# VERTEX-DISJOINT STARS IN GRAPHS 

Katsuhiro Ota<br>Department of Mathematics<br>Keio University<br>Yokohama, 223-8522 Japan<br>e-mail: ohta@math.keio.ac.jp


#### Abstract

In this paper, we give a sufficient condition for a graph to contain vertex-disjoint stars of a given size. It is proved that if the minimum degree of the graph is at least $k+t-1$ and the order is at least $(t+1) k+O\left(t^{2}\right)$, then the graph contains $k$ vertex-disjoint copies of a star $K_{1, t}$. The condition on the minimum degree is sharp, and there is an example showing that the term $O\left(t^{2}\right)$ for the number of uncovered vertices is necessary in a sense.


Keywords: stars, vertex-disjoint copies, minimum degree.
2000 Mathematics Subject Classification: 05C35, 05C70.

## 1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph $G$, we denote by $V(G), E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of $G$, respectively.

For a graph $F$ and a positive integer $k, k F$ denotes the vertex-disjoint union of $k$ copies of $F$. A spanning subgraph isomorphic to $k F$ for some integer $k$ is called an $F$-factor. There are several results concerning minimum degree conditions for a graph to have an $F$-factor for several specific graphs $F$. The result of Corrádi and Hajnal [3] implies that $\delta(G) \geq \frac{2}{3}|V(G)|$ suffices for the existence of a $K_{3}$-factor. (When we consider an $F$-factor of a graph $G$, we always assume that $|V(G)|$ is a multiple of $|V(F)|)$. Dirac [4] generalized this result by showing that if $\delta(G) \geq \frac{1}{2}(|V(G)|+k)$,
then $G$ contains $k$ vertex-disjoint triangles for any integer $k$ with $3 k \leq$ $|V(G)|$. Enomoto, Kaneko and Tuza $[7]$ proved for $F=P_{3}$ (the path of order three) that $\delta(G) \geq \frac{1}{3}|V(G)|$ is sufficient for the existence of an $F$-factor if we assume that $G$ is connected. Hajnal and Szemerédi [8] proved that for $F=$ $K_{t} \delta(G) \geq \frac{t-1}{t}|V(G)|$ suffices. More generally, Alon and Yuster [2] proved an asymptotic result, which states that $\delta(G) \geq\left(\frac{\chi(F)-1}{\chi(F)}+o(1)\right)|V(G)|$ assures the existence of an $F$-factor, where $\chi(F)$ denotes the chromatic number of $F$.

On the other hand, if we want to find $k$ vertex-disjoint copies of $F$ in a graph $G$ of order slightly larger than $k|F|$, and if $F$ admits a $\chi(F)$-coloring in which some color classes are tiny, then a much weaker condition may guarantee the existence. Komlós [9] (and Alon and Fischer [1] for bipartite case) have proved that the required minimum degree of $G$ is

$$
\frac{1}{\chi(F)-1}\left(\chi(F)-2+\frac{\alpha}{|V(F)|}\right)|V(G)|,
$$

where $\alpha$ is the smallest possible color class size in any $\chi(F)$-coloring of $F$.
In the case where $F$ is a star $K_{1, t}$, Alon and Yuster's result implies that $\delta(G) \geq\left(\frac{1}{2}+o(1)\right)|V(G)|$ is sufficient for the existence of a $K_{1, t}$-factor, and Komlós, Alon and Fischer's result implies that $\delta(G) \geq \frac{1}{t+1}|V(G)|$ is sufficient for the existence of $k$ copies of $K_{1, t}$ if $|V(G)|$ is large. In this paper, we prove the following theorem, in which the required minimum degree of $G$ does not depend on $|V(G)|$. The proof is given in the next section.

Theorem 1. Let $t$ be an integer with $t \geq 3$. If $G$ is a graph of order $n \geq(t+1) k+2 t^{2}-3 t-1$ with minimum degree at least $k+t-1$, then $G$ contains $k$ vertex-disjoint copies of $K_{1, t}$.

The minimum degree condition in the theorem cannot be replaced by any weaker condition even if the order of the graph is assumed much larger. To see this, let $H$ be a $t-1$ regular graph of order sufficiently large, and let $G$ be obtained from $H$ by adding $k-1$ new vertices which are joined to all other vertices. Then $\delta(G)=k+t-2$. Since any $K_{1, t}$ subgraph of $G$ must contain one of the new vertices, $G$ does not contain $k$ vertex-disjoint copies of $K_{1, t}$.

On the other hand, the following example shows that the term $O\left(t^{2}\right)$ for the number of uncovered vertices is necessary. Let $k_{1}+k_{2}+\cdots+k_{t}=k-1$ so that $\left|k_{i}-k_{j}\right| \leq 1$ for any $i$ and $j$. We define the graph $G$ to be the vertexdisjoint union of the complete graphs $K_{(t+1) k_{1}+t}, K_{(t+1) k_{2}+t}, \ldots, K_{(t+1) k_{t}+t}$.

Then, $|V(G)|=(t+1)(k-1)+t^{2}=(t+1) k+t^{2}-t-1$ and $\delta(G)=$ $(t+1)\left\lfloor\frac{k-1}{t}\right\rfloor+t-1 \geq k+t-1$ (if $k \gg t$ ). However, it is obvious that $G$ contains at most $k-1$ copies of $K_{1, t}$.

This example suggests that the same conclusion as in Theorem 1 follows if $|V(G)| \geq(t+1) k+t^{2}-t$. In fact, it is known to be true for $t \leq 3$. The case $t=1$ is an easy exercise. The case $t=3$ is proved in [6].

Theorem 2 [6]. If $G$ is a graph with $|V(G)| \geq 4 k+6$ and $\delta(G) \geq k+2$, then $G$ contains $k$ vertex-disjoint copies of $K_{1,3}$.

The case $t=2$ can be proved in the following way. We use the following theorem due to Enomoto [5].

Theorem 3 [5]. Let $G$ be a connected graph of order $n$ and $n=n_{1}+\cdots+n_{k}$ with $n_{i} \geq 2(1 \leq i \leq k)$. If $\delta(G) \geq k$, then $V(G)$ can be partitioned into $V_{1}, \ldots, V_{k}$ so that for each $i,\left|V_{i}\right|=n_{i}$ and $V_{i}$ induces a subgraph without isolated vertices.

Corollary 4. Let $G$ be a connected graph of order $n$, and $k$ be an integer with $3 k \leq n$. If

$$
\delta(G) \geq \begin{cases}k, & \text { if } n-3 k \text { is even } \\ k+1, & \text { if } n-3 k \text { is odd }\end{cases}
$$

then $G$ contains $k$ vertex-disjoint copies of $P_{3}$.
Proof. If $n-3 k$ is even, then put $n_{1}=\cdots=n_{k-1}=3$ and $n_{k}=n-3 k+3$, and apply Theorem 3. If $n-3 k$ is odd, then by deleting one vertex from $G$ so that the resulting graph is connected, we can apply the previous case.

Now we can prove the following theorem.

Theorem 5. Let $G$ be a graph of order $n$ with $n \geq 3 k+2$. If $\delta(G) \geq k+1$, then $G$ contains $k$ vertex-disjoint copies of $P_{3}$.

Proof. If $G$ is connected, or if $G$ has a component of order at least $3 k$, then the result follows immediately from Corollary 4. Suppose that $G$ is disconnected and each component is order less than $3 k$. Note that by Corollary 4, each component $C$ of $G$ contains $\lfloor|V(C)| / 3\rfloor$ vertex-disjoint copies of $P_{3}$. Also, since $\delta(G) \geq k+1$, each component has at least $k+2$ vertices.

If $G$ has at least three components, then $G$ contains at least $3\left\lfloor\frac{k+2}{3}\right\rfloor \geq k$ copies of $P_{3}$, and we are done. If $G$ consists of two components of orders $n_{1}$ and $n_{2}$, then the number of vertex-disjoint copies of $P_{3}$ in $G$ is at least $\left\lfloor\frac{n_{1}}{3}\right\rfloor+\left\lfloor\frac{n_{2}}{3}\right\rfloor \geq\left\lceil\frac{n_{1}+n_{2}-4}{3}\right\rceil \geq\left\lceil\frac{3 k+2-4}{3}\right\rceil=k$.

However, for the general case of the stronger statement, we need more crucial argument than the one used in this paper.

## 2. Proof of Theorem 1

Let $t$ be an integer with $t \geq 3$, and let $G$ be a graph of order at least $(t+1) k+2 t^{2}-3 t-1$ and minimum degree at least $k+t-1$.

We use the following notation and terminology. For $S \subset V(G)$, we write $\langle S\rangle$ for the subgraph of $G$ induced by $S$. For disjoint vertex sets $S$ and $T$, we denote the set of edges joining $S$ and $T$ by $E(S, T)$.

We consider a partition $V(G)=X \cup Y \cup Z$ satisfying the following conditions:
(a) $|X|=(t+1) p$ and $X$ contains $p$ vertex-disjoint copies of $K_{1, t}$, say $C_{1}$, $C_{2}, \ldots, C_{p}$.
(b) The vertices of $Y$ can be labelled $y_{1}, y_{2}, \ldots, y_{q}$ so that for each $r(1 \leq$ $r \leq q),\left|N_{G}\left(y_{r}\right) \cap Z\right| \geq r t+(2 t-1)$.
Note that $X=Y=\emptyset$ and $Z=V(G)$ satisfy the above conditions with $p=q=0$. We choose such a partition so that $p+q$ is maximum, and subject to this condition, $q$ is maximum possible.

Claim 1. For any subset $A \subset Z$ with $|A| \leq 2 t-1, G-X-A$ contains $q$ vertex-disjoint copies of $K_{1, t}$. In particular, $G$ contains $p+q$ vertex-disjoint copies of $K_{1, t}$.

Proof. By the condition (b), it follows that $\left|N_{G}\left(y_{r}\right) \cap(Z-A)\right| \geq r t+2 t-$ $1-|A| \geq r t$ for each $1 \leq r \leq q$. Therefore we can complete to take $q$ stars in $\langle Y \cup(Z-A)\rangle$ whose centers are $y_{1}, y_{2}, \cdots, y_{q}$, respectively.

Claim 2. For any subset $A \subset Z$ with $|A| \leq 2 t-1,\langle X \cup A\rangle$ does not contain $p+1$ vertex-disjoint copies of $K_{1, t}$.

Proof. Suppose that $\langle X \cup A\rangle$ contains $(p+1) K_{1, t}$. Then by Claim 1, $G$ contains $(p+q+1) K_{1, t}$. Let $V(G)=X^{\prime} \cup Y^{\prime} \cup Z^{\prime}$ be a partition such that $X^{\prime}$
is the set of vertices contained in $(p+q+1) K_{1, t}$ and $Y^{\prime}=\emptyset$. This partition satisfies the condition (a) and (b), and contradicts the maximality of $p+q$. In particular, we have the following.

Claim 3. The maximum degree of $\langle Z\rangle$ is less than $t$.
Let $a$ be the center of any star $C_{i}$ in $X$. If $\left|N_{G}(a) \cap Z\right| \geq t q+2 t-1$, then we put $X^{\prime}=X-V\left(C_{i}\right), Y^{\prime}=Y \cup\{a\}$ with $y_{q+1}=a$, and $Z^{\prime}=$ $Z \cup\left(V\left(C_{i}\right)-\{a\}\right)$. Then $\left\langle X^{\prime}\right\rangle$ contains $p-1$ vertex-disjoint $K_{1, t}$ 's, and $\left|N_{G}\left(y_{q+1}\right) \cap Z^{\prime}\right|=\left|N_{G}(a) \cap Z\right|+t \geq t(q+1)+2 t-1$. This contradicts the maximality of $q$. Hence we have

$$
\begin{equation*}
\left|N_{G}(a) \cap Z\right| \leq t q+2 t-2 . \tag{1}
\end{equation*}
$$

By a similar argument, for each leaf $b$ of any star $C_{i}$ in $X$, we have

$$
\begin{equation*}
\left|N_{G}(b) \cap Z\right| \leq t q+3 t-3 . \tag{2}
\end{equation*}
$$

Claim 4. For each $1 \leq i \leq p,\left|E\left(C_{i}, Z\right)\right| \leq \max \left\{t q+t^{2}+t-2,2 t^{2}-2 t\right\}$.
Proof. Let $a$ be the center and $b_{1}, b_{2}, \ldots, b_{t}$ be the leaves of $C_{i}$.
Case 1. $|E(a, Z)| \geq t+1$.
In this case, each $b_{j}$ is adjacent to at most $t-1$ vertices in $Z$. For otherwise, we can take $A \subset N\left(b_{j}\right) \cap Z$ with $|A|=t$ and $z \in N(a) \cap Z-A$ so that $\left\langle\left\{b_{j}\right\} \cup A\right\rangle$ and $\left\langle\left(V\left(C_{i}\right)-\left\{b_{j}\right\}\right) \cup\{z\}\right\rangle$ contain a $K_{1, t}$. This contradicts Claim 2. Hence $\left|E\left(b_{j}, Z\right)\right| \leq t-1$ for all $j(1 \leq j \leq t)$. Then, since $|E(a, Z)| \leq t q+2 t-2$ by (1),

$$
\begin{aligned}
\left|E\left(V\left(C_{i}\right), Z\right)\right| & =|E(a, Z)|+\sum_{j=1}^{t}\left|E\left(b_{j}, Z\right)\right| \\
& \leq t q+2 t-2+t(t-1)=t q+t^{2}+t-2 .
\end{aligned}
$$

Case 2. $1 \leq|E(a, Z)| \leq t$.
If $\left|E\left(b_{j}, Z\right)\right| \geq t+1$ for some $j(1 \leq j \leq t)$, then we can take $z \in N(a) \cap Z$ and $A \subset N\left(b_{j}\right) \cap Z-\{z\}$ with $|A|=t$ so that $\left\langle V\left(C_{i}\right) \cup A \cup\{z\}\right\rangle$ contains $2 K_{1, t}$, a contradiction. Hence $\left|E\left(b_{j}, Z\right)\right| \leq t$ for all $j$. Thus, $\left|E\left(V\left(C_{i}\right), Z\right)\right| \leq$ $(t+1) t \leq 2 t^{2}-2 t$, since $t \geq 3$.

Case 3. $|E(a, Z)|=0$.
If each leaf of $C_{i}$ is adjacent to at most $2 t-2$ vertices in $Z$, then we have $\left|E\left(V\left(C_{i}\right), Z\right)\right| \leq 2 t^{2}-2 t$. Hence we may assume that there exists a vertex $b_{h}(1 \leq h \leq t)$ with $\left|E\left(b_{h}, Z\right)\right| \geq 2 t-1$. If $\left|E\left(b_{j}, Z\right)\right| \geq t-1$ for some $j$ with $j \neq h$, then we can take $A \subset N\left(b_{j}\right) \cap Z$ with $|A|=t-1$ and $A^{\prime} \subset N\left(b_{h}\right) \cap Z-A$ with $\left|A^{\prime}\right|=t$ so that $\left\langle\left\{b_{h}, b_{j}, a\right\} \cup A \cup A^{\prime}\right\rangle$ contains $2 K_{1, t}$ 's, a contradiction. Hence $\left|E\left(b_{j}, Z\right)\right| \leq t-2$ for all $j \neq h$. Since $\left|E\left(b_{h}, Z\right)\right| \leq t q+3 t-3$ by (2), we have $\left|E\left(V\left(C_{i}\right), Z\right)\right| \leq t q+3 t-3+(t-1)(t-2)<t q+t^{2}+t-2$.
This completes the proof of Claim 4.
Now, we shall estimate the number of edges joining $X$ and $Z$ in two ways, by assuming that $G$ does not contain $k$ vertex-disjoint $K_{1, t}$ 's. By Claim 3 , each vertex in $Z$ is adjacent to at least $(k+t-1)-(t-1)-q=k-q$ vertices of $X$. Hence,

$$
|E(X, Z)| \geq(k-q)|Z|=(k-q)(n-(t+1) p-q) .
$$

On the other hand, it follows from Claim 4 that

$$
\begin{aligned}
|E(X, Z)| & =\sum_{i=1}^{p}\left|E\left(V\left(C_{i}\right), Z\right)\right| \\
& \leq p \cdot \max \left\{t q+t^{2}+t-2,2 t^{2}-2 t\right\}
\end{aligned}
$$

If $t q+t^{2}+t-2 \geq 2 t^{2}-2 t$, or equivalently if $q \geq t-2$, then

$$
(k-q)(n-(t+1) p-q) \leq|E(X, Z)| \leq p\left(t q+t^{2}+t-2\right)
$$

and hence

$$
(k-q)(n-q) \leq p\left((t+1) k+t^{2}+t-2-q\right)
$$

By Claim 1, we may assume that $p+q \leq k-1$. Thus the above inequality implies that

$$
(k-q)(n-q) \leq(k-1-q)\left((t+1) k+t^{2}+t-2-q\right)
$$

and hence

$$
\begin{aligned}
n & \leq(t+1) k+t^{2}+t-2-\frac{(t+1) k+t^{2}+t-2-q}{k-q} \\
& =(t+1) k+2 t^{2}-3 t-1-\left(t^{2}-4 t+3+\frac{(t-1) k+t^{2}+t-2+q}{k-q}\right) \\
& <(t+1) k+2 t^{2}-3 t-1
\end{aligned}
$$

This contradicts the assumption that $n \geq(t+1) k+2 t^{2}-3 t-1$.
If $q \leq t-3$, then since $t q+t^{2}+t-2<2 t^{2}-2 t$,

$$
(k-q)(n-(t+1) p-q) \leq|E(X, Z)| \leq p\left(2 t^{2}-2 t\right),
$$

and hence

$$
(k-q)(n-q) \leq p\left((t+1)(k-q)+2 t^{2}-2 t\right) .
$$

Since $p+q \leq k-1$,

$$
\begin{aligned}
(k-q)(n-q) & \leq(k-1-q)\left((t+1)(k-q)+2 t^{2}-2 t\right), \\
n & \leq(t+1) k+2 t^{2}-3 t-1-t q-\frac{2 t^{2}-2 t}{k-q} \\
& <(t+1) k+2 t^{2}-3 t-1 .
\end{aligned}
$$

This is a contradiction.
This completes the proof of Theorem 1.

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Received 27 September 2000
Revised 19 March 2001

