VERTEX-DISJOINT STARS IN GRAPHS

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Abstract

In this paper, we give a sufficient condition for a graph to contain vertex-disjoint stars of a given size. It is proved that if the minimum degree of the graph is at least $k + t - 1$ and the order is at least $(t + 1)k + O(t^2)$, then the graph contains $k$ vertex-disjoint copies of a star $K_{1,t}$. The condition on the minimum degree is sharp, and there is an example showing that the term $O(t^2)$ for the number of uncovered vertices is necessary in a sense.

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1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph $G$, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of $G$, respectively.

For a graph $F$ and a positive integer $k$, $kF$ denotes the vertex-disjoint union of $k$ copies of $F$. A spanning subgraph isomorphic to $kF$ for some integer $k$ is called an $F$-factor. There are several results concerning minimum degree conditions for a graph to have an $F$-factor for several specific graphs $F$. The result of Corrádi and Hajnal [3] implies that $\delta(G) \geq \frac{2}{3}|V(G)|$ suffices for the existence of a $K_3$-factor. (When we consider an $F$-factor of a graph $G$, we always assume that $|V(G)|$ is a multiple of $|V(F)|$). Dirac [4] generalized this result by showing that if $\delta(G) \geq \frac{1}{2}(|V(G)| + k)$,
then \( G \) contains \( k \) vertex-disjoint triangles for any integer \( k \) with \( 3k \leq |V(G)| \). Enomoto, Kaneko and Tuza [7] proved for \( F = P_3 \) (the path of order three) that \( \delta(G) \geq \frac{1}{3}|V(G)| \) is sufficient for the existence of an \( F \)-factor if we assume that \( G \) is connected. Hajnal and Szemerédi [8] proved that for \( F = K_t \) \( \delta(G) \geq \frac{1}{t-1}|V(G)| \) suffices. More generally, Alon and Yuster [2] proved an asymptotic result, which states that \( \delta(G) \geq \left( \frac{\chi(F)-1}{\chi(F)} + o(1) \right)|V(G)| \) assures the existence of an \( F \)-factor, where \( \chi(F) \) denotes the chromatic number of \( F \).

On the other hand, if we want to find \( k \) vertex-disjoint copies of \( F \) in a graph \( G \) of order slightly larger than \( k|F| \), and if \( F \) admits a \( \chi(F) \)-coloring in which some color classes are tiny, then a much weaker condition may guarantee the existence. Komlós [9] (and Alon and Fischer [1] for bipartite case) have proved that the required minimum degree of \( G \) is

\[
\frac{1}{\chi(F)-1} \left( \chi(F) - 2 + \frac{\alpha}{|V(F)|} \right)|V(G)|,
\]

where \( \alpha \) is the smallest possible color class size in any \( \chi(F) \)-coloring of \( F \).

In the case where \( F \) is a star \( K_{1,t} \), Alon and Yuster’s result implies that \( \delta(G) \geq (\frac{1}{t} + o(1))|V(G)| \) is sufficient for the existence of a \( K_{1,t} \)-factor, and Komlós, Alon and Fischer’s result implies that \( \delta(G) \geq \frac{1}{t-1}|V(G)| \) is sufficient for the existence of \( k \) copies of \( K_{1,t} \) if \( |V(G)| \) is large. In this paper, we prove the following theorem, in which the required minimum degree of \( G \) does not depend on \( |V(G)| \). The proof is given in the next section.

**Theorem 1.** Let \( t \) be an integer with \( t \geq 3 \). If \( G \) is a graph of order \( n \geq (t+1)k + 2t^2 - 3t - 1 \) with minimum degree at least \( k + t - 1 \), then \( G \) contains \( k \) vertex-disjoint copies of \( K_{1,t} \).

The minimum degree condition in the theorem cannot be replaced by any weaker condition even if the order of the graph is assumed much larger. To see this, let \( H \) be a \( t - 1 \) regular graph of order sufficiently large, and let \( G \) be obtained from \( H \) by adding \( k - 1 \) new vertices which are joined to all other vertices. Then \( \delta(G) = k + t - 2 \). Since any \( K_{1,t} \) subgraph of \( G \) must contain one of the new vertices, \( G \) does not contain \( k \) vertex-disjoint copies of \( K_{1,t} \).

On the other hand, the following example shows that the term \( O(t^2) \) for the number of uncovered vertices is necessary. Let \( k_1 + k_2 + \cdots + k_t = k - 1 \) so that \( |k_i - k_j| \leq 1 \) for any \( i \) and \( j \). We define the graph \( G \) to be the vertex-disjoint union of the complete graphs \( K_{(t+1)k_1+t}, K_{(t+1)k_2+t}, \ldots, K_{(t+1)k_t+t} \).
Then, $|V(G)| = (t + 1)(k - 1) + t^2 = (t + 1)k + t^2 - t - 1$ and $\delta(G) = (t + 1)\left\lfloor \frac{k - 1}{t} \right\rfloor + t - 1 \geq k + t - 1$ (if $k \gg t$). However, it is obvious that $G$ contains at most $k - 1$ copies of $K_{1,t}$.

This example suggests that the same conclusion as in Theorem 1 follows if $|V(G)| \geq (t + 1)k + t^2 - t$. In fact, it is known to be true for $t \leq 3$. The case $t = 1$ is an easy exercise. The case $t = 3$ is proved in [6].

Theorem 2 [6]. If $G$ is a graph with $|V(G)| \geq 4k + 6$ and $\delta(G) \geq k + 2$, then $G$ contains $k$ vertex-disjoint copies of $K_{1,3}$.

The case $t = 2$ can be proved in the following way. We use the following theorem due to Enomoto [5].

Theorem 3 [5]. Let $G$ be a connected graph of order $n$ and $n = n_1 + \cdots + n_k$ with $n_i \geq 2$ ($1 \leq i \leq k$). If $\delta(G) \geq k$, then $V(G)$ can be partitioned into $V_1, \ldots, V_k$ so that for each $i$, $|V_i| = n_i$ and $V_i$ induces a subgraph without isolated vertices.

Corollary 4. Let $G$ be a connected graph of order $n$, and $k$ be an integer with $3k \leq n$. If

$$\delta(G) \geq \begin{cases} k, & \text{if } n - 3k \text{ is even,} \\ k + 1, & \text{if } n - 3k \text{ is odd,} \end{cases}$$

then $G$ contains $k$ vertex-disjoint copies of $P_3$.

Proof. If $n - 3k$ is even, then put $n_1 = \cdots = n_{k-1} = 3$ and $n_k = n - 3k + 3$, and apply Theorem 3. If $n - 3k$ is odd, then by deleting one vertex from $G$ so that the resulting graph is connected, we can apply the previous case. ■

Now we can prove the following theorem.

Theorem 5. Let $G$ be a graph of order $n$ with $n \geq 3k + 2$. If $\delta(G) \geq k + 1$, then $G$ contains $k$ vertex-disjoint copies of $P_3$.

Proof. If $G$ is connected, or if $G$ has a component of order at least $3k$, then the result follows immediately from Corollary 4. Suppose that $G$ is disconnected and each component is order less than $3k$. Note that by Corollary 4, each component $C$ of $G$ contains $\lfloor |V(C)|/3 \rfloor$ vertex-disjoint copies of $P_3$. Also, since $\delta(G) \geq k + 1$, each component has at least $k + 2$ vertices.
If $G$ has at least three components, then $G$ contains at least $3\lceil \frac{k+2}{3} \rceil \geq k$ copies of $P_3$, and we are done. If $G$ consists of two components of orders $n_1$ and $n_2$, then the number of vertex-disjoint copies of $P_3$ in $G$ is at least $\lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2}{3} \rceil \geq \lceil \frac{3k+2-4}{3} \rceil = k$. \hfill \blacksquare

However, for the general case of the stronger statement, we need more crucial argument than the one used in this paper.

2. Proof of Theorem 1

Let $t$ be an integer with $t \geq 3$, and let $G$ be a graph of order at least $(t+1)k + 2t^2 - 3t - 1$ and minimum degree at least $k + t - 1$.

We use the following notation and terminology. For $S \subset V(G)$, we write $\langle S \rangle$ for the subgraph of $G$ induced by $S$. For disjoint vertex sets $S$ and $T$, we denote the set of edges joining $S$ and $T$ by $E(S,T)$.

We consider a partition $V(G) = X \cup Y \cup Z$ satisfying the following conditions:

(a) $|X| = (t+1)p$ and $X$ contains $p$ vertex-disjoint copies of $K_{1,t}$, say $C_1, C_2, \ldots, C_p$.

(b) The vertices of $Y$ can be labelled $y_1, y_2, \ldots, y_q$ so that for each $r$ ($1 \leq r \leq q$), $|\mathcal{N}_G(y_r) \cap Z| \geq rt + (2t - 1)$.

Note that $X = Y = \emptyset$ and $Z = V(G)$ satisfy the above conditions with $p = q = 0$. We choose such a partition so that $p + q$ is maximum, and subject to this condition, $q$ is maximum possible.

**Claim 1.** For any subset $A \subset Z$ with $|A| \leq 2t - 1$, $G - X - A$ contains $q$ vertex-disjoint copies of $K_{1,t}$. In particular, $G$ contains $p + q$ vertex-disjoint copies of $K_{1,t}$.

**Proof.** By the condition (b), it follows that $|\mathcal{N}_G(y_r) \cap (Z - A)| \geq rt + 2t - 1 - |A| \geq rt$ for each $1 \leq r \leq q$. Therefore we can complete to take $q$ stars in $\langle Y \cup (Z - A) \rangle$ whose centers are $y_1, y_2, \ldots, y_q$, respectively. \hfill \blacksquare

**Claim 2.** For any subset $A \subset Z$ with $|A| \leq 2t - 1$, $\langle X \cup A \rangle$ does not contain $p + 1$ vertex-disjoint copies of $K_{1,t}$.

**Proof.** Suppose that $\langle X \cup A \rangle$ contains $(p + 1)K_{1,t}$. Then by Claim 1, $G$ contains $(p + q + 1)K_{1,t}$. Let $V(G) = X' \cup Y' \cup Z'$ be a partition such that $X'$
is the set of vertices contained in \((p + q + 1)K_{1,t}\) and \(Y' = \emptyset\). This partition satisfies the condition (a) and (b), and contradicts the maximality of \(p + q\).

In particular, we have the following.

**Claim 3.** The maximum degree of \(\langle Z \rangle\) is less than \(t\).

Let \(a\) be the center of any star \(C_i\) in \(X\). If \(|N_G(a) \cap Z| \geq tq + 2t - 1\), then we put \(X' = X - V(C_i), Y' = Y \cup \{a\}\) with \(y_{q+1} = a\), and \(Z' = Z \cup (V(C_i) - \{a\})\). Then \((X')\) contains \(p - 1\) vertex-disjoint \(K_{1,t}\)'s, and \(|N_G(y_{q+1}) \cap Z'| = |N_G(a) \cap Z| + t \geq t(q + 1) + 2t - 1\). This contradicts the maximality of \(q\). Hence we have

\[
(1) \quad |N_G(a) \cap Z| \leq tq + 2t - 2.
\]

By a similar argument, for each leaf \(b\) of any star \(C_i\) in \(X\), we have

\[
(2) \quad |N_G(b) \cap Z| \leq tq + 3t - 3.
\]

**Claim 4.** For each \(1 \leq i \leq p\), \(|E(C_i, Z)| \leq \max\{tq + t^2 + t - 2, 2t^2 - 2t\}\).

**Proof.** Let \(a\) be the center and \(b_1, b_2, \ldots, b_i\) be the leaves of \(C_i\).

**Case 1.** \(|E(a, Z)| \geq t + 1\).

In this case, each \(b_j\) is adjacent to at most \(t - 1\) vertices in \(Z\). For otherwise, we can take \(A \subset N(b_j) \cap Z\) with \(|A| = t\) and \(z \in N(a) \cap Z - A\) so that \(\langle \{b_j\} \cup A\rangle\) and \(\langle (V(C_i) - \{b_j\}) \cup \{z\}\rangle\) contain a \(K_{1,t}\). This contradicts \(N_G(z) \cap Z\) = \(|N_G(a) \cap Z| + t \geq t(q + 1) + 2t - 1\).

Then, since \(|E(a, Z)| \leq tq + 2t - 2\) by (1),

\[
|E(V(C_i), Z)| = |E(a, Z)| + \sum_{j=1}^{t} |E(b_j, Z)| \\
\leq tq + 2t - 2 + t(t - 1) = tq + t^2 + t - 2.
\]

**Case 2.** \(1 \leq |E(a, Z)| \leq t\).

If \(|E(b_j, Z)| \geq t + 1\) for some \(j\) \((1 \leq j \leq t)\), then we can take \(z \in N(a) \cap Z\) and \(A \subset N(b_j) \cap Z - \{z\}\) with \(|A| = t\) so that \(\langle V(C_i) \cup A \cup \{z\}\rangle\) contains \(2K_{1,t}\), a contradiction. Hence \(|E(b_j, Z)| \leq t\) for all \(j\). Thus, \(|E(V(C_i), Z)| \leq (t + 1)t \leq 2t^2 - 2t\), since \(t \geq 3\).
Case 3. $|E(a, Z)| = 0$.

If each leaf of $C_i$ is adjacent to at most $2t - 2$ vertices in $Z$, then we have $|E(V(C_i), Z)| \leq 2t^2 - 2t$. Hence we may assume that there exists a vertex $b_h$ ($1 \leq h \leq t$) with $|E(b_h, Z)| \geq t - 1$. If $|E(b_j, Z)| \geq t - 1$ for some $j$ with $j \neq h$, then we can take $A \subset N(b_j) \cap Z$ with $|A| = t - 1$ and $A' \subset N(b_h) \cap Z - A$ with $|A'| = t$ so that $\{b_h, b_j, a\} \cup A \cup A'$ contains $2K_t$-s, a contradiction. Hence $|E(b_j, Z)| \leq t - 2$ for all $j \neq h$. Since $|E(b_h, Z)| \leq tq + 3t - 3$ by (2), we have $|E(V(C_i), Z)| \leq tq + 3t - 3 + (t - 1)(t - 2) < tq + t^2 + t - 2$.

This completes the proof of Claim 4.

Now, we shall estimate the number of edges joining $X$ and $Z$ in two ways, by assuming that $G$ does not contain $k$ vertex-disjoint $K_{1,t}$-s. By Claim 3, each vertex in $Z$ is adjacent to at least $(k + t - 1) - (t - 1) - q = k - q$ vertices of $X$. Hence,

$$|E(X, Z)| \geq (k - q)|Z| = (k - q)(n - (t + 1)p - q).$$

On the other hand, it follows from Claim 4 that

$$|E(X, Z)| = \sum_{i=1}^{p} |E(V(C_i), Z)| \leq p \cdot \max\{tq + t^2 + t - 2, 2t^2 - 2t\}.$$

If $tq + t^2 + t - 2 \geq 2t^2 - 2t$, or equivalently if $q \geq t - 2$, then

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(tq + t^2 + t - 2),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)k + t^2 + t - 2 - q).$$

By Claim 1, we may assume that $p + q \leq k - 1$. Thus the above inequality implies that

$$(k - q)(n - q) \leq (k - 1 - q)((t + 1)k + t^2 + t - 2 - q),$$

and hence

$$n \leq (t + 1)k + t^2 + t - 2 - \frac{(t + 1)k + t^2 + t - 2 - q}{k - q}$$

$$= (t + 1)k + 2t^2 - 3t - 1 - \left(t^2 - 4t + 3 + \frac{(t - 1)k + t^2 + t - 2 + q}{k - q}\right)$$

$$< (t + 1)k + 2t^2 - 3t - 1.$$
This contradicts the assumption that $n \geq (t + 1)k + 2t^2 - 3t - 1$.

If $q \leq t - 3$, then since $tq + t^2 + t < 2t^2 - 2t$,

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(2t^2 - 2t),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)(k - q) + 2t^2 - 2t).$$

Since $p + q \leq k - 1$,

$$(k - q)(n - q) \leq (k - 1 - q)((t + 1)(k - q) + 2t^2 - 2t),$$

$$n \leq (t + 1)k + 2t^2 - 3t - 1 - tq - \frac{2t^2 - 2t}{k - q}$$

$$< (t + 1)k + 2t^2 - 3t - 1.$$ 

This is a contradiction.

This completes the proof of Theorem 1.

References


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