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A σ_3 TYPE CONDITION FOR HEAVY CYCLES IN WEIGHTED GRAPHS

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Abstract

A weighted graph is a graph in which each edge e is assigned a non-negative number $w(e)$, called the weight of e. The weight of a cycle is the sum of the weights of its edges. The weighted degree $d^w(v)$ of a vertex v is the sum of the weights of the edges incident with v. In this paper, we prove the following result: Suppose G is a 2-connected weighted graph which satisfies the following conditions: 1. The weighted degree sum of any three independent vertices is at least m; 2. $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$; 3. In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$. This generalizes a theorem of Fournier and Fraisse on the existence of long cycles in k-connected unweighted graphs in the case $k = 2$. Our proof of the above result also suggests a new proof to the theorem of Fournier and Fraisse in the case $k = 2$.

Keywords: weighted graph, (long, heavy, Hamilton) cycle, weighted degree, (weighted) degree sum.

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1. Terminology and Notation

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let $G = (V, E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a non-negative number $w(e)$, called the *weight* of e. For any subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

An *optimal cycle* is one with maximum weight. For each vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in H that are adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v), d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively. An (x, y)-*path* is a path connecting the two vertices x and y. The distance between two vertices x and y, denoted by $d(x, y)$, is the length of a shortest (x, y)-path. If u and v are two vertices on a path P, $P[u, v]$ denotes the segment of P from u to v. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. For a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k independent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k independent vertices. Instead of $\sigma_1(G)$ and $\sigma_1^w(G)$, we use the notations $\delta(G)$ and $\delta^w(G)$, respectively.

2. Results

There have been many results on the existence of long cycles in graphs. The following three theorems are well-known.

Theorem A (Dirac [5]). *Let G be a 2-connected graph such that $\delta(G) \geq r$. Then G contains either a Hamilton cycle or a cycle of length at least $2r$.*

Theorem B (Pósa [7]). *Let G be a 2-connected graph such that $\sigma_2(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of length at least s.*

Theorem C (Fournier and Fraisse [6]). *Let G be a k-connected graph where $2 \le k < \alpha(G)$, such that $\sigma_{k+1}(G) \ge m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.*

It is easy to see that Theorem B generalizes Theorem A, and Theorem C in turn generalizes Theorem B.

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a cycle is simply the length of the cycle.

Theorem A and Theorem B were generalized to weighted graphs by the following two theorems, respectively.

Theorem 1 (Bondy and Fan [3]). *Let G be 2-connected weighted graph such that $\delta^w(G) \ge r$. Then either G contains a cycle of weight at least $2r$ or every optimal cycle is a Hamilton cycle.*

Theorem 2 (Bondy et al. [2]). *Let G be 2-connected weighted graph such that $\sigma_2^w(G) \ge s$. Then G contains either a Hamilton cycle or a cycle of weight at least s.*

A natural question is whether Theorem C also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 1. *Let G be a k-connected weighted graph where $2 \le k < \alpha(G)$, such that $\sigma_{k+1}^w(G) \ge m$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$?*

If the answer to the question of this problem is positive, then the result would be best possible and it would generalize Theorem C and Theorem 2.

It seems very difficult to settle this problem, even for the case $k = 2$. In the next section, we prove that the answer to the case $k = 2$ of Problem 1 is positive if we add some extra conditions. These extra conditions were motivated by a recent generalization of a theorem of Fan to weighted graphs (cf. [8]). Our result is an analogue and also a generalization of Theorem C to weighted graphs in the case $k = 2$.

Theorem 3. *Let G be a 2-connected weighted graph which satisfies the following conditions:*

1. *The weighted degree sum of any three independent vertices is at least m;*

2. *$w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$;*
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

3. Proof of Theorem 3

Let G be a 2-connected weighted graph satisfying the conditions of Theorem 3. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least $2m/3$.

Choose a path $P = v_1v_2 \cdots v_p$ in G such that

(a) P is as long as possible;

(b) $w(P)$ is as large as possible, subject to (a);

(c) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a) and (b).

From the choice of P, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$.

Claim 1. *There exists no cycle of length p.*

Proof. Suppose there exists a cycle C of length p. Since G contains no Hamilton cycle and G is connected, we can find a vertex $u \in V(G)\backslash V(C)$ and a path Q from u to a vertex $v \in V(C)$, such that Q is internally disjoint from C. The subgraph $C \cup Q$ of G contains a path longer than P, contradicting the choice of P in (a). ■

Claim 2. *$v_1v_p \notin E(G)$.*

Proof. If $v_1v_p \in E(G)$, then we can find a cycle $C = v_1v_2 \cdots v_pv_1$ of length p, contradicting Claim 1. ■

Claim 3. *If $v_i \in N(v_1)$, then $v_{i-1} \notin N(v_p)$.*

Proof. Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$. Then we can form a cycle $C = v_1v_iv_{i+1} \cdots v_pv_{i-1}v_{i-2} \cdots v_1$ with length p, again contradicting Claim 1. ■

Claim 4. *If $v_i \in N(v_1)$, then $w(v_{i-1}v_i) \geq w(v_1v_i)$. If $v_j \in N(v_p)$, then $w(v_jv_{j+1}) \geq w(v_jv_p)$.*

Proof. If $v_i \in N(v_1)$, the path $P' = v_{i-1}v_{i-2}\cdots v_1 v_i \cdots v_p$ has the same length as P. So, because of (b), we must have $w(P) \geq w(P')$, hence $w(v_{i-1}v_i) \geq w(v_1 v_i)$. The second assertion can be proved similarly. ∎

Since G is 2-connected, by Lemma 1 of [1], there is a sequence of internally disjoint paths $P_1, P_2, \ldots, P_m$ such that

(1) P_k has end vertices x_k and y_k, and $V(P_k) \cap V(P) = \{x_k, y_k\}$ for $k = 1, 2, \ldots, m$;

(2) $v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p$, where the inequalities denote the order of the vertices on P.
By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that

(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;

(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by C_k the cycle $P_k \cup P[x_k, y_k]$ for $k = 1, 2, \ldots, m$, and let C be the cycle whose edge set is the symmetric difference of the edge sets of these cycles C_k. By (3), (4) and Claim 3 we have for all $v_i \in N(v_1)\backslash\{y_1\}$ and $v_j \in N(v_p)\backslash\{x_m\}$ that $v_{i-1}v_i$, $v_j v_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_j v_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1 y_1$ and $P_m = x_m v_p$. Using Claim 4, this shows that

$$\begin{aligned} w(C) &\geq \sum_{v_i \in N(v_1)\backslash\{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p)\backslash\{x_m\}} w(v_j v_{j+1}) \\ &\quad + w(v_1 y_1) + w(x_m y_p) \\ &\geq \sum_{v_i \in N(v_1)} w(v_1 v_i) + \sum_{v_j \in N(v_p)} w(v_j v_p) \\ &= d^w(v_1) + d^w(v_p). \end{aligned}$$

Without loss of generality, we can assume that $d^w(v_1) \leq w(C)/2$.

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1) \cap V(P)$ such that k is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p-1$.

Now we consider two cases.

Case 1. There exists a vertex $v_i \in V(P)$ such that $v_1v_i \in E(G)$ but $v_1v_{i-1} \notin E(G)$ for some i with $3 \le i \le k$.
By Claim 3 we know that $v_{i-1}v_p \notin E(G)$, so the three vertices v_1, v_{i-1} and v_p are independent. From Condition 2 of the theorem and the fact $d(v_1, v_{i-1}) = 2$ we know that $v_{i-1}v_{i-2} \cdots v_1v_i \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we have $d^w(v_{i-1}) \le d^w(v_1) \le w(C)/2$. With $d^w(v_1)+d^w(v_p) \le w(C)$, we have $d^w(v_1)+d^w(v_{i-1})+d^w(v_p) \le 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

Case 2. $v_1v_i \in E(G)$ for all i with $3 \le i \le k$.

Case 2.1. $w(v_1v_{i-1}) = w(v_1v_i) = w(v_{i-1}v_i) = w^*$ for all i with $3 \le i \le k$.
For every i with $2 \le i \le k-1$, v_i can not be adjacent to any vertex outside P. Otherwise, there will be a path of length p, contradicting the choice of P in (a). Since G is 2-connected, there must be an edge $v_jv_s \in E(G)$ with $j < k < s$. Choose $v_jv_s \in E(G)$ such that $j < k < s$ and s is as large as possible. From Claim 3 we have $s < p$.

Case 2.1.1. $s \ge k+2$.
By the choice of v_k we know that $v_1v_{s-1} \notin E(G)$. If $v_{s-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1} \cdots v_{s-1}v_p \cdots v_sv_j \cdots v_1$ of length p, contradicting Claim 1. So, the three vertices v_1, v_{s-1} and v_p are independent. By the choice of v_k, we have $d(v_1, v_s) = 2$. If $v_jv_{s-1} \in E(G)$, then $d(v_1, v_{s-1}) = 2$. Then it follows from Condition 2 of the theorem that $w(v_jv_{s-1}) = w(v_jv_s) = w(v_1v_j) = w^*$, and from Condition 3 of the theorem we get $w(v_{s-1}v_s) = w^*$. If $v_jv_{s-1} \notin E(G)$, then $d(v_jv_{s-1}) = 2$. This implies that $w(v_{s-1}v_s) = w(v_jv_s) = w^*$. Thus, in both cases the path $v_{s-1}v_{s-2} \cdots v_{j+1}v_1 \cdots v_jv_s \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we know that $d^w(v_{s-1}) \le d^w(v_1) \le w(C)/2$. With $d^w(v_1)+d^w(v_p) \le w(C)$, we have $d^w(v_1)+d^w(v_{s-1})+d^w(v_p) \le 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

Case 2.1.2. $s = k+1$.
By Claim 3 we may assume that $k+1 < p$. From the 2-connectedness of G and the choice of v_s, there must be an edge $v_kv_t \in E(G)$ such that $t \ge k+2$. By the choice of v_k, we know that $v_1v_{t-1} \notin E(G)$. On the other hand, if $v_{t-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1} \cdots v_kv_t \cdots v_pv_{t-1} \cdots v_{k+1}$

$v_j \cdots v_1$ of length p, contradicting Claim 1. So, the three vertices v_1, v_{t-1} and v_p are independent.

If $v_k v_{t-1} \in E(G)$, then from Condition 2 of the theorem we have $w(v_k v_{t-1}) = w(v_k v_t) = w(v_1 v_k) = w^*$, and from Condition 3 of the theorem, the edge $v_{t-1}v_t$ has weight w^*. If $v_k v_{t-1} \notin E(G)$, then from Condition 2 of the theorem we also get $w(v_{t-1}v_t) = w^*$. Thus, in both cases the path $v_{t-1}v_{t-2} \cdots v_{k+1}v_j \cdots v_1\ v_{j+1} \cdots v_k v_t \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), $d(v_{t-1}) \leq d^w(v_1) \leq w(C)/2$. With $d^w(v_1) + d^w(v_p) \leq w(C)$, we have $d^w(v_1) + d^w(v_{t-1}) + d^w(v_p) \leq 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

This completes the proof of Case 2.1.

Case 2.2. There is some vertex v_i with $3 \leq i \leq k$ such that $w(v_1 v_{i-1})$, $w(v_1 v_i)$ and $w(v_{i-1}v_i)$ are all different.
In this case, choose vertex v_j such that $w(v_1 v_{j-1})$, $w(v_1 v_j)$ and $w(v_{j-1}v_j)$ are all different, and j is as large as possible. Denote the weight of $v_1 v_j$, $v_{j-1}v_j$ and $v_1 v_{j-1}$ by w_1, w_2 and w_3, respectively. It follows from Condition 3 (or Condition 2 if $j = k$) that $w(v_{j-1}v_j) = w_2 \neq w_1 = w(v_j v_{j+1})$, and from Condition 2 of the theorem that $v_{j-1}v_{j+1} \in E(G)$. If $j < k$, then the weight of the edge $v_{j-1}v_{j+1}$ is different from the weight w_1 of the edge $v_{j+1}v_{j+2}$ since there is a triangle $v_1 v_{j-1} v_{j+1} v_1$ and $w(v_1 v_{j-1}) = w_3 \neq w_1 = w(v_1 v_{j+1})$. With the same argument, we can prove that $v_{j-1}v_i \in E(G)$ for all i with $j \leq i \leq k+1$. By the choice of v_k, we have that $w(v_{j-1}v_{k+1}) = w_3$.

Suppose first that $v_k v_{k+2} \in E(G)$. Then $d(v_1, v_{k+2}) = 2$. This shows that $w(v_k v_{k+2}) = w(v_1 v_k) = w_1$. From $w(v_k v_{k+1}) = w(v_k v_{k+2}) = w_1$ and Condition 3 of the theorem we know that $w(v_{k+1}v_{k+2}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ since the two edges $v_{j-1}v_{k+1}$ and $v_{k+1}v_{k+2}$ have different weights. Again, by the fact $d(v_1, v_{k+2}) = 2$, we obtain that $w(v_{j-1}v_{k+2}) = w(v_1 v_{j-1}) = w_3$. This leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ in which $w(v_{j-1}v_{k+1}) = w(v_{j-1}v_{k+2}) = w_3$ and $w(v_{k+1}v_{k+2}) = w_1$, contradicting Condition 3 of the theorem. Hence $v_k v_{k+2} \notin E(G)$. Thus $d(v_k, v_{k+2}) = 2$. This implies that $w(v_{k+1}v_{k+2}) = w(v_k v_{k+1}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ and $w(v_{j-1}v_{k+2}) = w_3$. This also leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ which is impossible by Condition 3 of the theorem.

The proof of the theorem is complete. ■

4. Remarks

The proof of Theorem C in [6] is very complicated. It is clear that our proof of Theorem 3 provides a simpler proof for Theorem C in the case $k = 2$. We do not know whether the extra conditions in Theorem 3 are necessary. The results in [8] indicate that for some generalizations of long cycle results to weighted graphs one cannot avoid such additional conditions. We do not believe that there is an analogous generalization of Theorem C for the case $k \neq 2$.

References

[1] J.A. Bondy, *Large cycles in graphs*, Discrete Math. **1** (1971) 121–132.

[2] J.A. Bondy, H.J. Broersma, J. van den Heuvel and H.J. Veldman, *Heavy cycles in weighted graphs*, to appear in Discuss. Math. Graph Theory.

[3] J.A. Bondy and G. Fan, *Optimal paths and cycles in weighted graphs*, Ann. Discrete Math. **41** (1989) 53–69.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan London and Elsevier, New York, 1976).

[5] G.A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (3) (1952) 69–81.

[6] I. Fournier and P. Fraisse, *On a conjecture of Bondy*, J. Combin. Theory (B) **39** (1985) 17–26.

[7] L. Pósa, *On the circuits of finite graphs*, Magyar Tud. Math. Kutató Int. Közl. **8** (1963) 355–361.

[8] S. Zhang, X. Li and H.J. Broersma, *A Fan type condition for heavy cycles in weighted graphs*, to appear in Graphs and Combinatorics.

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