Discussiones Mathematicae
Graph Theory 21 (2001) 179–185

VERTEX-DISJOINT STARS IN GRAPHS

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Abstract

In this paper, we give a sufficient condition for a graph to contain vertex-disjoint stars of a given size. It is proved that if the minimum degree of the graph is at least $k + t - 1$ and the order is at least $(t + 1)k + O(t^2)$, then the graph contains k vertex-disjoint copies of a star $K_{1,t}$. The condition on the minimum degree is sharp, and there is an example showing that the term $O(t^2)$ for the number of uncovered vertices is necessary in a sense.

Keywords: stars, vertex-disjoint copies, minimum degree.

2000 Mathematics Subject Classification: 05C35, 05C70.

1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively.

For a graph F and a positive integer k, kF denotes the vertex-disjoint union of k copies of F. A spanning subgraph isomorphic to kF for some integer k is called an *F-factor*. There are several results concerning minimum degree conditions for a graph to have an F-factor for several specific graphs F. The result of Corrádi and Hajnal [3] implies that $\delta(G) \geq \frac{2}{3}|V(G)|$ suffices for the existence of a K_3-factor. (When we consider an F-factor of a graph G, we always assume that $|V(G)|$ is a multiple of $|V(F)|$). Dirac [4] generalized this result by showing that if $\delta(G) \geq \frac{1}{2}(|V(G)| + k)$, then G contains k vertex-disjoint triangles for any integer k with $3k \leq |V(G)|$. Enomoto, Kaneko and Tuza [7] proved for $F = P_3$ (the path of order three) that $\delta(G) \geq \frac{1}{3}|V(G)|$ is sufficient for the existence of an F-factor if we assume that G is connected. Hajnal and Szemerédi [8] proved that for $F = K_t$ $\delta(G) \geq \frac{t-1}{t}|V(G)|$ suffices. More generally, Alon and Yuster [2] proved an asymptotic result, which states that $\delta(G) \geq \Big(\frac{\chi(F)-1}{\chi(F)} + o(1)\Big)|V(G)|$ assures the existence of an F-factor, where $\chi(F)$ denotes the chromatic number of F.

On the other hand, if we want to find k vertex-disjoint copies of F in a graph G of order slightly larger than $k|F|$, and if F admits a $\chi(F)$-coloring in which some color classes are tiny, then a much weaker condition may guarantee the existence. Komlós [9] (and Alon and Fischer [1] for bipartite case) have proved that the required minimum degree of G is

$$\frac{1}{\chi(F)-1}\Big(\chi(F) - 2 + \frac{\alpha}{|V(F)|}\Big)|V(G)|,$$

where α is the smallest possible color class size in any $\chi(F)$-coloring of F.

In the case where F is a star $K_{1,t}$, Alon and Yuster's result implies that $\delta(G) \geq (\frac{1}{2} + o(1))|V(G)|$ is sufficient for the existence of a $K_{1,t}$-factor, and Komlós, Alon and Fischer's result implies that $\delta(G) \geq \frac{1}{t+1}|V(G)|$ is sufficient for the existence of k copies of $K_{1,t}$ if $|V(G)|$ is large. In this paper, we prove the following theorem, in which the required minimum degree of G does not depend on $|V(G)|$. The proof is given in the next section.

Theorem 1. *Let t be an integer with $t \geq 3$. If G is a graph of order $n \geq (t+1)k + 2t^2 - 3t - 1$ with minimum degree at least $k + t - 1$, then G contains k vertex-disjoint copies of $K_{1,t}$.*

The minimum degree condition in the theorem cannot be replaced by any weaker condition even if the order of the graph is assumed much larger. To see this, let H be a $t-1$ regular graph of order sufficiently large, and let G be obtained from H by adding $k-1$ new vertices which are joined to all other vertices. Then $\delta(G) = k + t - 2$. Since any $K_{1,t}$ subgraph of G must contain one of the new vertices, G does not contain k vertex-disjoint copies of $K_{1,t}$.

On the other hand, the following example shows that the term $O(t^2)$ for the number of uncovered vertices is necessary. Let $k_1 + k_2 + \cdots + k_t = k - 1$ so that $|k_i - k_j| \leq 1$ for any i and j. We define the graph G to be the vertex-disjoint union of the complete graphs $K_{(t+1)k_1+t}, K_{(t+1)k_2+t}, \ldots, K_{(t+1)k_t+t}$.

Then, $|V(G)| = (t+1)(k-1) + t^2 = (t+1)k + t^2 - t - 1$ and $\delta(G) = (t+1)\lfloor \frac{k-1}{t} \rfloor + t - 1 \geq k + t - 1$ (if $k \gg t$). However, it is obvious that G contains at most $k-1$ copies of $K_{1,t}$.

This example suggests that the same conclusion as in Theorem 1 follows if $|V(G)| \geq (t+1)k + t^2 - t$. In fact, it is known to be true for $t \leq 3$. The case $t = 1$ is an easy exercise. The case $t = 3$ is proved in [6].

Theorem 2 [6]**.** *If G is a graph with $|V(G)| \geq 4k + 6$ and $\delta(G) \geq k + 2$, then G contains k vertex-disjoint copies of $K_{1,3}$.*

The case $t = 2$ can be proved in the following way. We use the following theorem due to Enomoto [5].

Theorem 3 [5]**.** *Let G be a connected graph of order n and $n = n_1 + \cdots + n_k$ with $n_i \geq 2$ $(1 \leq i \leq k)$. If $\delta(G) \geq k$, then $V(G)$ can be partitioned into $V_1, \ldots, V_k$ so that for each i, $|V_i| = n_i$ and V_i induces a subgraph without isolated vertices.*

Corollary 4. *Let G be a connected graph of order n, and k be an integer with $3k \leq n$. If*

$$\delta(G) \geq \begin{cases} k, & \text{if } n - 3k \text{ is even,} \\ k+1, & \text{if } n - 3k \text{ is odd,} \end{cases}$$

then G contains k vertex-disjoint copies of P_3.

Proof. If $n - 3k$ is even, then put $n_1 = \cdots = n_{k-1} = 3$ and $n_k = n - 3k + 3$, and apply Theorem 3. If $n - 3k$ is odd, then by deleting one vertex from G so that the resulting graph is connected, we can apply the previous case. ■

Now we can prove the following theorem.

Theorem 5. *Let G be a graph of order n with $n \geq 3k + 2$. If $\delta(G) \geq k + 1$, then G contains k vertex-disjoint copies of P_3.*

Proof. If G is connected, or if G has a component of order at least $3k$, then the result follows immediately from Corollary 4. Suppose that G is disconnected and each component is order less than $3k$. Note that by Corollary 4, each component C of G contains $\lfloor |V(C)|/3 \rfloor$ vertex-disjoint copies of P_3. Also, since $\delta(G) \geq k + 1$, each component has at least $k + 2$ vertices.

If G has at least three components, then G contains at least $3\lfloor\frac{k+2}{3}\rfloor \geq k$ copies of P_3, and we are done. If G consists of two components of orders n_1 and n_2, then the number of vertex-disjoint copies of P_3 in G is at least $\lfloor\frac{n_1}{3}\rfloor + \lfloor\frac{n_2}{3}\rfloor \geq \lceil\frac{n_1+n_2-4}{3}\rceil \geq \lceil\frac{3k+2-4}{3}\rceil = k$. ■

However, for the general case of the stronger statement, we need more crucial argument than the one used in this paper.

2. Proof of Theorem 1

Let t be an integer with $t \geq 3$, and let G be a graph of order at least $(t+1)k + 2t^2 - 3t - 1$ and minimum degree at least $k + t - 1$.

We use the following notation and terminology. For $S \subset V(G)$, we write $\langle S\rangle$ for the subgraph of G induced by S. For disjoint vertex sets S and T, we denote the set of edges joining S and T by $E(S,T)$.

We consider a partition $V(G) = X \cup Y \cup Z$ satisfying the following conditions:

(a) $|X| = (t+1)p$ and X contains p vertex-disjoint copies of $K_{1,t}$, say C_1, $C_2, \ldots, C_p$.

(b) The vertices of Y can be labelled $y_1, y_2, \ldots, y_q$ so that for each r ($1 \leq r \leq q$), $|N_G(y_r) \cap Z| \geq rt + (2t-1)$.

Note that $X = Y = \emptyset$ and $Z = V(G)$ satisfy the above conditions with $p = q = 0$. We choose such a partition so that $p + q$ is maximum, and subject to this condition, q is maximum possible.

Claim 1. For any subset $A \subset Z$ with $|A| \leq 2t-1$, $G - X - A$ contains q vertex-disjoint copies of $K_{1,t}$. In particular, G contains $p+q$ vertex-disjoint copies of $K_{1,t}$.

Proof. By the condition (b), it follows that $|N_G(y_r) \cap (Z - A)| \geq rt + 2t - 1 - |A| \geq rt$ for each $1 \leq r \leq q$. Therefore we can complete to take q stars in $\langle Y \cup (Z - A)\rangle$ whose centers are $y_1, y_2, \cdots, y_q$, respectively. ■

Claim 2. For any subset $A \subset Z$ with $|A| \leq 2t-1$, $\langle X \cup A\rangle$ does not contain $p+1$ vertex-disjoint copies of $K_{1,t}$.

Proof. Suppose that $\langle X \cup A\rangle$ contains $(p+1)K_{1,t}$. Then by Claim 1, G contains $(p+q+1)K_{1,t}$. Let $V(G) = X' \cup Y' \cup Z'$ be a partition such that X'

is the set of vertices contained in $(p+q+1)K_{1,t}$ and $Y' = \emptyset$. This partition satisfies the condition (a) and (b), and contradicts the maximality of $p+q$. ■

In particular, we have the following.

Claim 3. The maximum degree of $\langle Z \rangle$ is less than t. ■

Let a be the center of any star C_i in X. If $|N_G(a) \cap Z| \geq tq + 2t - 1$, then we put $X' = X - V(C_i)$, $Y' = Y \cup \{a\}$ with $y_{q+1} = a$, and $Z' = Z \cup (V(C_i) - \{a\})$. Then $\langle X' \rangle$ contains $p-1$ vertex-disjoint $K_{1,t}$'s, and $|N_G(y_{q+1}) \cap Z'| = |N_G(a) \cap Z| + t \geq t(q+1) + 2t - 1$. This contradicts the maximality of q. Hence we have

$$|N_G(a) \cap Z| \leq tq + 2t - 2. \tag{1}$$

By a similar argument, for each leaf b of any star C_i in X, we have

$$|N_G(b) \cap Z| \leq tq + 3t - 3. \tag{2}$$

Claim 4. For each $1 \leq i \leq p$, $|E(C_i, Z)| \leq \max\{tq + t^2 + t - 2, 2t^2 - 2t\}$.

Proof. Let a be the center and $b_1, b_2, \ldots, b_t$ be the leaves of C_i.

Case 1. $|E(a, Z)| \geq t + 1$.
In this case, each b_j is adjacent to at most $t-1$ vertices in Z. For otherwise, we can take $A \subset N(b_j) \cap Z$ with $|A| = t$ and $z \in N(a) \cap Z - A$ so that $\langle \{b_j\} \cup A \rangle$ and $\langle (V(C_i) - \{b_j\}) \cup \{z\} \rangle$ contain a $K_{1,t}$. This contradicts Claim 2. Hence $|E(b_j, Z)| \leq t - 1$ for all j $(1 \leq j \leq t)$. Then, since $|E(a, Z)| \leq tq + 2t - 2$ by (1),

$$\begin{aligned} |E(V(C_i), Z)| &= |E(a, Z)| + \sum_{j=1}^{t} |E(b_j, Z)| \\ &\leq tq + 2t - 2 + t(t-1) \;=\; tq + t^2 + t - 2. \end{aligned}$$

Case 2. $1 \leq |E(a, Z)| \leq t$.
If $|E(b_j, Z)| \geq t + 1$ for some j $(1 \leq j \leq t)$, then we can take $z \in N(a) \cap Z$ and $A \subset N(b_j) \cap Z - \{z\}$ with $|A| = t$ so that $\langle V(C_i) \cup A \cup \{z\} \rangle$ contains $2K_{1,t}$, a contradiction. Hence $|E(b_j, Z)| \leq t$ for all j. Thus, $|E(V(C_i), Z)| \leq (t+1)t \leq 2t^2 - 2t$, since $t \geq 3$.

Case 3. $|E(a,Z)| = 0$.
If each leaf of C_i is adjacent to at most $2t-2$ vertices in Z, then we have $|E(V(C_i),Z)| \le 2t^2-2t$. Hence we may assume that there exists a vertex b_h $(1 \le h \le t)$ with $|E(b_h,Z)| \ge 2t-1$. If $|E(b_j,Z)| \ge t-1$ for some j with $j \ne h$, then we can take $A \subset N(b_j) \cap Z$ with $|A| = t-1$ and $A' \subset N(b_h) \cap Z - A$ with $|A'| = t$ so that $\langle \{b_h, b_j, a\} \cup A \cup A' \rangle$ contains $2K_{1,t}$'s, a contradiction. Hence $|E(b_j,Z)| \le t-2$ for all $j \ne h$. Since $|E(b_h,Z)| \le tq+3t-3$ by (2), we have $|E(V(C_i),Z)| \le tq+3t-3+(t-1)(t-2) < tq+t^2+t-2$.
This completes the proof of Claim 4. ∎

Now, we shall estimate the number of edges joining X and Z in two ways, by assuming that G does not contain k vertex-disjoint $K_{1,t}$'s. By Claim 3, each vertex in Z is adjacent to at least $(k+t-1)-(t-1)-q = k-q$ vertices of X. Hence,

$$|E(X,Z)| \;\ge\; (k-q)|Z| \;=\; (k-q)(n-(t+1)p-q).$$

On the other hand, it follows from Claim 4 that

$$\begin{aligned} |E(X,Z)| &= \sum_{i=1}^{p} |E(V(C_i),Z)| \\ &\le p \cdot \max\{tq+t^2+t-2, 2t^2-2t\}. \end{aligned}$$

If $tq+t^2+t-2 \ge 2t^2-2t$, or equivalently if $q \ge t-2$, then

$$(k-q)(n-(t+1)p-q) \;\le\; |E(X,Z)| \;\le\; p(tq+t^2+t-2),$$

and hence

$$(k-q)(n-q) \;\le\; p((t+1)k+t^2+t-2-q).$$

By Claim 1, we may assume that $p+q \le k-1$. Thus the above inequality implies that

$$(k-q)(n-q) \;\le\; (k-1-q)((t+1)k+t^2+t-2-q),$$

and hence

$$\begin{aligned} n &\le (t+1)k+t^2+t-2-\frac{(t+1)k+t^2+t-2-q}{k-q} \\ &= (t+1)k+2t^2-3t-1-\left(t^2-4t+3+\frac{(t-1)k+t^2+t-2+q}{k-q}\right) \\ &< (t+1)k+2t^2-3t-1. \end{aligned}$$

This contradicts the assumption that $n \geq (t+1)k + 2t^2 - 3t - 1$.

If $q \leq t-3$, then since $tq + t^2 + t - 2 < 2t^2 - 2t$,

$$(k-q)(n-(t+1)p-q) \;\leq\; |E(X,Z)| \;\leq\; p(2t^2-2t),$$

and hence

$$(k-q)(n-q) \;\leq\; p((t+1)(k-q)+2t^2-2t).$$

Since $p + q \leq k - 1$,

$$\begin{aligned}(k-q)(n-q) &\leq (k-1-q)((t+1)(k-q)+2t^2-2t), \\ n &\leq (t+1)k + 2t^2 - 3t - 1 - tq - \frac{2t^2-2t}{k-q} \\ &< (t+1)k + 2t^2 - 3t - 1.\end{aligned}$$

This is a contradiction.

This completes the proof of Theorem 1. ■

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Received 27 September 2000
Revised 19 March 2001