

## VERTEX-DISJOINT STARS IN GRAPHS

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### Abstract

In this paper, we give a sufficient condition for a graph to contain vertex-disjoint stars of a given size. It is proved that if the minimum degree of the graph is at least  $k + t - 1$  and the order is at least  $(t + 1)k + O(t^2)$ , then the graph contains  $k$  vertex-disjoint copies of a star  $K_{1,t}$ . The condition on the minimum degree is sharp, and there is an example showing that the term  $O(t^2)$  for the number of uncovered vertices is necessary in a sense.

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## 1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph  $G$ , we denote by  $V(G)$ ,  $E(G)$  and  $\delta(G)$  the vertex set, the edge set and the minimum degree of  $G$ , respectively.

For a graph  $F$  and a positive integer  $k$ ,  $kF$  denotes the vertex-disjoint union of  $k$  copies of  $F$ . A spanning subgraph isomorphic to  $kF$  for some integer  $k$  is called an  $F$ -factor. There are several results concerning minimum degree conditions for a graph to have an  $F$ -factor for several specific graphs  $F$ . The result of Corrádi and Hajnal [3] implies that  $\delta(G) \geq \frac{2}{3}|V(G)|$  suffices for the existence of a  $K_3$ -factor. (When we consider an  $F$ -factor of a graph  $G$ , we always assume that  $|V(G)|$  is a multiple of  $|V(F)|$ ). Dirac [4] generalized this result by showing that if  $\delta(G) \geq \frac{1}{2}(|V(G)| + k)$ ,

then  $G$  contains  $k$  vertex-disjoint triangles for any integer  $k$  with  $3k \leq |V(G)|$ . Enomoto, Kaneko and Tuza [7] proved for  $F = P_3$  (the path of order three) that  $\delta(G) \geq \frac{1}{3}|V(G)|$  is sufficient for the existence of an  $F$ -factor if we assume that  $G$  is connected. Hajnal and Szemerédi [8] proved that for  $F = K_t$   $\delta(G) \geq \frac{t-1}{t}|V(G)|$  suffices. More generally, Alon and Yuster [2] proved an asymptotic result, which states that  $\delta(G) \geq \left(\frac{\chi(F)-1}{\chi(F)} + o(1)\right)|V(G)|$  assures the existence of an  $F$ -factor, where  $\chi(F)$  denotes the chromatic number of  $F$ .

On the other hand, if we want to find  $k$  vertex-disjoint copies of  $F$  in a graph  $G$  of order slightly larger than  $k|F|$ , and if  $F$  admits a  $\chi(F)$ -coloring in which some color classes are tiny, then a much weaker condition may guarantee the existence. Komlós [9] (and Alon and Fischer [1] for bipartite case) have proved that the required minimum degree of  $G$  is

$$\frac{1}{\chi(F)-1} \left( \chi(F) - 2 + \frac{\alpha}{|V(F)|} \right) |V(G)|,$$

where  $\alpha$  is the smallest possible color class size in any  $\chi(F)$ -coloring of  $F$ .

In the case where  $F$  is a star  $K_{1,t}$ , Alon and Yuster's result implies that  $\delta(G) \geq (\frac{1}{2} + o(1))|V(G)|$  is sufficient for the existence of a  $K_{1,t}$ -factor, and Komlós, Alon and Fischer's result implies that  $\delta(G) \geq \frac{1}{t+1}|V(G)|$  is sufficient for the existence of  $k$  copies of  $K_{1,t}$  if  $|V(G)|$  is large. In this paper, we prove the following theorem, in which the required minimum degree of  $G$  does not depend on  $|V(G)|$ . The proof is given in the next section.

**Theorem 1.** *Let  $t$  be an integer with  $t \geq 3$ . If  $G$  is a graph of order  $n \geq (t+1)k + 2t^2 - 3t - 1$  with minimum degree at least  $k + t - 1$ , then  $G$  contains  $k$  vertex-disjoint copies of  $K_{1,t}$ .*

The minimum degree condition in the theorem cannot be replaced by any weaker condition even if the order of the graph is assumed much larger. To see this, let  $H$  be a  $t-1$  regular graph of order sufficiently large, and let  $G$  be obtained from  $H$  by adding  $k-1$  new vertices which are joined to all other vertices. Then  $\delta(G) = k + t - 2$ . Since any  $K_{1,t}$  subgraph of  $G$  must contain one of the new vertices,  $G$  does not contain  $k$  vertex-disjoint copies of  $K_{1,t}$ .

On the other hand, the following example shows that the term  $O(t^2)$  for the number of uncovered vertices is necessary. Let  $k_1 + k_2 + \cdots + k_t = k-1$  so that  $|k_i - k_j| \leq 1$  for any  $i$  and  $j$ . We define the graph  $G$  to be the vertex-disjoint union of the complete graphs  $K_{(t+1)k_1+t}, K_{(t+1)k_2+t}, \dots, K_{(t+1)k_t+t}$ .

Then,  $|V(G)| = (t+1)(k-1) + t^2 = (t+1)k + t^2 - t - 1$  and  $\delta(G) = (t+1)\lfloor \frac{k-1}{t} \rfloor + t - 1 \geq k + t - 1$  (if  $k \gg t$ ). However, it is obvious that  $G$  contains at most  $k-1$  copies of  $K_{1,t}$ .

This example suggests that the same conclusion as in Theorem 1 follows if  $|V(G)| \geq (t+1)k + t^2 - t$ . In fact, it is known to be true for  $t \leq 3$ . The case  $t = 1$  is an easy exercise. The case  $t = 3$  is proved in [6].

**Theorem 2** [6]. *If  $G$  is a graph with  $|V(G)| \geq 4k + 6$  and  $\delta(G) \geq k + 2$ , then  $G$  contains  $k$  vertex-disjoint copies of  $K_{1,3}$ .*

The case  $t = 2$  can be proved in the following way. We use the following theorem due to Enomoto [5].

**Theorem 3** [5]. *Let  $G$  be a connected graph of order  $n$  and  $n = n_1 + \dots + n_k$  with  $n_i \geq 2$  ( $1 \leq i \leq k$ ). If  $\delta(G) \geq k$ , then  $V(G)$  can be partitioned into  $V_1, \dots, V_k$  so that for each  $i$ ,  $|V_i| = n_i$  and  $V_i$  induces a subgraph without isolated vertices.*

**Corollary 4.** *Let  $G$  be a connected graph of order  $n$ , and  $k$  be an integer with  $3k \leq n$ . If*

$$\delta(G) \geq \begin{cases} k, & \text{if } n - 3k \text{ is even,} \\ k + 1, & \text{if } n - 3k \text{ is odd,} \end{cases}$$

*then  $G$  contains  $k$  vertex-disjoint copies of  $P_3$ .*

**Proof.** If  $n - 3k$  is even, then put  $n_1 = \dots = n_{k-1} = 3$  and  $n_k = n - 3k + 3$ , and apply Theorem 3. If  $n - 3k$  is odd, then by deleting one vertex from  $G$  so that the resulting graph is connected, we can apply the previous case. ■

Now we can prove the following theorem.

**Theorem 5.** *Let  $G$  be a graph of order  $n$  with  $n \geq 3k + 2$ . If  $\delta(G) \geq k + 1$ , then  $G$  contains  $k$  vertex-disjoint copies of  $P_3$ .*

**Proof.** If  $G$  is connected, or if  $G$  has a component of order at least  $3k$ , then the result follows immediately from Corollary 4. Suppose that  $G$  is disconnected and each component is order less than  $3k$ . Note that by Corollary 4, each component  $C$  of  $G$  contains  $\lfloor |V(C)|/3 \rfloor$  vertex-disjoint copies of  $P_3$ . Also, since  $\delta(G) \geq k + 1$ , each component has at least  $k + 2$  vertices.

If  $G$  has at least three components, then  $G$  contains at least  $3\lfloor \frac{k+2}{3} \rfloor \geq k$  copies of  $P_3$ , and we are done. If  $G$  consists of two components of orders  $n_1$  and  $n_2$ , then the number of vertex-disjoint copies of  $P_3$  in  $G$  is at least  $\lfloor \frac{n_1}{3} \rfloor + \lfloor \frac{n_2}{3} \rfloor \geq \lceil \frac{n_1+n_2-4}{3} \rceil \geq \lceil \frac{3k+2-4}{3} \rceil = k$ . ■

However, for the general case of the stronger statement, we need more crucial argument than the one used in this paper.

## 2. Proof of Theorem 1

Let  $t$  be an integer with  $t \geq 3$ , and let  $G$  be a graph of order at least  $(t+1)k + 2t^2 - 3t - 1$  and minimum degree at least  $k + t - 1$ .

We use the following notation and terminology. For  $S \subset V(G)$ , we write  $\langle S \rangle$  for the subgraph of  $G$  induced by  $S$ . For disjoint vertex sets  $S$  and  $T$ , we denote the set of edges joining  $S$  and  $T$  by  $E(S, T)$ .

We consider a partition  $V(G) = X \cup Y \cup Z$  satisfying the following conditions:

- (a)  $|X| = (t+1)p$  and  $X$  contains  $p$  vertex-disjoint copies of  $K_{1,t}$ , say  $C_1, C_2, \dots, C_p$ .
- (b) The vertices of  $Y$  can be labelled  $y_1, y_2, \dots, y_q$  so that for each  $r$  ( $1 \leq r \leq q$ ),  $|N_G(y_r) \cap Z| \geq rt + (2t - 1)$ .

Note that  $X = Y = \emptyset$  and  $Z = V(G)$  satisfy the above conditions with  $p = q = 0$ . We choose such a partition so that  $p + q$  is maximum, and subject to this condition,  $q$  is maximum possible.

**Claim 1.** For any subset  $A \subset Z$  with  $|A| \leq 2t - 1$ ,  $G - X - A$  contains  $q$  vertex-disjoint copies of  $K_{1,t}$ . In particular,  $G$  contains  $p + q$  vertex-disjoint copies of  $K_{1,t}$ .

**Proof.** By the condition (b), it follows that  $|N_G(y_r) \cap (Z - A)| \geq rt + 2t - 1 - |A| \geq rt$  for each  $1 \leq r \leq q$ . Therefore we can complete to take  $q$  stars in  $\langle Y \cup (Z - A) \rangle$  whose centers are  $y_1, y_2, \dots, y_q$ , respectively. ■

**Claim 2.** For any subset  $A \subset Z$  with  $|A| \leq 2t - 1$ ,  $\langle X \cup A \rangle$  does not contain  $p + 1$  vertex-disjoint copies of  $K_{1,t}$ .

**Proof.** Suppose that  $\langle X \cup A \rangle$  contains  $(p + 1)K_{1,t}$ . Then by Claim 1,  $G$  contains  $(p + q + 1)K_{1,t}$ . Let  $V(G) = X' \cup Y' \cup Z'$  be a partition such that  $X'$

is the set of vertices contained in  $(p+q+1)K_{1,t}$  and  $Y' = \emptyset$ . This partition satisfies the condition (a) and (b), and contradicts the maximality of  $p+q$ . ■

In particular, we have the following.

**Claim 3.** The maximum degree of  $\langle Z \rangle$  is less than  $t$ . ■

Let  $a$  be the center of any star  $C_i$  in  $X$ . If  $|N_G(a) \cap Z| \geq tq + 2t - 1$ , then we put  $X' = X - V(C_i)$ ,  $Y' = Y \cup \{a\}$  with  $y_{q+1} = a$ , and  $Z' = Z \cup (V(C_i) - \{a\})$ . Then  $\langle X' \rangle$  contains  $p-1$  vertex-disjoint  $K_{1,t}$ 's, and  $|N_G(y_{q+1}) \cap Z'| = |N_G(a) \cap Z| + t \geq t(q+1) + 2t - 1$ . This contradicts the maximality of  $q$ . Hence we have

$$(1) \quad |N_G(a) \cap Z| \leq tq + 2t - 2.$$

By a similar argument, for each leaf  $b$  of any star  $C_i$  in  $X$ , we have

$$(2) \quad |N_G(b) \cap Z| \leq tq + 3t - 3.$$

**Claim 4.** For each  $1 \leq i \leq p$ ,  $|E(C_i, Z)| \leq \max\{tq + t^2 + t - 2, 2t^2 - 2t\}$ .

**Proof.** Let  $a$  be the center and  $b_1, b_2, \dots, b_t$  be the leaves of  $C_i$ .

*Case 1.*  $|E(a, Z)| \geq t + 1$ .

In this case, each  $b_j$  is adjacent to at most  $t-1$  vertices in  $Z$ . For otherwise, we can take  $A \subset N(b_j) \cap Z$  with  $|A| = t$  and  $z \in N(a) \cap Z - A$  so that  $\langle \{b_j\} \cup A \rangle$  and  $\langle (V(C_i) - \{b_j\}) \cup \{z\} \rangle$  contain a  $K_{1,t}$ . This contradicts Claim 2. Hence  $|E(b_j, Z)| \leq t-1$  for all  $j$  ( $1 \leq j \leq t$ ). Then, since  $|E(a, Z)| \leq tq + 2t - 2$  by (1),

$$\begin{aligned} |E(V(C_i), Z)| &= |E(a, Z)| + \sum_{j=1}^t |E(b_j, Z)| \\ &\leq tq + 2t - 2 + t(t-1) = tq + t^2 + t - 2. \end{aligned}$$

*Case 2.*  $1 \leq |E(a, Z)| \leq t$ .

If  $|E(b_j, Z)| \geq t+1$  for some  $j$  ( $1 \leq j \leq t$ ), then we can take  $z \in N(a) \cap Z$  and  $A \subset N(b_j) \cap Z - \{z\}$  with  $|A| = t$  so that  $\langle V(C_i) \cup A \cup \{z\} \rangle$  contains  $2K_{1,t}$ , a contradiction. Hence  $|E(b_j, Z)| \leq t$  for all  $j$ . Thus,  $|E(V(C_i), Z)| \leq (t+1)t \leq 2t^2 - 2t$ , since  $t \geq 3$ .

*Case 3.*  $|E(a, Z)| = 0$ .

If each leaf of  $C_i$  is adjacent to at most  $2t - 2$  vertices in  $Z$ , then we have  $|E(V(C_i), Z)| \leq 2t^2 - 2t$ . Hence we may assume that there exists a vertex  $b_h$  ( $1 \leq h \leq t$ ) with  $|E(b_h, Z)| \geq 2t - 1$ . If  $|E(b_j, Z)| \geq t - 1$  for some  $j$  with  $j \neq h$ , then we can take  $A \subset N(b_j) \cap Z$  with  $|A| = t - 1$  and  $A' \subset N(b_h) \cap Z - A$  with  $|A'| = t$  so that  $\langle \{b_h, b_j, a\} \cup A \cup A' \rangle$  contains  $2K_{1,t}$ 's, a contradiction. Hence  $|E(b_j, Z)| \leq t - 2$  for all  $j \neq h$ . Since  $|E(b_h, Z)| \leq tq + 3t - 3$  by (2), we have  $|E(V(C_i), Z)| \leq tq + 3t - 3 + (t - 1)(t - 2) < tq + t^2 + t - 2$ .

This completes the proof of Claim 4.  $\blacksquare$

Now, we shall estimate the number of edges joining  $X$  and  $Z$  in two ways, by assuming that  $G$  does not contain  $k$  vertex-disjoint  $K_{1,t}$ 's. By Claim 3, each vertex in  $Z$  is adjacent to at least  $(k + t - 1) - (t - 1) - q = k - q$  vertices of  $X$ . Hence,

$$|E(X, Z)| \geq (k - q)|Z| = (k - q)(n - (t + 1)p - q).$$

On the other hand, it follows from Claim 4 that

$$\begin{aligned} |E(X, Z)| &= \sum_{i=1}^p |E(V(C_i), Z)| \\ &\leq p \cdot \max\{tq + t^2 + t - 2, 2t^2 - 2t\}. \end{aligned}$$

If  $tq + t^2 + t - 2 \geq 2t^2 - 2t$ , or equivalently if  $q \geq t - 2$ , then

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(tq + t^2 + t - 2),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)k + t^2 + t - 2 - q).$$

By Claim 1, we may assume that  $p + q \leq k - 1$ . Thus the above inequality implies that

$$(k - q)(n - q) \leq (k - 1 - q)((t + 1)k + t^2 + t - 2 - q),$$

and hence

$$\begin{aligned} n &\leq (t + 1)k + t^2 + t - 2 - \frac{(t + 1)k + t^2 + t - 2 - q}{k - q} \\ &= (t + 1)k + 2t^2 - 3t - 1 - \left( t^2 - 4t + 3 + \frac{(t - 1)k + t^2 + t - 2 + q}{k - q} \right) \\ &< (t + 1)k + 2t^2 - 3t - 1. \end{aligned}$$

This contradicts the assumption that  $n \geq (t+1)k + 2t^2 - 3t - 1$ .

If  $q \leq t - 3$ , then since  $tq + t^2 + t - 2 < 2t^2 - 2t$ ,

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(2t^2 - 2t),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)(k - q) + 2t^2 - 2t).$$

Since  $p + q \leq k - 1$ ,

$$\begin{aligned} (k - q)(n - q) &\leq (k - 1 - q)((t + 1)(k - q) + 2t^2 - 2t), \\ n &\leq (t + 1)k + 2t^2 - 3t - 1 - tq - \frac{2t^2 - 2t}{k - q} \\ &< (t + 1)k + 2t^2 - 3t - 1. \end{aligned}$$

This is a contradiction.

This completes the proof of Theorem 1. ■

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