial ordering, and by $\mathbf{D}^d$ the dual lattice of $\mathbf{D}$ (i.e., $m \leq n$ in $\mathbf{D}^d$ if n divides m). Further, denote by $\mathbf{3}$ the three-element chain and by $\oplus$ the linear (ordinal) sum (i.e., $\mathbf{P} \oplus \mathbf{Q}$ is defined by taking the following order relation on $P \cup Q$: $x \leq y$ iff $x \leq y$ in $\mathbf{P}$ or in $\mathbf{Q}$ or $x \in P$, $y \in Q$).

We will say that a variety $\mathbf{V}$ is tournament-free if $\mathcal{T}_n \notin \mathbf{V}$ for each $n \geq 3$ (i.e., if it contains no nontrivial tournament).

Theorem 2.1. (Main result) *The lattice $\mathbf{L}_1$ of all tournament-free varieties of orgraphs is isomorphic with the lattice $\mathbf{3} \oplus \mathbf{D}^d$ (see Figure 2). Moreover, the lattice $\mathbf{L}_1$ is an ideal of the lattice $\mathbf{L}(I, S, \Gamma)$.*

In Figure 2 the generators are used to denote the corresponding varieties. In the remaining part of the paper we are successively proving the main Theorem 2.1.

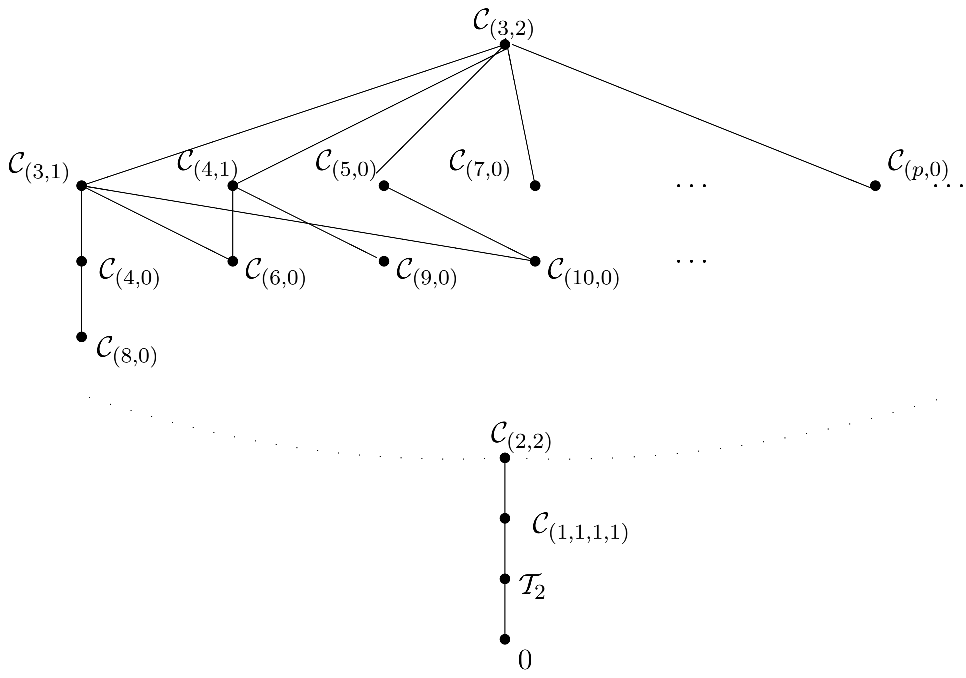

Figure 2

Proposition 2.2. *The only atom of the lattice* $\mathbf{L}(I,S,\Gamma)$ *is the variety* $\mathbf{T} = V(\mathcal{T}_2)$ *of all w-trees.*

The proof is straightforward.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be w-circles and let $\mathcal{G} = \mathcal{C}_1 \cup^f \mathcal{C}_2$. Suppose that the w-circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are glued in a w-path $\mathcal{W} = w_1 \dots w_k$. Denote by $\mathcal{C}^{\mathcal{W}}$ the circle obtained from $\mathcal{G}$ by deletion of the vertices $w_2, \dots, w_{k-1}$, if $k > 2$, and obtained by deletion of the edge $w_1 w_2$ if $k = 2$.

Lemma 2.3. *Let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be w-circles and let* $\mathcal{G} = \mathcal{C}_1 \cup^f \mathcal{C}_2$. *Suppose that the w-circles* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are glued in a w-path* $\mathcal{W} = w_1 \dots w_k$. *The characteristic of the w-circle* $\mathcal{C}^{\mathcal{W}}$ *is* $ch(\mathcal{C}^{\mathcal{W}}) = ch(\mathcal{C}_1) + ch(\mathcal{C}_2)$ *or* $ch(\mathcal{C}^{\mathcal{W}}) = |ch(\mathcal{C}_1) - ch(\mathcal{C}_2)|$.

Proof. Let the w-circle $\mathcal{C}_1$ be of the type $\mathcal{C}_{(p_1,n_1,p_2,n_2,\dots,p_k,n_k)}$ and $\mathcal{P}_1, \dots, \mathcal{P}_k$ be all its paths from vertices v_i of type v_0^2 to vertices v_j of type v_2^0 for $i < j$ and $\mathcal{N}_1, \dots, \mathcal{N}_k$ be its opposite paths. Put

$$E^p(\mathcal{C}_1) = E(\mathcal{P}_1) \cup \dots \cup E(\mathcal{P}_k),$$

$$E^n(\mathcal{C}_1) = E(\mathcal{N}_1) \cup \cdots \cup E(\mathcal{N}_k),$$

and let analogously be defined $E^p(\mathcal{C}_2)$ and $E^n(\mathcal{C}_2)$. Further, let us denote by x the cardinality of the set $E^p(\mathcal{C}_1) \cap E(\mathcal{W})$ and by y the cardinality of the set $E^n(C_1) \cap E(W)$.

One can easily check that (with respect to the symmetry) either

$$\begin{aligned} ch(\mathcal{C}^{\mathcal{W}}) &= \big| (|E^p(\mathcal{C}_1)| - x) + (|E^p(\mathcal{C}_2)| - y) - ((|E^n(\mathcal{C}_1)| - y) \\ &\qquad + (|E^n(\mathcal{C}_2)| - x)) \big| \\ &= \big| |E^p(\mathcal{C}_1)| - |E^n(\mathcal{C}_1)| + |E^p(\mathcal{C}_2)| - |E^n(\mathcal{C}_2)| \big| \end{aligned}$$

or

$$\begin{aligned} ch(\mathcal{C}^{\mathcal{W}}) &= |(|E^p(\mathcal{C}_1)| - x + |E^n(\mathcal{C}_2)| - y) - (|E^n(\mathcal{C}_1)| - y + |E^p(\mathcal{C}_2)| - x)| \\ &= \big| |E^p(\mathcal{C}_1)| - |E^n(\mathcal{C}_1)| - (|E^p(\mathcal{C}_2)| - |E^n(\mathcal{C}_2)|) \big| \end{aligned}$$

and the lemma follows. ■

Proposition 2.4. *The only variety covering the variety* $\mathbf{T}$ *of all* w*-trees is the variety* $\mathbf{V}_1 = \mathbf{V}(\mathcal{C}_{(1,1,1,1)})$ *generated by the* w*-circle* $\mathcal{C}_{(1,1,1,1)}$. *An orgraph* $\mathcal{G}$ *belongs to the variety* $\mathbf{V}_1$ *if and only if every* w*-circle in* $\mathcal{G}$ *is of the type* $\mathcal{C}_{(1,1,\ldots,1)}$.

Proof. 1. If $\mathbf{V} \supsetneqq \mathbf{T}$ then there exists an orgraph $\mathcal{G} \in \mathbf{V}$ containing a w-circle $\mathcal{C}$ as an induced suborgraph. Thus, $\mathcal{C} \in \mathbf{V}$ and we can show that the variety $\mathbf{V}$ also contains a w-circle $\mathcal{C}_1$ with the induced suborgraph in Figure 3.

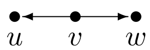

Figure 3

By the suborgraph identification of two copies of $\mathcal{C}_1$ in the w-path $\mathcal{C}_1 - v$ we obtain an orgraph which contains the w-circle $\mathcal{C}_{(1,1,1,1)}$ as an induced suborgraph. Therefore $\mathcal{C}_{(1,1,1,1)} \in \mathbf{V}$. It implies $\mathbf{V}_1 \subseteq \mathbf{V}$, i.e., the variety $\mathbf{V}_1 = \mathbf{V}(\mathcal{C}_{(1,1,1,1)})$ is the only variety covering the variety $\mathbf{T}$ of all w-trees.

2. Obviously, every w-circle of the type $\mathcal{C}_{(1,\ldots,1)}$ belongs to $\mathbf{V}_1$. Hence if every w-circle of an orgraph $\mathcal{G}$ is of the type $\mathcal{C}_{(1,\ldots,1)}$ then $\mathcal{G} \in \mathbf{V}_1$ by Lemma 1.2.

Now, we will show that every w-circle of any orgraph from $\mathbf{V}_1$ is of the type $\mathcal{C}_{(1,1,\dots,1)}$. It is sufficient to show that if $\mathcal{G} = \mathcal{G}_1 \cup^f \mathcal{G}_2$, $f : \mathcal{G}'_1 \to \mathcal{G}'_2$ and every w-circle of both orgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is of the type $\mathcal{C}_{(1,1,\dots,1)}$ then every w-circle of $\mathcal{G}$ is of the type $\mathcal{C}_{(1,1,\dots,1)}$, too.

Let $\mathcal{C} = v_1v_2\dots v_nv_1$ be a w-circle of the orgraph $\mathcal{G}$. If the w-circle $\mathcal{C}$ is a suborgraph of the orgraph $\mathcal{G}_1$ or $\mathcal{G}_2$ then $\mathcal{C}$ is of the type $\mathcal{C}_{(1,1,\dots,1)}$ by the assumption. Let $\mathcal{C}$ be a suborgraph neither of $\mathcal{G}_1$ nor $\mathcal{G}_2$. Let $v_i, v_{i+1}, \dots, v_{i+j} \in V(\mathcal{C})$; we say that $v_i \curvearrowright v_{i+j}$ is a *jump* in $\mathcal{G}$ if $v_i \in V(\mathcal{G}_1) - V(\mathcal{G}'_1)$, $v_{i+1}, \dots, v_{i+j-1} \in V(\mathcal{G}'_1)$, $v_{i+j} \in V(\mathcal{G}_2) - V(\mathcal{G}'_2)$ or $v_i \in V(\mathcal{G}_2) - V(\mathcal{G}'_2)$, $v_{i+1}, \dots, v_{i+j-1} \in V(\mathcal{G}'_1)$, $v_{i+j} \in V(\mathcal{G}_1) - V(\mathcal{G}'_1)$ (see Figure 4).

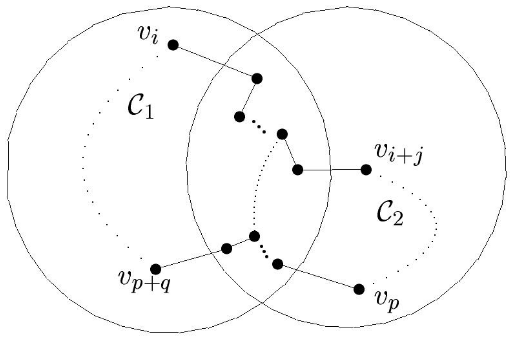

Figure 4

We proceed by induction on the number of jumps of the w-circle $\mathcal{C}$. Firstly, we suppose that there are only two jumps (in $\mathcal{G}$) $v_i \curvearrowright v_{i+j}$ and $v_p \curvearrowright v_{p+q}$, $i < p$. Since $\mathcal{G}_1 \cup^f \mathcal{G}_2$ is the suborgraph identification under a weakly connected suborgraph $\mathcal{G}'_1$ there exists a w-path $v_{i+1}w_1w_2\dots w_kv_{p+q-1}$ in $\mathcal{G}'_1$. If this w-path is disjoint with the circle $\mathcal{C}$ we get a w-circle $\mathcal{C}_1$ of the orgraph $\mathcal{G}_1$ and a w-circle $\mathcal{C}_2$ of the orgraph $\mathcal{G}_2$ which both contain the mentioned w-path or its part (see Figure 4). Both w-circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are of the type $\mathcal{C}_{(1,1,\dots,1)}$, hence the w-circle $\mathcal{C}$ is of the same type, too. If the mentioned w-path is not disjoint with the circle $\mathcal{C}$ we get w-circles $\mathcal{C}_1, \dots, \mathcal{C}_k$ such that each of them belongs to either $\mathcal{G}_1$ or $\mathcal{G}_2$ and each of them contains a part of the w-path $v_{i+1}w_1w_2\dots w_kv_{p+q-1}$. Since we can get the circle $\mathcal{C}$ by successive gluing the w-circles $\mathcal{C}_1, \dots, \mathcal{C}_k$ of the type $\mathcal{C}_{(1,1,\dots,1)}$, the w-circle $\mathcal{C}$ is of the same type, too.

Assuming the statement for w-circles with less than $2r$ jumps, we can prove it for $2r$ jumps. Without loss of generality we can assume that

$v_i \curvearrowright v_{i+j}$, $v_p \curvearrowright v_{p+q}$ are jumps and that for each jump $v_k \curvearrowright v_{k+s}$ of the w-circle $\mathcal{C}$, $i \leq k \leq p$ holds. Analogously as in the case two jumps we can get w-circles $\mathcal{C}_1, \dots, \mathcal{C}_k$ having less than $2r$ jumps and each of them belongs to either $\mathcal{G}_1$ or $\mathcal{G}_2$. By the assumption $\mathcal{C}_1, \dots, \mathcal{C}_k$ are of the type $\mathcal{C}_{(1,1,\dots,1)}$, therefore the w-circle $\mathcal{C}$ is also of the same type. ■

Proposition 2.5. *The variety* $\mathbf{V}_1$ *is covered only by the variety* $\mathbf{V}_2 = \mathbf{V}(\mathcal{C}_{(2,2)})$ *generated by the* w*-circle* $\mathcal{C}_{(2,2)}$. *The variety* $\mathbf{V}_2$ *contains an orgraph* $\mathcal{G}$ *if and only if characteristics of all* w*-circles of the orgraph* $\mathcal{G}$ *are zero.*

Proof. 1. If $\mathbf{V} > \mathbf{V}_1$ holds then the variety $\mathbf{V}$ contains an orgraph $\mathcal{G}$ containing a w-circle different from $\mathcal{C}_{(1,1,\dots,1)}$ as an suborgraph. Hence the variety $\mathbf{V}$ contains a w-circle $\mathcal{C}$ with the induced suborgraph in Figure 5.

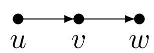

Figure 5

By the suborgraph identification of two copies of the w-circle $\mathcal{C}$ we can get the orgraph with the induced suborgraph $\mathcal{C}_{(2,2)}$.

(2a) We will show that an orgraph $\mathcal{G}$ belongs to $\mathbf{V}_2$ if characteristics of all its w-circles are zero.

Firstly we show by induction that every w-circle with the characteristic zero belongs to the variety $\mathbf{V}_2$. Let every w-circle with the characteristic 0 of a length less $2n$, $n \geq 3$, belong to the variety $\mathbf{V}_2$. Let the characteristic of a w-circle $\mathcal{C}$ be 0 and the length of $\mathcal{C}$ be $2n$. If $\mathcal{C} = \mathcal{C}_{(1,\dots,1)}$ then $\mathcal{C} \in \mathbf{V}_2$. If $\mathcal{C}$ is not of this type then $\mathcal{C}$ contains the suborgraph depicted in Figure 6. We distinguish two cases.

Figure 6

Figure 7

Figure 8

Let the w-circle $\mathcal{C}$ contain the suborgraph in Figure 7. The variety $\mathbf{V}_2$ contains the orgraph in Figure 9, hence $\mathbf{V}_2$ contains its induced suborgraph $\mathcal{C}_{(3,3)}$. By the suborgraph identification of the w-circle $\mathcal{C}$ with the orgraph

$\mathcal{C}_{(3,3)}$ under the w-path in Figure 7 we get the orgraph in Figure 10. We can get this orgraph also by the suborgraph identification of the w-circle $\mathcal{C}_{(3,3)}$ with a w-circle $\mathcal{C}'$ of the length $2n-2$ with the characteristic zero (Figure 10). By the induction assumption $\mathcal{C}' \in \mathbf{V}_2$ hence $\mathcal{C} \in \mathbf{V}_2$, too.

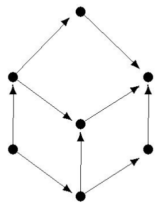

Figure 9

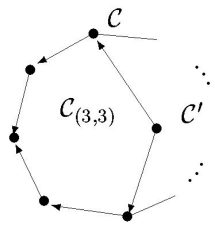

Figure 10

If the w-circle $\mathcal{C}$ contains the suborgraph in Figure 8 we can analogously show that $\mathcal{C} \in \mathbf{V}_2$.

If the characteristic of every w-circle of an orgraph $\mathcal{G}$ is zero then $\mathcal{G}$ contains neither the w-circle $\mathcal{C}_{(3,0)}$ nor the w-circle $\mathcal{C}_{(2,1)}$. Therefore no tournament $\mathcal{T}_n$, $n \geq 3$, is an suborgraph of $\mathcal{G}$. Therefore $\mathcal{G} \in \mathbf{V}_2$ by Lemma 1.2.

(2b) We can prove (similarly to the last part of the proof of Proposition 2.4) that the characteristic of any w-circle $\mathcal{C}$ of an orgraph $\mathcal{G} \in \mathbf{V}_2$ is zero (by induction under the number of jumps of the w-circle $\mathcal{C}$). ∎

Lemma 2.6. *Let* $\mathbf{V}$ *be a variety for which* $\mathbf{V} \geq \mathbf{V}_2$. *Let* $\mathcal{C}$ *be a* w*-circle different from* w*-circles of the type* $\mathcal{C}_{(1,1,\dots,1)}$. *If* $\mathcal{C} \in \mathbf{V}$ *and* $\mathcal{C}'$ *is a* w*-circle for which*

$$ch(\mathcal{C}') = ch(\mathcal{C}), \quad \mathcal{C}' \neq \mathcal{C}_{(3,0)} \text{ and } \mathcal{C}' \neq \mathcal{C}_{(2,1)}$$

then $\mathcal{C}' \in \mathbf{V}$, *too.*

Proof. (a) Let $\mathcal{C} \neq \mathcal{C}_{(3,0)}$ and $\mathcal{C} \neq \mathcal{C}_{(2,1)}$. The w-circles $\mathcal{C}_{(2,2,1,1)}$ and $\mathcal{C}_{(2,1,1,2)}$ belong to $\mathbf{V}$ by Proposition 2.5.

The suborgraph identification of $\mathcal{C}_{(2,2,1,1)}$ with a w-circle enables us to obtain a w-circle of the same characteristic and — longer than the mentioned w-circle if we glue in the w-path

 or

— shorter than the mentioned w-circle if we glue in the w-path or or

Suborgraph identifications with $\mathcal{C}_{(2,2,1,1)}$ and $\mathcal{C}_{(2,1,1,2)}$, respectively, also enables us to change the configurations of the arcs

 with

and

 with

Therefore we can obtain the w-circle $\mathcal{C}'$ from the w-circle $\mathcal{C}$ by suborgraph identifications with the w-circles $\mathcal{C}_{(2,2,1,1)}$, $\mathcal{C}_{(2,1,1,2)}$ and taking induced suborgraphs.

(b) If $\mathcal{C} = \mathcal{C}_{(3,0)}$ we obtain $\mathcal{C}_{(4,1)}$ as the induced suborgraph of the orgraph in Figure 11, if $\mathcal{C} = \mathcal{C}_{(2,1)}$ we obtain $\mathcal{C}_{(3,2)}$ as the induced suborgraph of the orgraph in Figure 12 and the rest of the proof runs as in the case (a).

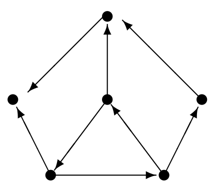

Figure 11

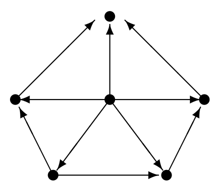

Figure 12

$\blacksquare$

Corollary 2.7. *The variety* $\mathbf{V}_2$ *is generated by any* w*-circle* $\mathcal{C}$ *with the characteristic zero and different from the type* $\mathcal{C}_{(1,1\ldots,1)}$. *If* n *is positive integer,* p*,* q *nonnegative integers and neither* $\mathcal{C}_{(n+p,p)}$ *nor* $\mathcal{C}_{(n+q,q)}$ *is a tournament then* $\langle\mathcal{C}_{(n+p,p)}\rangle = \langle\mathcal{C}_{(n+q,q)}\rangle$.

Lemma 2.8. *Any variety* $\mathbf{V} \geq \mathbf{V}_1$ *which does not contain a tournament* $\mathcal{T}_n$*,* $n > 3$*, is generated by some set of circles.*

Proof. The statement holds if $\mathbf{V} = \mathbf{V}_1$ or $\mathbf{V} = \mathbf{V}_2$. Let $\mathbf{V} > \mathbf{V}_2$, $\mathcal{G} \in \mathbf{V}$ and $\mathcal{C} = v_1v_2\ldots v_nv_1$ be a w-circle of the orgraph $\mathcal{G}$. If $v_iv_j \notin E(\mathcal{G})$ for every $j > i+1$ then $\mathcal{C}$ is an induced suborgraph of $\mathcal{G}$, hence $\mathcal{C} \in \mathbf{V}$. Let $i+1 < j$ and $v_iv_j \in E(\mathcal{G})$. There are two possibilities:

If $v_{i-1}v_iv_{i+1}$ is not a path we consider a suborgraph identification of the orgraph $\mathcal{G}$ with the w-circle $\mathcal{C}_{(1,1,1,1)}$ under the weak path (v_{i-1}, v_i, v_{i+1}).

If $v_{i-1}v_iv_{i+1}$ is the path we consider a suborgraph identification of the orgraph $\mathcal{G}$ with the w-circle $\mathcal{C}_{(2,2)}$ under the path $v_{i-1}v_iv_{i+1}$.

We obtain an orgraph $\mathcal{G}' \in \mathbf{V}$ containing a w-circle $\mathcal{C}' = v_1 \dots v_{i-1}v_i'v_{i+1} \dots v_nv_1$ isomorphic with $\mathcal{C}$, but $v_i'v_j$ is not an arc of $\mathcal{G}'$. After finite number of the analogous transformations we get an orgraph $\mathcal{G}_1 \in \mathbf{V}$ which contains a w-circle $\mathcal{C}_1$ isomorphic with the w-circle $\mathcal{C}$ and $\mathcal{C}_1$ is the induced suborgraph of $\mathcal{G}_1$. Hence the variety $\mathbf{V}$ contains a w-circle isomorphic with $\mathcal{C}$, so again $\mathcal{C} \in \mathbf{V}$.

Let $S_{\mathcal{G}}$ be a set of all w-circles of the orgraph $\mathcal{G}$. From the previous part and Lemma 1.2 follows $\langle \mathcal{G} \rangle = \langle S_{\mathcal{G}} \rangle$. If we denote $M = \bigcup_{\mathcal{G} \in \mathbf{V}} S_{\mathcal{G}}$ then $\mathbf{V} = \langle M \rangle$. ∎

Proposition 2.9. *Let $\mathbf{V}$ be a variety generated by a w-circle $\mathcal{C}_{(n,0)}$, $n \geq 3$, or by $\mathcal{C}_{(3,1)}$, or by $\mathcal{C}_{(3,2)}$. Let $\mathcal{G}$ be an orgraph containing no nontrivial tournament (as an induced suborgraph). Then $\mathcal{G} \in \mathbf{V}$ iff the characteristic of each w-circle of the orgraph $\mathcal{G}$ is a multiple of the characteristic of the generating w-circle.*

Proof. Let $\mathbf{V} = \langle \mathcal{C}_{(n,0)} \rangle$, $n \geq 3$. Firstly, we show that if a characteristic of a w-circle $\mathcal{C}$ is a multiple of the integer n then $\mathcal{C} \in \mathbf{V}$. By Lemma 2.6 it is sufficient to prove the statement in the case when $\mathcal{C}$ is a circle.

Let x be a nonnegative integer and let $\mathcal{C}$ be a circle with the characteristic $n.x$. If $x = 0$ then $\mathcal{C} \in \mathbf{V}$ by Proposition 2.5. We now turn to the case $x > 0$ (i.e., $\mathcal{C} = \mathcal{C}_{(n.x,0)}$).

According to Lemma 2.6 we have $\mathcal{C}_{(n+1,1)} \in \mathbf{V}$. The suborgraph identification $\mathcal{C}_{(n+1,1)}$ with itself (see Figure 13) in a w-path of the length two gives an orgraph which contains the induced suborgraph $\mathcal{C}_{(2n,0)}$. So, $\mathcal{C}_{(2n,0)} \in \mathbf{V}$. Similarly,

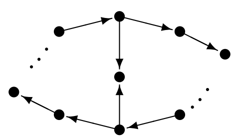

Figure 13

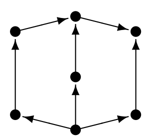

Figure 14

$\mathcal{C}_{(2n+1,1)} \in \mathbf{V}$ (by Lemma 2.6) and the suborgraph identification $\mathcal{C}_{(2n+1,1)}$ with $\mathcal{C}_{(n+1,1)}$ in a w-path of the length 2 gives an orgraph which contains the induced suborgraph $\mathcal{C}_{(3n,0)}$. We can continue in this fashion to obtain the circle $\mathcal{C}_{(n.x,0)}$.

It follows that if $\mathcal{G}$ contains only w-circles of characteristics which are multiples of integer n then $\mathcal{G} \in \mathbf{V}$ by Lemma 1.2.

If $\mathcal{G} \in \mathbf{V}$ then analogously to the last part of Proposition 2.4 we can show (with respect to Lemma 2.3) that each characteristic of a w-circle of $\mathcal{G}$ is a multiple of the characteristic of the generating w-circle.

The proofs in the cases $\mathbf{V} = \langle \mathcal{C}_{(3,1)} \rangle$ and $\mathbf{V} = \langle \mathcal{C}_{(3,2)} \rangle$ are similar. But in the case $\mathbf{V} = \langle \mathcal{C}_{(3,2)} \rangle$ we also use gluing by Figure 14. ■

Proposition 2.10. *Let n, m be any positive integers and p, q, r any non-negative integers. If neither $\mathcal{C}_{(n+p,p)}$ nor $\mathcal{C}_{(m+q,q)}$ is nontrivial tournament then*

(a) $\langle \mathcal{C}_{(n+p,p)} \rangle \bigvee \langle \mathcal{C}_{(m+q,q)} \rangle = \langle \mathcal{C}_{(D(n,m)+r,r)} \rangle$ *if* $D(n,m) > 3$,
(b) $\langle \mathcal{C}_{(n+p,p)} \rangle \bigvee \langle \mathcal{C}_{(m+q,q)} \rangle = \langle \mathcal{C}_{(4+r,1+r)} \rangle$ *if* $D(n,m) = 3$,
(c) $\langle \mathcal{C}_{(n+p,p)} \rangle \bigvee \langle \mathcal{C}_{(m+q,q)} \rangle = \langle \mathcal{C}_{(3+r,1+r)} \rangle$ *if* $D(n,m) = 2$,
(d) $\langle \mathcal{C}_{(n+p,p)} \rangle \bigvee \langle \mathcal{C}_{(m+q,q)} \rangle = \langle \mathcal{C}_{(3+r,2+r)} \rangle$ *if* $D(n,m) = 1$,

where $D(n,m)$ denotes the greatest common divisor of the integers n, m.

Proof. By Proposition 2.9 and Corollary 2.7 follows $\langle \mathcal{C}_{(n+p,p)} \rangle, \langle \mathcal{C}_{(m+q,q)} \rangle \subseteq \langle \mathcal{C}_{(D(n,m)+r,r)} \rangle$, hence $\langle \mathcal{C}_{(n+p,p)} \rangle \bigvee \langle \mathcal{C}_{(m+q,q)} \rangle \subseteq \langle \mathcal{C}_{(D(n,m)+r,r)} \rangle$, too.

For $d = D(n,m)$ there exist positive integers x, y such that $d = nx - my$ or $d = -nx + my$. Without loss of generality we can assume that $d = nx - my$. By Proposition 2.9 we get that $\mathcal{C}_{(n.x+r,r)} \in \langle \mathcal{C}_{(n+p,p)} \rangle$ and $\mathcal{C}_{(m.y+r,r)} \in \langle \mathcal{C}_{(m+q,q)} \rangle$ and so $\mathcal{C}_{(nx-m.y+r,r)} \in \langle \mathcal{C}_{(n+p,p)}, \mathcal{C}_{(m+q,q)} \rangle$ by Lemma 2.3 and Corollary 2.7. It yields $\langle \mathcal{C}_{(d+r,r)} \rangle \subseteq \langle \mathcal{C}_{(n+p,p)}, \mathcal{C}_{(m+q,q)} \rangle = \langle \mathcal{C}_{(n+p,p)} \rangle \bigvee \langle \mathcal{C}_{(m+q,q)} \rangle$. ■

Proposition 2.11. *Let $\mathbf{V} > \mathbf{T}$ be a variety which contains no nontrivial tournament. Then $\mathbf{V}$ is generated by exactly one w-circle which is one of $\mathcal{C}_{(1,1,1,1)}$ or $\mathcal{C}_{(2,2)}$ or $\mathcal{C}_{(3,2)}$ or $\mathcal{C}_{(3,1)}$ or $\mathcal{C}_{(4,1)}$ or $\mathcal{C}_{(n,0)}$, where $n > 3$.*

Proof. If $\mathbf{V} = \mathbf{V}_1$ or $\mathbf{V} = \mathbf{V}_2$ then the statement is true by Proposition 2.4 and Proposition 2.5, respectively. Let $\mathbf{V} > \mathbf{V}_2$; by Lemma 2.8 every variety which contains no nontrivial tournament is generated by a set $\mathbb{S}$ of circles. Let us denote by $N_{\mathbb{S}}$ the set of characteristics of circles belonging to $\mathbb{S}$ and by d the greatest common divisor of integers from $N_{\mathbb{S}}$. It is clear that there exists a finite set $d_1, d_2, \dots, d_k$ of integers from the set $N_{\mathbb{S}}$ such that $d = D(d_1, d_2, \dots, d_k)$.

Let $d > 3$. The inclusion $\mathbf{V} = \langle \mathbb{S} \rangle \supseteq \langle \mathcal{C}_{(d_1,0)} \rangle \bigvee \cdots \bigvee \langle \mathcal{C}_{(d_k,0)} \rangle$ is obvious and hence we have $\mathbf{V} \supseteq \langle \mathcal{C}_{(d,0)} \rangle$ by Proposition 2.10. According to Proposition 2.9 it follows $\langle \mathcal{C} \rangle \subseteq \langle \mathcal{C}_{(d,0)} \rangle$ for each circle $\mathcal{C} \in \mathbb{S}$, therefore $\langle \mathbb{S} \rangle \subseteq \langle \mathcal{C}_{(d,0)} \rangle$.

The statement can be checked for $d \in \{1, 2, 3\}$, too. ■

Corollary 2.12. *The interval* $[\langle \mathcal{C}_{2,2)} \rangle, \langle \mathcal{C}_{(3,2)} \rangle]$ *of the lattice* $\mathbf{L}(I, S, \Gamma)$ *of varieties is dually isomorphic to the lattice of all nonnegative integers with the relation of divisibility as the partial ordering* (*i.e., isomorphic to the lattice* $\mathbf{D}^d$).

Proof. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be varieties generated by w-circles $\mathcal{C}_{(n,0)}$ and $\mathcal{C}_{(m,0)}$, $n, m > 3$, respectively. From the previous assertions follows that

$$\langle \mathcal{C}_{(n,0)} \rangle \subseteq \langle \mathcal{C}_{(m,0)} \rangle \iff m|n.$$

For any w-circle $\mathcal{C}$ such that $\mathcal{C} \neq \mathcal{C}_{(3,0)}$ and $\mathcal{C} \neq \mathcal{C}_{(2,1)}$ we get $\mathcal{C} \in \langle \mathcal{C}_{(3,2)} \rangle$, hence $\langle \mathcal{C} \rangle \subseteq \langle \mathcal{C}_{(3,2)} \rangle$. Similarly, $\langle \mathcal{C}_{(n,0)} \rangle \subseteq \langle \mathcal{C}_{(3,1)} \rangle$ if $n > 2$ is an even number and $\langle \mathcal{C}_{(n,0)} \rangle \subseteq \langle \mathcal{C}_{(4,1)} \rangle$ if $n > 3$ and n is a multiple of the number 3. ■

Proposition 2.13. *The lattice* $\mathbf{L}_1$ *of all tournament-free varieties is isomorphic with the lattice* $\mathbf{3} \oplus \mathbf{D}^d$.

Proof. If a variety $\mathbf{V}$ contains the tournament $\mathcal{C}_{(3,0)}$ then $\mathbf{V}$ also contains the w-circle $\mathcal{C}_{(4,1)}$ and therefore $\mathbf{V} > \langle \mathcal{C}_{(4,1)} \rangle$. When a tournament $\mathcal{T}_n$, $n > 3$, belongs to $\mathbf{V}$ then the w-circle $\mathcal{C}_{(2,1)}$ is its induced suborgraph and therefore $\mathcal{C}_{(3,2)} \in \mathbf{V}$, too. It implies $\mathbf{V} > \langle \mathcal{C}_{(3,2)} \rangle$. Thus the set of all tournament-free varieties (with set inclusion as the partial ordering) is (by previous considerations) the lattice $\mathbf{3} \oplus \mathbf{D}^d$, and this lattice is an ideal of the lattice $\mathbf{L}(I, S, \Gamma)$. ■

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