# TOTAL DOMINATION EDGE CRITICAL GRAPHS WITH MAXIMUM DIAMETER 

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#### Abstract

Denote the total domination number of a graph $G$ by $\gamma_{t}(G)$. A graph $G$ is said to be total domination edge critical, or simply $\gamma_{t^{-}}$ critical, if $\gamma_{t}(G+e)<\gamma_{t}(G)$ for each edge $e \in E(\bar{G})$. For $3_{t}$-critical graphs $G$, that is, $\gamma_{t}$-critical graphs with $\gamma_{t}(G)=3$, the diameter of $G$ is either 2 or 3 . We characterise the $3_{t}$-critical graphs $G$ with $\operatorname{diam} G=3$.


## 1. Introduction

Let $G=(V, E)$ be a graph with order $|V|=n$. The open neighbourhood of a vertex $v$ is the set of vertices adjacent to $v$, that is, $N(v)=\{w \mid v w \in E(G)\}$, and the closed neighbourhood of $v$ is $N[v]=N(v) \cup\{v\}$. For $S \subseteq V(G)$ we define the open and closed neighbourhoods $N(S)$ and $N[S]$ of $S$ by $N(S)=$ $\bigcup_{v \in S} N(v)$ and $N[S]=\bigcup_{v \in S} N[v]$, respectively. The private neighbourhood of $x \in S, S \subseteq V(G)$, consists of all vertices in the closed neighbourhood of $x$ but not in the closed neighbourhood of $S-\{x\}$, and is denoted by $p n(x, S)$, that is, $p n(x, S)=N[x]-N[S-\{x\}]$. If $v \in p n(x, S)$, then $v$ is called a private neighbour of $x$ relative to $S$, or simply a private neighbour of $x$, if confusion is unlikely. If $G$ is a graph with $\operatorname{diam} G=k$ and $d(u, v)=k$, then
we say that $u$ and $v$ are diametrical vertices. A shortest $u-v$ path in $G$ is a diametrical path. Two subsets $X$ and $Y$ of $V$ are called diametrical sets if $d(x, y)=\operatorname{diam} G$ for each $x \in X$ and $y \in Y$. If $X$ and $Y$ are diametrical sets, then $(X, Y)$ is a maximal diametrical pair if for each $z \in V-(X \cup Y)$, $d(x, z)<\operatorname{diam} G$ for some $x \in X$ and $d(y, z)<\operatorname{diam} G$ for some $y \in Y$.

For sets $S, X \subseteq V$, if $N[S]=X(N(S)=X$, respectively), we say that $S$ dominates $X$, written $S \succ X$ ( $S$ totally dominates $X$, respectively, written $S \succ_{t} X$ ). If $S=\{s\}$ or $X=\{x\}$, we also write $s \succ X, S \succ_{t} x$, etc. If $S \succ V\left(S \succ_{t} V\right.$, respectively), we say that $S$ is a dominating set (total dominating set) of $G$, and we also write $S \succ G\left(S \succ_{t} G\right.$, respectively). The cardinality of a minimum dominating (minimum total dominating) set of $G$ is called the domination number (total domination number) of $G$ and is denoted by $\gamma(G)\left(\gamma_{t}(G)\right.$, respectively); if $S$ is a minimum dominating (minimum total dominating) set, we also call $S$ a $\gamma$-set $\left(\gamma_{t}\right.$-set) of $G$. We note that the parameter $\gamma_{t}(G)$ is only defined for graphs $G$ without isolated vertices. Domination-related concepts not defined here can be found in [2].

The addition of an edge to a graph can change the domination number by at most one. Sumner and Blitch $[5,6]$ studied domination edge critical graphs $G$, that is, graphs $G$ for which $\gamma(G)=\gamma(G)-1$ for each $e \in E(\bar{G})$. We consider the same concept for total domination. A graph $G$ is total domination edge critical or just $\gamma_{t}$-critical if $\gamma_{t}(G+e)<\gamma_{t}(G)$ for any edge $e \in E(\bar{G}) \neq \emptyset$. It is shown in [3] that the addition of an edge to a graph can change the total domination number by at most two.

Proposition 1 [3]. For any edge $e \in E(\bar{G})$,

$$
\gamma_{t}(G)-2 \leq \gamma_{t}(G+e) \leq \gamma_{t}(G)
$$

Graphs $G$ with the property $\gamma_{t}(G+e)=\gamma_{t}(G)-2$ for any $e \in E(\bar{G})$ are called supercritical and are characterised in [4].

In this paper, we restrict our attention to $3_{t}$-critical graphs $G$, that is, $\gamma_{t}$-critical graphs $G$ with $\gamma_{t}(G)=3$. Note that since $\gamma_{t}(G) \geq 2$ for any graph $G$, the addition of an edge to a $3_{t}$-critical graph reduces the total domination number by exactly one. Also, observe that any graph $G$ with $\gamma_{t}(G)=3$ is connected. Sharp bounds on the diameter of a $3_{t}$-critical graph are determined in [3].

Proposition 2 [3]. If $G$ is a $3_{t}$-critical graph, then
$2 \leq \operatorname{diam} G \leq 3$.

The graphs in Figures 1 and 2 illustrate sharpness of these bounds. Our goal is to investigate the $3_{t}$-critical graphs with diameter three.


Figure 1. A $3_{t}$-critical graph $G$ with diam $G=2$

## 2. $3_{t}$-Critical Graphs with Diameter Three

In [3] the authors showed that any $3_{t}$-critical graph $G$ with a cutvertex has exactly one cutvertex and it is adjacent to an endvertex. Moreover, they proved that such graphs $G$ have diam $G=3$ and are the only $3_{t}$-critical graphs with an endvertex. Figure 2 illustrates a $3_{t}$-critical graph with an endvertex.


Figure 2. A $3_{t}$-critical graph with an endvertex

Theorem 3 [3]. A graph $G$ with a cutvertex $v$ is $3_{t}$-critical if and only if $v$ is adjacent to an endvertex $x$, and for $W=N(v)-\{x\}$ and $Y=V-N[v]$,
(1) $\langle W\rangle$ is complete and $|W| \geq 2$,
(2) $\langle Y\rangle$ is complete and $|Y| \geq 2$,
and
(3) every vertex in $W$ is adjacent to $|Y|-1$ vertices in $Y$ and every vertex in $Y$ is adjacent to at least one vertex in $W$.

We begin with a straightforward but useful observation.

Observation 4. For any $3_{t}$-critical graph $G$ and non-adjacent vertices $u$ and $v$, either
(1) $\{u, v\}$ dominates $G$
or
(2) (without loss of generality) $\{u, w\}$ dominates $G-v$, but not $v$, for some $w \in N(u)$. In this case, we write $u w \mapsto v$.

Next we develop some structural properties of $3_{t}$-critical graphs $G$ with diam $G=3$. Although it is possible in a $3_{t}$-critical graph $G$ of diameter two for every pair of nonadjacent vertices to dominate $G$ (see Figure 1, for example), we now show this is not possible if $\operatorname{diam} G=3$.

Proposition 5. If $G$ is a $3_{t}$-critical graph with diam $G=3$, then $G$ has a pair of nonadjacent vertices that does not dominate $G$.

Proof. Let $G$ be a $3_{t}$-critical graph with $\operatorname{diam} G=3$ and suppose that every pair of nonadjacent vertices of $G$ dominates $G$. Let $x$ and $y$ be diametrical vertices of $G$ where $x, a, b, y$ is a shortest $x-y$ path. Since $\{x, b\} \succ G$, every neighbour of $y$ is also dominated by $b$. Similarly, every neighbour of $x$ is dominated by $a$. Hence $\{a, b\}$ is a total dominating set of $G$, contradicting the fact that $\gamma_{t}(G)=3$.

Also, it is possible for a $3_{t}$-critical graph $G$ with $\operatorname{diam} G=2$ to have the property that for every pair of nonadjacent vertices $u$ and $v$, there is a vertex $x$ such that $u x \mapsto v$, and there is a vertex $y$ such that $v y \mapsto u$. See Figure 3 for an example. We now show that a $3_{t}$-critical graph with diameter three cannot have this property.


Figure 3. A $3_{t}$-critical graph with $\operatorname{diam} G=2$
Proposition 6. If $G$ is a $3_{t}$-critical graph with diam $G=3$, then $G$ has a pair of nonadjacent vertices $u$ and $v$ such that $u x \mapsto v$, for some $x \in V$, but there is no vertex $y$ such that $v y \mapsto u$.

Proof. Let $G$ be a $3_{t}$-critical graph with diameter three. Let $x$ and $y$ be diametrical vertices of $G$ where $x, a, b, y$ is a shortest $x-y$ path. By the proof of Proposition 5, at least one of $\{x, b\}$ and $\{a, y\}$ does not dominate $G$. Assume then, without loss of generality, that $\{x, b\}$ does not dominate $G$. If $x w \mapsto b$, then $w \in N(x)$ by Observation 4 and $w \in N(y)$ to dominate $y$, thus $d(x, y) \leq 2$, a contradiction. Hence the only possibility is that $b w \mapsto x$.

It is useful to know more about the diametrical sets of vertices of a $3_{t}$-critical graph with diameter three.

Theorem 7. If $G$ is a $3_{t}$-critical graph with diam $G=3$, then $G$ has a unique maximal diametrical pair $(X, Y)$. Moreover, $X$ (say) has cardinality one and $\langle Y\rangle$ is complete.

Proof. Let $G$ be a $3_{t}$-critical graph with $\operatorname{diam} G=3$. The proof of the theorem is a direct consequence of the following three lemmas.

Lemma 8. For any maximal diametrical pair $\left(Y_{1}, Y_{2}\right)$ of $G,\left\langle Y_{i}\right\rangle$ is complete for each $i$ and $\left|Y_{i}\right|=1$ for at least one $i$.

Proof. Let $\left(Y_{1}, Y_{2}\right)$ be a maximal diametrical pair of $G$. First we show that if $\left|Y_{i}\right| \geq 2$, then $\left\langle Y_{i}\right\rangle$ is complete. Let $x \in Y_{1}$ and $\{y, z\} \subseteq Y_{2}$ and
suppose that $y z \notin E(G)$. Since $\{y, z\} \nsucc G$, we may assume without loss of generality that $y w \mapsto z$ for some vertex $w$, contradicting the fact that $d(x, y)=3$. Hence $\left\langle Y_{2}\right\rangle$ is complete. A similar argument shows that $\left\langle Y_{1}\right\rangle$ is complete.

Next we show that $\left|Y_{i}\right|=1$ for at least one $i$. Suppose to the contrary that both $Y_{1}$ and $Y_{2}$ have cardinality at least two. Let $x \in Y_{1}$ and $y \in Y_{2}$ and consider $\{x, y\}$. Since $\left|Y_{i}\right| \geq 2$ for $i \in\{1,2\}$, there is no vertex $w$ such that $x w \mapsto y$ or $y w \mapsto x$. It follows that $\{x, y\} \succ G$. This is the case for every $x \in Y_{1}$ and every $y \in Y_{2}$. Let $A(B$, respectively) be the set of vertices that are distance one from every vertex of $Y_{1}\left(Y_{2}\right.$, respectively). If both $\langle A\rangle$ and $\langle B\rangle$ are complete, then $\gamma_{t}(G)=2$, a contradiction. Thus let $a, b \in A$ where $a b \notin E(G)$. Consider $\{a, y\}$. Since neither $a$ nor $y$ is adjacent to $b$, $\{a, y\} \nsucc G$. Hence, $y c \mapsto a$ or $a c \mapsto y$. Since no vertex in $N[y]$ dominates $Y_{1}$, $a c \mapsto y$. Therefore, $c$ dominates $Y_{2}-\{y\}$. Furthermore, since $\{x, y\} \succ G$, $c$ is adjacent to $x$, implying that $y$ is the only vertex at distance three from $x$, contradicting the fact that $\left|Y_{i}\right|>1$ for $i \in\{1,2\}$.
Consider the maximal diametrical pair $(\{x\}, Y)$ of $G$. Note that by Lemma 8 and the definition of maximal diametrical pair, $Y=\{y \in V \mid d(x, y)=3\}$.

Lemma 9. For every vertex $u \in V-\{x\}, d(u, y) \leq 2$ for every $y \in Y$.
Proof. If $|Y|=1$, then $x$ is the only vertex at distance three from $Y$. Assume then that $|Y| \geq 2$. Let $y, z \in Y$ and suppose there is a vertex $u$ such that $d(u, y)=3$ and $d(u, z)=2$; note that $u \neq x$. Let $u a b y$ be a $u-y$ path and let $u c z$ be a $u-z$ path ( $c$ may equal $a)$. Note that $c y \notin E(G)$. Since neither $x$ nor $y$ is adjacent to $c, x w \mapsto y$ or $y w \mapsto x$. If $x w \mapsto y$, then $d(x, z)=2$, contradicting that $z \in Y$ and that $\{x\}$ and $Y$ are diametrical sets. If $y w \mapsto x$, then $d(u, y)=2$, again a contradiction.

Lemma 10. $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$.
Proof. Consider any maximal diametrical pair $(\{u\}, W)$ of $G$. If $u=x$, then $W=\{w \in V \mid d(u, w)=3\}=\{w \in V \mid d(x, w)=3\}=Y$ and we are done. If $u \in Y$, then $d(x, u)=3$, i.e., $x \in W$ and by Lemma 9, $d(u, z) \leq 2$ for each $z \in V-\{x\}$. Hence $W=\{x\}$ and since $(\{u\},\{x\})$ is a maximal diametrical pair, it follows that $Y=\{u\}$ and the result follows. Hence we may assume that $u \notin Y \cup\{x\}$. It follows from Lemma 9 that $W \cap(Y \cup\{x\})=\emptyset$.

Consider any $w \in W$ and suppose firstly that $\{u, w\} \succ G$. Note that no vertex is adjacent to $x$ as well as to a vertex in $Y$. Hence either $u x \in E(G)$ and $w y \in E(G)$ for each $y \in Y$, or $w x \in E(G)$ and $u y \in E(G)$ for each $y \in Y$. Suppose the former case holds and consider an arbitrary vertex $y \in Y$. By Lemma $9, d(u, y)=2$ and $d(w, x)=2$. Let uay and $w b x$ be a $u-y$ path and a $w-x$ path, respectively and note that $\{u b, y b\} \cap E(G)=\emptyset$. Thus $\{u, y\} \nsucc G$ and so $u c \mapsto y$ or $y c \mapsto u$ for some vertex $c$. If $u c \mapsto y$, then $c w \in E(G)$ and so $d(u, w)=2$, a contradiction since $u$ and $w$ are diametrical vertices. If $y c \mapsto u$, then $d(x, y)=2$, also a contradiction. Similarly, the case $w x \in E(G)$ and $u y \in E(G)$ for each $y \in Y$ is impossible. We conclude that $\{u, w\} \nsucc G$.

Thus there is some vertex $d$ such that $\{u, w, d\}$ is independent. Since neither $d$ nor $u$ is adjacent to $w, u c \mapsto d$ or $d c \mapsto u$. If $u c \mapsto d$, then $d(u, w)=2$, a contradiction. Thus we may assume that $d c \mapsto u$. Then without loss of generality, $d \in N(Y)$ and $c \in N(x)$. Now we consider $\{x, d\}$. Since $d$ is not adjacent to $u$ or $w$, and $x$ cannot be adjacent to both $u$ and $w, x d$ is not a dominating edge for $G+x d$. Then $x s \mapsto d$ or $d s \mapsto x$. If $x s \mapsto d$, then $d(x, y)=2$, a contradiction. If $d s \mapsto x$, then $s$ is adjacent to both $u$ and $w$, contradicting the fact that $d(u, w)=3$. Hence $(\{x\}, Y)$ is the unique diametrical pair of $G$.

## 3. Characterisation

In the rest of this paper we characterise the $3_{t}$-critical graphs with diameter three. We introduce more notation to simplify the characterisation. Let $G$ be a graph with $\operatorname{diam} G=3$ and let $(\{x\}, Y)$ be a maximal diametrical pair of $G$. Let $A=N(x), B=\{b \mid b \notin Y$ and $b \succ Y\}$, and $C=V-(A \cup B \cup Y \cup\{x\})$. Note that at least one of $B$ and $C$ is not empty. Let $\mathcal{F}$ be the family of all graphs $G$ with $\operatorname{diam} G=3$ and the maximal diametrical pair $(\{x\}, Y)$. Then $\mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2} \cup \mathcal{F}_{3} \cup \mathcal{F}_{4}$, where
$G \in \mathcal{F}_{1}$ if $C=\emptyset$ and $|Y| \geq 2$,
$G \in \mathcal{F}_{2}$ if $C=\emptyset$ and $|Y|=1$,
$G \in \mathcal{F}_{3}$ if $B=\emptyset$,
$G \in \mathcal{F}_{4}$ if $B \neq \emptyset$ and $C \neq \emptyset$.
To characterise the $3_{t}$-critical graphs with diameter 3, we characterise the $3_{t}$-critical graphs in each family $\mathcal{F}_{i}, 1 \leq i \leq 4$. We begin with a lemma.

Lemma 11. Let $G \in \mathcal{F}$ be $3_{t}$-critical with $|Y| \geq 2$. If either $B=\emptyset$ or $C=\emptyset$, then $\langle A\rangle$ is complete.

Proof. Let $G \in \mathcal{F}$ with $|Y| \geq 2$ and suppose that $\langle A\rangle$ is not complete. First assume that $C=\emptyset$. Let $u, v \in A$ with $u v \notin E(G)$. Consider $\{u, y\}$ for some vertex $y \in Y$. Since neither $u$ nor $y$ is adjacent to $v, u w \mapsto y$ or $y w \mapsto u$ for some vertex $w$. If $u w \mapsto y$, then $w \in A \cup\{x\}$ since $w \notin N(y)$. But then $Y-\{y\}$ is not dominated by $\{u, w\}$, a contradiction. If $y w \mapsto u$, then $d(x, y) \leq 2$, again a contradiction. Next assume that $B=\emptyset$. Since $\{u, v\} \nsucc G$, we may assume, without loss of generality, that $u w \mapsto v$. But this implies that $w \succ Y$, contradicting the fact that $B=\emptyset$.

Lemma 11 requires that the graph $G$ has a diametrical set $Y$ with cardinality greater than one. (See Figure 4(b)). The graph in Figure 4(a) is an example of a graph with a diametrical set $Y$ such that $|Y|=1$ and $\langle A\rangle$ complete. However, the condition of the lemma is necessary as can be seen by the $3_{t}$-critical graph in Figure 5 that has $|Y|=1$ and $\langle A\rangle$ is not complete.


Figure 4. Two $3_{t}$-critical graphs with diameter three


$$
|X|=|Y|=1
$$

Figure 5. A $3_{t}$-critical graph with $\langle A\rangle$ not complete

We first characterise the $3_{t}$-critical graphs $G \in \mathcal{F}_{1}$.
Theorem 12. A graph $G \in \mathcal{F}_{1}$ is $3_{t}$-critical if and only if the following conditions hold:
(1) $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y\rangle$ is complete.
(2) $\langle A\rangle$ is complete.
(3) For every nonadjacent pair $u, v \in B$, there is a vertex $a \in A$ such that $u a \mapsto v$. Also, no pair of adjacent vertices dominates $G$.
(4) For every vertex $b \in B$, there is a vertex $d \in B \cup Y$ such that $b d \mapsto x$.
(5) For every pair $\{a, b\}$ of nonadjacent vertices where $a \in A$ and $b \in B$, $\{a, b\} \succ G$ or $a w \mapsto b$ for some $w \in B$.

Proof. Let $G \in \mathcal{F}_{1}$ be $3_{t}$-critical. By Theorem $10,(\{x\}, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y\rangle$ is complete.

Since $C=\emptyset$, it follows that $\{x, y\} \succ G$ for every $y \in Y$. From Lemma 11 we have that $\langle A\rangle$ is complete. Furthermore, since $(\{x\}, Y)$ is a maximal diametrical pair, each $b \in B$ is adjacent to at least one vertex $a \in A$. If there is a vertex $b \in B$ that dominates $B$, then $\{a, b\} \succ_{t} G$ for an $a \in A$, contradicting the fact that $\gamma_{t}(G)=3$. Let $u, v \in B$ with $u v \notin E(G)$. Obviously, $\{u, v\} \nsucc x$, so without loss of generality, there is a vertex $a \in A$ such that $a u \mapsto v$. Since $\gamma_{t}(G)=3$, no pair of adjacent vertices dominates $G$. To show that (4) holds, let $b$ be any vertex in $B$. Since there is at least one vertex in $B$ not adjacent to $b,\{x, b\} \nsucc G$. No vertex in $N[x]$ dominates $Y$, so $b d \mapsto x$ for some $d \in B \cup Y$. Condition (5) follows directly from Observation 4 and the fact that if $b w \mapsto a$, then $w \in A$ to dominate $x$; hence $w \succ a$ since $\langle A\rangle$ is complete, a contradiction.

Conversely, let $G \in \mathcal{F}_{1}$ such that the stated properties hold. Since no pair of adjacent vertices dominates $G, \gamma_{t}(G) \geq 3$. Further, $\{a, b, y\}$ is a $\gamma_{t}$-set for every $a \in A, b \in B, y \in Y$ where $a b \in E(G)$, implying that $\gamma_{t}(G) \leq 3$. Hence $\gamma_{t}(G)=3$. To show that $G$ is $3_{t}$-critical we consider first $\{x, y\}$ for $y \in Y$. Since $C=\emptyset,\{x, y\} \succ G$. Similarly, $\{a, y\} \succ G$ for every $a \in A$. We next consider $\{x, b\}$. Since condition (4) holds, there is a vertex $d \in B \cup Y$ such that $b d \mapsto x$. We also consider $\{a, b\}, a \in A$ and $b \in B$. Property (5) implies that either $\{a, b\} \succ G$ or there is a vertex $w \in B$ such that $a w \mapsto b$. Finally we consider $\{b, c\}$, where $b, c \in B$. Since condition (3) holds, there is a vertex $a \in A$ such that $a b \mapsto c$. Thus $G$ is $3_{t}$-critical.

Note that $\{x, y\} \succ G$ for every $y \in Y$. We state this result as a corollary.
Corollary 13. If $G \in \mathcal{F}_{1}$ is $3_{t}$-critical, then $\gamma(G)=2$.
We now give a more descriptive characterisation of the $3_{t}$-critical graphs $G \in \mathcal{F}_{1}$ with $\delta(G)=2$. We first show that if $\delta(G)=2$, then $\operatorname{deg}(x)=2$. Recall that $\langle A\rangle$ is complete.

Lemma 14. If $G \in \mathcal{F}_{1}$ and $G$ is $3_{t}$-critical with $\delta(G)=2$, then $\operatorname{deg}(x)=2$ and $\operatorname{deg}(v) \geq 3$ for all $v \in V(G)-\{x\}$.

Proof. Let $G \in \mathcal{F}_{1}$ be $3_{t}$-critical. Since $G$ has no cutvertices (Theorem 3), $|A|,|B| \geq 2$. Every vertex $b \in B$ is adjacent to some vertex $a \in A$ and to every vertex $y \in Y$. Thus $\operatorname{deg}(b) \geq 3$ for every $b \in B$, since $|Y| \geq 2$. By Theorem 10, $\langle Y\rangle$ is complete. Therefore $\operatorname{deg}(y) \geq 3$ for each $y \in Y$. Finally, every vertex $a \in A$ has at least one neighbour in $A$, implying that $\operatorname{deg}(a) \geq 3$.

We use the following notation for the characterisation. Let $A=N(x)=$ $\left\{x_{1}, x_{2}\right\}$ and $B_{1}=\left(N\left(x_{1}\right) \cap N\left(x_{2}\right)\right)-\{x\}, B_{2}=N\left(x_{1}\right)-\left(B_{1} \cup\left\{x, x_{2}\right\}\right)$, and $B_{3}=N\left(x_{2}\right)-\left(B_{1} \cup\left\{x, x_{1}\right\}\right)$. Recall that $C=\emptyset$ and hence $B=B_{1} \cup B_{2} \cup B_{3}$.

We need the following lemmas for the characterisation. To simplify notation we refer to the $3_{t}$-critical graphs $G \in \mathcal{F}_{1}$ with $\delta(G)=2$ as family $\mathcal{G}_{2}$.

Lemma 15. If $G \in \mathcal{G}_{2}$ and $B_{i} \neq \emptyset$, then $\left\langle B_{i}\right\rangle$ is complete for $i \in\{1,2,3\}$.
Proof. Let $G \in \mathcal{G}_{2}$ and assume that $B_{i} \neq \emptyset$. Suppose that $u, v \in B_{i}$ and $u v \notin E(G)$. Since neither $u$ nor $v$ dominates $x$, without loss of generality, $u w \mapsto v$. Then $w \in N(u) \cap N(x)$. But since $u$ and $v$ are in $B_{i}, v \in N(w)$, contradicting that $u w \mapsto v$.

Lemma 16. If $G \in \mathcal{G}_{2}$ and $B_{1} \neq \emptyset$, then each vertex in $B_{1}$ dominates exactly $\left|B_{i}\right|-1$ vertices in $B_{i}$ for $i \in\{2,3\}$.

Proof. It is easy to see that no vertex $b \in B_{1}$ dominates $B_{2}$ or $B_{3}$. Suppose, without loss of generality, a vertex $b \in B_{1}$ is not adjacent to two vertices in $B_{2}$, say $u$ and $v$, and consider $\{b, u\}$. Since neither $b$ nor $u$ dominates $x,\{b, u\} \nsucc G$. Furthermore, $u x_{1} \nvdash b$ since $x_{1} \in N(b)$. Hence $b x_{2} \mapsto u$, implying that $v \in B_{3}$, a contradiction.

Lemma 17. If $G \in \mathcal{G}_{2}$, then $\left|B_{i}\right| \geq 2$ for $i \in\{2,3\}$.

Proof. Let $G \in \mathcal{G}_{2}$. Since $(\{x\}, Y)$ is a maximal diametrical pair, each $a \in A$ is adjacent to some $b \in B$. Hence $B_{1} \cup B_{i} \neq \emptyset$ for $i \in\{2,3\}$. If $B_{2}=\emptyset$ (or $B_{3}=\emptyset$, respectively), then $\left\{x_{2}, b_{3}\right\} \succ_{t} G$ for $b_{3} \in B_{1} \cup B_{3}\left(\left\{x_{1}, b_{2}\right\} \succ_{t} G\right.$ for $b_{2} \in B_{1} \cup B_{2}$, respectively). Hence neither $B_{2}$ nor $B_{3}$ is empty. Suppose without loss of generality that $\left|B_{2}\right|=1$, say $B_{2}=\left\{b_{2}\right\}$. By Lemma $16, b_{2}$ is not adjacent to any vertex in $B_{1}$. Also, $b_{2}$ is not adjacent to any vertex in $B_{3}$, for otherwise $\left\{x_{2}, b_{3}\right\} \succ_{t} G$ for some $b_{3} \in B_{3} \cup N\left(b_{2}\right)$. Now consider $\left\{b_{2}, x\right\}$. Since $\left\{b_{2}, x\right\} \nsucc B_{3} \neq \emptyset$ and $\left\{x, x_{i}\right\} \nsucc Y$, there exists a vertex $w$ such that $b_{2} w \mapsto x$. But no vertex adjacent to $b_{2}$ dominates $x_{2}$ as well as $B_{3}$, a contradiction. Hence $\left|B_{i}\right| \geq 2$ for $i \in\{2,3\}$.

Lemma 18. If $G \in \mathcal{G}_{2}$, then $\overline{\left\langle B_{2} \cup B_{3}\right\rangle}$ is the disjoint union of non-trivial stars.

Proof. Note that $\overline{\left\langle B_{2} \cup B_{3}\right\rangle}$ has no isolates, for if $u \in B_{2}$ (say) dominates $B_{3}$, then $\left\{u, x_{1}\right\} \succ_{t} G$, contradicting the fact that $\gamma_{t}(G)=3$. Assume without loss of generality that a vertex $u \in B_{2}$ is not adjacent to vertices $b_{1}, \ldots, b_{k} \in B_{3}$, where $k \geq 2$ and where $b_{1}$ (say) is not adjacent to $v \in B_{2}$, $v \neq u$. Since $\left\{u, b_{1}\right\} \nsucc x$, we may assume without loss of generality that $u w \mapsto b_{1}$ for some vertex $w$. Then $w=x_{1}$ to dominate $x$, but $\left\{u, x_{1}\right\} \nsucc b_{2}$, a contradiction. The result follows since $\left\langle B_{i}\right\rangle$ is complete for $i=2,3$.

Theorem 19. A graph $G \in \mathcal{G}_{2}$ if and only if the following conditions hold:
(1) $(\{x\}, Y)$ is the unique maximal diametrical pair and $\langle Y\rangle$ is complete.
(2) $\operatorname{deg}(x)=2$ and $\langle A\rangle$ is complete.
(3) $B_{1}=\emptyset$ or $\left\langle B_{1}\right\rangle$ is complete.
(4) $\left|B_{i}\right| \geq 2$ and $\left\langle B_{i}\right\rangle$ for $i \in\{2,3\}$ is complete.
(5) $\overline{\left\langle B_{2} \cup B_{3}\right\rangle}$ is the disjoint union of non-trivial stars.
(6) If $B_{1} \neq \emptyset$, then every vertex in $B_{1}$ dominates exactly $\left|B_{i}\right|-1$ vertices in $B_{i}$ for $i \in\{2,3\}$. Also, if $u \in B_{2}\left(u \in B_{3}\right.$, respectively $)$ does not dominate $B_{1}$, then there is a vertex $v \in B_{1} \cup B_{3}\left(v \in B_{1} \cup B_{2}\right.$, respectively) such that $\{u, v\} \succ_{t} B$.

Proof. Let $G \in \mathcal{G}_{2}$. By Theorem 12, $(\{x\}, Y)$ is the unique maximal diametrical pair of $G,\langle Y\rangle$ is complete, and $\langle A\rangle$ is complete. By Lemma 14, $\operatorname{deg}(x)=2$. By Lemmas 15, 17, and 18, conditions (3), (4), and (5) hold. Assume without loss of generality that $u \in B_{2}$ does not dominate $B_{1}$. Since
$\{x, u\} \nsucc G$ and $\left\{x, x_{i}\right\} \nsucc Y$, it follows that $u v \mapsto x$ for some $v$. To dominate $x_{2}$ but not $x, v \in B_{1} \cup B_{3}$, and clearly $\{u, v\} \succ_{t} B$. Thus by Lemma 16 , condition (6) holds.

Conversely, let $G$ be graph such that all the conditions of the theorem hold. There is no edge $u v \in E(G)$ such that $\{u, v\} \succ G$. Hence $\gamma_{t}(G) \geq 3$. The path $x_{1}, x_{2}, b_{i}$, for $b_{i} \in B$, is a total dominating set. Therefore $\gamma_{t}(G)=3$.

To show that $G$ is $\gamma_{t}$-critical we first consider $\{x, y\}$ for any $y \in Y$. Since $C=\emptyset,\{x, y\} \succ G$ for every $y \in Y$. Next consider $\{x, b\}$ for any $b \in B_{1}$. Since $b \succ A \cup Y, b y \mapsto x$ for any $y \in Y$. Now consider $\{x, u\}$ for any $u \in B_{2}$. If $u$ is not adjacent to any vertex in $B_{3}$, then by (5), every $c \in B_{2}-\{u\} \neq \emptyset$ is adjacent to all vertices in $B_{3}$, i.e., $\left\{x_{1}, c\right\} \succ_{t} G$, a contradiction. So, if $B_{1}=\emptyset$ or $u \succ B_{1}$, let $v \in B_{3}$ be adjacent to $u$. Clearly, $u v \mapsto x$. If $u \nsucc B_{1}$, then by (6) there is a vertex $v \in B_{1} \cup B_{3}$ such that $\{u, v\} \succ_{t} B$ and it is easy to see that $u v \mapsto x$. The set $\{x, u\}$ for any $u \in B_{3}$ is dealt with in exactly the same way. Further, it is easy to see that $\left\{x_{1}, v\right\}$ and $\left\{x_{2}, u\right\}$ dominate $G$ for every $v \in B_{3}$ and every $u \in B_{2}$. Also, $\left\{x_{i}, y\right\} \succ G$ for $i=1,2$ and every $y \in Y$. By Condition (6) a vertex $b \in B_{1}$ dominates exactly $\left|B_{i}\right|-1$ vertices in $B_{i}, i=2,3$. Let $u \in B_{2}$ be non-adjacent to $b \in B_{1}$. Then $b x_{2} \mapsto u$. Similarly, $b x_{1} \mapsto v$, for $v \in B_{3}$ and $b v \notin E(G)$. Finally, we consider $\{u, v\}$ with $u \in B_{2}$ and $v \in B_{3}$, where $u v \notin E$. Since $\overline{\left\langle B_{2} \cup B_{3}\right\rangle}$ is the disjoint union of non-trivial stars, we may assume without loss of generality that $u$ has degree 1 in $\overline{\left\langle B_{2} \cup B_{3}\right\rangle}$. Then $u x_{1} \mapsto v$. It now follows that $G \in \mathcal{G}_{2}$.

For an example of a $3_{t}$-critical graph $G \in \mathcal{G}_{2}$, see Figure 6 .


Figure 6. A $3_{t}$-critical graph $G \in \mathcal{G}_{2}$
For $3_{t}$-critical graphs $G \in \mathcal{F}_{1}$, the cardinality of $Y$ is greater than one. A necessary condition for these graphs is that $\langle A\rangle$ is complete. However, when
the cardinality of $Y$ is equal to one, this condition is no longer required. Figure $4(\mathrm{a})$ is an example of $G \in \mathcal{F}_{2}$ and $3_{t}$-critical with $\langle A\rangle$ complete and Figure 5 is an example of a graph $G \in \mathcal{F}_{2}$ and $3_{t}$-critical with $|Y|=1$ and $\langle A\rangle$ not complete.

Theorem 20. A graph $G \in \mathcal{F}_{2}$ is $3_{t}$-critical if and only if the following conditions hold:
(1) $(\{x\},\{y\})$ is the unique diametrical pair of $G$.
(2) For each $a \in A$ and $b \in B$ with $a b \in E(G)$ there exists a vertex $w \notin$ $N(a) \cup N(b)$.
(3) For each $a, a^{\prime} \in A$, with $a a^{\prime} \notin E(G)$, there exists $b^{\prime} \in B$ such that $a b^{\prime} \mapsto a^{\prime}$. A similar statement holds for each $b, b^{\prime} \in B$ with $b b^{\prime} \notin E(G)$.
(4) For every $a \in A,\{a, y\} \succ G$ or there exists $a^{\prime} \in A$ such that $a a^{\prime} \mapsto y$. $A$ similar statement holds for every $b \in B$ and $\{x\}$.
(5) For each $a \in A$ and $b \in B$ with $a b \notin E(G),\{a, b\} \succ G$ or, without loss of generality, there exists $b^{\prime} \in B$ such that $a b^{\prime} \mapsto b$.

Proof. Let $G \in \mathcal{F}_{2}$ be $3_{t}$-critical. By Theorem $7(\{x\},\{y\})$ is the unique diametrical pair of $G$. Condition (2) follows from the fact that $\gamma_{t}(G)=3$. Since $\langle A\rangle$ and $\langle B\rangle$ cannot both be complete, let $a, a^{\prime} \in A$ with $a a^{\prime} \notin E(G)$. Neither $a$ nor $a^{\prime}$ is adjacent to $y$. Therefore without loss of generality there exists $b^{\prime} \in B$ such that $a b^{\prime} \mapsto a^{\prime}$. Let $a \in A$ be an arbitrary vertex. If $\{a, y\} \succ G$, then Condition (4) holds. Otherwise there exists $w$ such that $y w \mapsto a$ or $a w \mapsto y$. If $y w \mapsto a$, then $x \in N(w)$ implying $d(x, y)=2$, a contradiction. Hence $a w \mapsto y$ for some $w \in A$. A similar argument shows that for every $b \in B,\{b, x\} \succ G$ or there exists $b^{\prime} \in B$ such that $b b^{\prime} \mapsto x$. Let $a \in A$ and $b \in B$ with $a b \notin E(G)$. If $\{a, b\} \succ G$, then Condition (5) holds. Otherwise, without loss of generality, there exists $b^{\prime} \in B$ such that $a b^{\prime} \mapsto b$.

Conversely, let $G$ be a graph such that the stated conditions hold. By Condition (2) there is no edge that dominates $G$. Thus, $\gamma_{t}(G) \geq 3$. Consider $\{a, y\}$ for any $a \in A$. If $\{a, y\} \succ G$, then with $b \in N(a) \cap N(y),\{a, b, y\}$ is a total dominating set. If $\{a, y\} \nsucc G$, then by Condition (4) there exists $a^{\prime} \in A$ such that $a a^{\prime} \mapsto y$. Again with $b \in N(a) \cap N(y),\left\{a, a^{\prime}, b\right\}$ is a total dominating set, so $\gamma_{t}(G) \leq 3$. Hence $\gamma_{t}(G)=3$. That $G$ is $\gamma_{t}$-critical follows from the fact that $\{x, y\} \succ G$ and from Conditions (2) through (5).

Two additional lemmas are needed for the remaining characterisations.

Lemma 21. If $G \in \mathcal{F}$ is $3_{t}$-critical, then every vertex in $C$ is adjacent to exactly $|Y|-1$ vertices in $Y$.

Proof. By definition, there is no vertex in $C$ that dominates $Y$. Suppose there is a vertex $c \in C$ that is not adjacent to at least two vertices in $Y$, say $u$ and $v$. Clearly, $\{c, u\} \nsucc G$. Therefore $c w \mapsto u$ or $u w \mapsto c$ for some vertex $w$. If $c w \mapsto u$, then $w \in N(x)$ and $w \succ v$, contradicting the fact that $d(x, v)=3$. If $u w \mapsto c$, then $w \succ x$, again contradicting that $d(x, u)=3$.

It was shown in Theorem 7 that $\langle Y\rangle$ is complete. We now consider $\langle C\rangle$.
Lemma 22. If $G \in \mathcal{F}$ is $3_{t}$-critical and $C \neq \emptyset$, then $\langle C\rangle$ is complete.
Proof. Let $u, v \in C$ and $u v \notin E(G)$. Since $\{u, v\} \nsucc G$, assume without loss of generality that $u w \mapsto v$. By definition there is a vertex $y \in Y$ not adjacent to $u$. Therefore, $w \succ y$ and $w \succ x$. But this contradicts the fact that $d(x, y)=3$.
We now characterise the $3_{t}$-critical graphs in family $\mathcal{F}_{3}$.
Theorem 23. A graph $G \in \mathcal{F}_{3}$ is $3_{t}$-critical if and only if the following conditions hold:
(1) $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y\rangle$ is complete.
(2) $\langle A \cup C\rangle$ is complete.
(3) $|C| \geq 2,|Y| \geq 2$ and every vertex in $C$ is adjacent to exactly $|Y|-1$ vertices in $Y$.

Proof. Let $G \in \mathcal{F}_{3}$ be $3_{t}$-critical. From Theorem 7 we have that $(\{x\}, Y)$ is the unique maximal diametrical pair and $\langle Y\rangle$ is complete.

By Lemmas 11 and $22,\langle A\rangle$ and $\langle C\rangle$ are complete. We show that $\langle A \cup C\rangle$ is complete. Let $a \in A$ and $c \in C$ with $a c \notin E(G)$. Since there is at least one vertex in $Y$ not adjacent to $c,\{a, c\} \nsucc G$. The only possibility is that $a w \mapsto c$. Thus $w \succ Y$, contradicting the fact that $B=\emptyset$.

By Lemma 21, if $Y=\{y\}$ (say), then no vertex in $C$ is adjacent to $y$ and since $B=\emptyset$, it follows that $y$ is isolated in $G$, which is impossible. Hence $|Y| \geq 2$. Suppose that $|C|=1$. Since $|Y| \geq 2$, there is a vertex $y \in Y$ that is not adjacent to a vertex of $C$. But then $\operatorname{diam}(G)>3$, a contradiction. Hence $|C| \geq 2$.

For the necessity, let $G \in \mathcal{F}_{3}$ and assume that the conditions of the theorem hold. It is easy to see that there is no edge $a c \in E(G)$ such that $\{a, c\}$ dominates $G$. Thus $\gamma_{t}(G) \geq 3$. On the other hand, every shortest $y$ - $a$ path, $y \in Y$ and $a \in A$, is a total dominating set of cardinality three, implying that $\gamma_{t}(G)=3$. We now show that $G$ is $3_{t}$-critical. First consider $\{x, c\}$, for any $c \in C$. Since $c \succ A \cup C, c y \mapsto x$ for any $y \in Y$ adjacent to $c$. Next, consider $\{x, y\}$, for any $y \in Y$. Here it is also easy to see that $y c \mapsto x$ for any $c \in N(y) \cap C$. For any $a \in A$ and $y \in Y,\{a, y\} \succ G$. Finally we consider $\{c, y\}$ with $c y \notin E$. Since $y$ is the only vertex in $Y$ not adjacent to $c, c a \mapsto y$ for any $a \in A$.

Corollary 24. If $G \in \mathcal{F}_{3}$ is $3_{t}$-critical, then $\gamma(G)=2$.
See Figures 2 and 4(b) for examples of $3_{t}$-critical graphs in $\mathcal{F}_{3}$. Note that this family of $3_{t}$-critical graphs includes those graphs with minimum degree one characterised in Theorem 3 where $x$ is the endvertex of $G$.

Next we consider the family $\mathcal{F}_{4}$. See Figure 7 for an example.


Figure 7. A $3_{t}$-critical graph $G \in \mathcal{F}_{4}$
We now characterise the $3_{t}$-critical graphs $G \in F_{4}$ using the same notation as before.

Theorem 25. A graph $G \in F_{4}$ is $3_{t}$-critical if and only if the following conditions hold:
(1) $(x, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y\rangle$ is complete.
(2) $\langle C\rangle$ is complete and each $c \in C$ dominates exactly $|Y|-1$ vertices in $Y$.
(3) If $|Y| \geq 2$, then for every $y \in Y,\{x, y\} \succ G$ or there exists $w \in B \cup C$ such that $y w \mapsto x$. If $|Y|=1($ say $Y=\{y\})$, then $\{x, y\} \nsucc G$ and there exists $y^{\prime} \in B$ such that $y^{\prime} \succ A \cup C$ or $x^{\prime} \in A$ such that $x^{\prime} \succ B \cup C$.
(4) For every $c \in C$, there exists $w \in B \cup C \cup Y$ such that $c w \mapsto x$.
(5) For every $b \in B,\{x, b\} \succ G$ or there exists $w \in B \cup C \cup Y$ such that $b w \mapsto x$.
(6) For every $a \in A$ and $y \in Y,\{a, y\} \succ G$ or there exists $w \in A \cup C$ if $Y=\{y\}(w \in C$ if $|Y| \geq 2)$ such that aw $\mapsto y$.
(7) For each $a \in A$ and $c \in C$ with $a c \notin E(G)$, there exists $b \in B$ such that $a b \mapsto c$.
(8) For each $a \in A$ and $b \in B$ with $a b \notin E(G),\{a, b\} \succ G$ or there exists $a^{\prime} \in A$ such that $a^{\prime} b \mapsto a$ or $b^{\prime} \in B$ such that $a b^{\prime} \mapsto b$. For each $a b \in E(G)$ with $a \in A$ and $b \in B$, there exists $w \in A \cup B \cup C$ such that $w \notin(N(a) \cup N(b))$.
(9) For each $b \in B$ and $c \in C$ with $b c \notin E(G)$, there exists $a \in A$ such that $a b \mapsto c$.
(10) For each $c \in C$ and $y \in Y$ with $c y \notin E(G)$, there exists $a \in A$ such that $a c \mapsto y$.

Proof. Let $G \in \mathcal{F}_{4}$ be $3_{t}$-critical. Condition (1) follows directly from Theorem 7. By Lemma 22, $\langle C\rangle$ is complete. By Lemma 21, each vertex in $C$ is adjacent to exactly $|Y|-1$ vertices in $Y$.

Consider arbitrary $y \in Y$. If $\{x, y\} \succ G$, then $|Y| \geq 2$ since $C \neq \emptyset$ and $y$ must dominate $C$. Hence Condition (3) holds in this case. Therefore we may assume that $\{x, y\}$ does not dominate $G$. Since $G$ is $3_{t}$-critical, $x w \mapsto y$ or $y w \mapsto x$. If $x w \mapsto y$, then $w \in A$ implying that $w \succ B \cup C$ and that $Y=\{y\}$. Thus if $|Y| \geq 2$, then $y w \mapsto x$ and we have shown that Condition (3) holds if $|Y| \geq 2$. Therefore we may assume that $|Y|=1$. Now $G$ has the unique maximal diametrical pair $(\{x\},\{y\})$ and neither $x$ nor $y$ dominates any vertex in $C$. Hence $x x^{\prime} \mapsto y$ with $x^{\prime} \in A$ or $y y^{\prime} \mapsto x$ with $y^{\prime} \in B$, and Condition (3) follows.

Condition (4) follows from the fact that each $c \in C$ dominates at most $|Y|-1$ vertices in $Y$ and there is no $x^{\prime} \in A$ such that $x x^{\prime} \mapsto c$ for any $c \in C$.

Let $b$ be an arbitrary vertex in $B$. If $\{x, b\} \succ G$, then Condition (5) holds. Otherwise $x x^{\prime} \mapsto b$ for $x^{\prime} \in A$ or $b b^{\prime} \mapsto x$ for $b^{\prime} \in B \cup C \cup Y$. If $x x^{\prime} \mapsto b$, then $x^{\prime} \succ Y$ implying $d(x, y)=2$, a contradiction. Hence $b b^{\prime} \mapsto x$.

If for $a \in A$ and $y \in Y,\{a, y\} \succ G$, then Condition (6) holds. Otherwise $y y^{\prime} \mapsto a$ for $y^{\prime} \in N(y)$ or $a a^{\prime} \mapsto y$ for $a^{\prime} \in A \cup C$. If $y y^{\prime} \mapsto a$, then $x \in N\left(y^{\prime}\right)$ implying $d(x, y)<3$, a contradiction. Hence $a a^{\prime} \mapsto y$.

Consider $\{a, c\}$ where $a \in A$ and $c \in C$ are not adjacent. Since neither $a$ nor $c$ dominates $Y,\{a, c\} \nsucc G$. Therefore, $c a^{\prime} \mapsto a$ with $a^{\prime} \in A$ (to dominate $x$ ) or $a b^{\prime} \mapsto c$ with $b^{\prime} \in B$ (to dominate $Y$ ). If $c a^{\prime} \mapsto a$, then $c \succ Y$, contradicting that each $c \in C$ dominates at most $|Y|-1$ vertices in $Y$. Hence $a b^{\prime} \mapsto c$ and Condition (7) holds.

Condition (8) follows directly from the definition of $3_{t}$-critical graphs. If $b \in B$ and $c \in C$ with $b c \notin E(G)$, then $\{b, c\} \nsucc G$ since neither $b$ nor $c$ is adjacent to $x$. Since there is no $c^{\prime} \in N(c)$ such that $c c^{\prime} \mapsto b, b a^{\prime} \mapsto c$ with $a^{\prime} \in A$. Hence Condition (9) holds.

Finally we consider $\{c, y\}$ with $c y \notin E(G)$. Again since neither $c$ nor $y$ is adjacent to $x,\{c, y\} \nsucc G$. Also, since $y$ has no neighbour $y^{\prime}$ such that $y^{\prime} \succ x, c a^{\prime} \mapsto y$ with $a^{\prime} \in A$.

Let $G$ be a graph such that the stated properties hold. By Condition (8) there is no $a b \in E(G)$ with $a \in A$ and $b \in B$ such that $\{a, b\} \succ G$, and since no other edge dominates $G, \gamma_{t}(G) \geq 3$. By Condition (10), there is $a \in A$ for every $c \in C$ such that $a c \mapsto y$ for some $y \in Y$. Further, each $a \in A$ is adjacent to some $b \in B$ since ( $\{x\}, Y$ ) is the unique maximal diametrical pair. Therefore, $\{a, b, c\}$ is a total dominating set of $G$, implying that $\gamma_{t}(G) \leq 3$. Hence $\gamma_{t}(G)=3$. That $G$ is $\gamma_{t}$-critical, follows from Conditions (3) through (10).
Finally we consider a subclass of the family $\mathcal{F}_{4}$.
Lemma 26. If $G \in \mathcal{F}_{4}$ is $3_{t}$-critical and $\langle A\rangle$ is not complete, then every $y \in Y$ dominates at most $|C|-1$ vertices in $C$.

Proof. Let $u, v \in A$ with $u v \notin E(G)$ and suppose there is a vertex $y \in Y$ such that $y \succ C$. Consider $\{u, y\}$. Since $\{u, y\} \nsucc G$ and there is no vertex $c \in C$ such that $u c \mapsto y$, there must be a vertex $w \in N(y)$ such that $y w \mapsto u$. But then $d(y, x) \leq 2$, contradicting $\operatorname{diam}(G)=3$.

Lemma 27. If $G \in \mathcal{F}_{4}$ is $3_{t}$-critical and $\langle A\rangle$ is not complete, then $|C| \geq|Y|$.
Proof. Let $|C|=k$ and $|Y|=p$. Since every vertex in $C$ is adjacent to exactly $|Y|-1$ vertices in $Y$, there are exactly $k(p-1)$ edges from $C$ to $Y$. By Lemma 26, every $y \in Y$ dominates at most $|C|-1$ vertices in $C$.

Therefore there are at most $p(k-1)-s$ edges from $Y$ to $C, s \geq 0$. Thus

$$
p(k-1)-s=k(p-1)
$$

hence

$$
k-s=p
$$

and it follows that $k \geq p$.
Restricting our attention to the graphs $G \in \mathcal{F}_{4}$ with $\langle A\rangle$ not complete and $|Y|=|C|$, we are able to obtain a more concise and descriptive characterisation than the one given for the family $\mathcal{F}_{4}$.

Theorem 28. Let $G$ be a graph in $\mathcal{F}_{4}$ with $\langle A\rangle$ not complete and $|Y|=|C|$. Then $G$ is $3_{t}$-critical if and only if the following conditions hold:
(1) $(\{x\}, Y)$ is the unique maximal diametrical pair of $G$ and $\langle Y\rangle$ is complete.
(2) $\langle C\rangle$ is complete and $\langle C \cup Y\rangle$ is complete minus a perfect matching between $C$ and $Y$.
(3) Every vertex $c \in C$ dominates $A \cup B$.
(4) For every $a b \in E(\langle A \cup B\rangle)$, there is a vertex $a_{i} \in A$ or $b_{j} \in B$ not adjacent to $a$ and $b$ and if $a_{1}, a_{2}\left(b_{1}, b_{2}\right.$, respectively) are nonadjacent vertices in $A(B$, respectively $)$, then there is a vertex $b \in B(a \in A$, respectively) such that $a_{1} b \mapsto a_{2}\left(b_{1} a \mapsto b_{2}\right.$, respectively). Also for every $a \in A$ and $b \in B$ that are not adjacent, $\{a, b\} \succ G$ or there is a vertex $w$ such that $a w \mapsto b$ or $b w \mapsto a$.

Proof. Let $G \in \mathcal{F}_{4}$ with $\langle A\rangle$ not complete and $|Y|=|C|$ be $3_{t}$-critical. Condition (1) follows directly from Theorem 7. By Lemma 22, $\langle C\rangle$ is complete. By Lemmas 21 and 26, we have that each vertex in $C$ is adjacent to $|Y|-1$ vertices in $Y$ and if $\langle A\rangle$ is not complete, then each vertex in $Y$ is adjacent to at most $|C|-1$ vertices is $C$. Thus there are $|C|(|Y|-1)$ edges from $C$ to $Y$ and at most $|Y|(|C|-1)$ edges from $Y$ to $C$. Since $|C|=|Y|$, there are exactly $|Y|(|C|-1)$ edges from $Y$ to $C$ and so every vertex in $Y$ is adjacent to exactly $|C|-1$ vertices in $C$. Therefore, all edges minus a perfect matching are present between $C$ and $Y$.

To show that (3) holds, suppose that $a c \notin E(G), a \in A$ and $c \in C$. Consider $\{a, y\}$ where $y \in Y$ and $c y \notin E(G)$. Obviously, $\{a, y\} \nsucc G$. Since no vertex in $N[y]$ dominates $x$, it follows that $a z \mapsto y$ and $z \in C$. But this contradicts condition (2). Hence $c \succ A$ for each $c \in C$.

Now suppose that $c b \notin E(G), c \in C$ and $b \in B$, and consider $\{b, c\}$. Since neither $b$ nor $c$ is adjacent to $x,\{c, b\} \nsucc G$. Therefore $c w \mapsto b$ or $b w \mapsto c$ for $w \in A$ (to dominate $x$ ). But if $c w \mapsto b$, then $Y$ is not dominated, a contradiction. And if $b w \mapsto c$, then $w \notin N(c)$, contradicting the fact that every vertex $c \in C$ dominates $A$. Hence $c \succ B$ for each $c \in C$. Thus, condition (3) holds. Condition (4) follows from the fact that every $b \in B$ dominates $C \cup Y$ and every $a \in A$ dominates $C$.

Let $G \in \mathcal{F}_{4}$ with $\langle A\rangle$ not complete and $|Y|=|C|$ and assume that the conditions of the theorem hold. Since no pair of adjacent vertices dominate $G, \gamma_{t}(G) \geq 3$. Further, $\{a, b, c\}$, where $a \in A, b \in B$ and $c \in C$, is a total dominating set, so $\gamma_{t}(G)=3$. To show that $G$ is $3_{t}$-critical, we first consider $\{x, y\}$ for $y \in Y$. Then $c y \mapsto x$ where $c \in N(y)$. A similar argument holds for $\{x, c\}$. Next consider $\{x, b\}$ for $b \in B$. Then $b c \mapsto x$ for any $c \in C$. For $\{a, y\}, a c \mapsto y$ where $c \in C-N(y)$. It now follows from condition (4) that $G$ is $3_{t}$-critical.

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## References

[1] E. Cockayne, R. Dawes and S. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211-219.
[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).
[3] T.W. Haynes, C.M. Mynhardt and L.C. van der Merwe, Total domination edge critical graphs, Utilitas Math. 54 (1998) 229-240.
[4] T.W. Haynes, C.M. Mynhardt and L.C. van der Merwe, Criticality index of total domination, Congr. Numer. 131 (1998) 67-73.
[5] D.P. Sumner and P. Blitch, Domination critical graphs, J. Combin. Theory (B) 34 (1983) 65-76.
[6] D.P. Sumner and E. Wojcicka, Graphs critical with respect to the domination number, Domination in Graphs: Advanced Topics (Chapter 16), T.W. Haynes, S.T. Hedetniemi and P.J. Slater, eds. (Marcel Dekker, Inc., New York, 1998).

