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# A $\sigma_3$ TYPE CONDITION FOR HEAVY CYCLES IN WEIGHTED GRAPHS

Shenggui Zhang* and Xueliang Li

*Department of Applied Mathematics*
*Northwestern Polytechnical University*
*Xi'an, Shaanxi 710072, P.R. China*

and

Hajo Broersma

*Faculty of Mathematical Sciences*
*University of Twente*
*P.O. Box 217*
*7500 AE Enschede, The Netherlands*

**Abstract**

A weighted graph is a graph in which each edge $e$ is assigned a non-negative number $w(e)$, called the weight of $e$. The weight of a cycle is the sum of the weights of its edges. The weighted degree $d^w(v)$ of a vertex $v$ is the sum of the weights of the edges incident with $v$. In this paper, we prove the following result: Suppose $G$ is a 2-connected weighted graph which satisfies the following conditions: 1. The weighted degree sum of any three independent vertices is at least $m$; 2. $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$; 3. In every triangle $T$ of $G$, either all edges of $T$ have different weights or all edges of $T$ have the same weight. Then $G$ contains either a Hamilton cycle or a cycle of weight at least $2m/3$. This generalizes a theorem of Fournier and Fraisse on the existence of long cycles in $k$-connected unweighted graphs in the case $k = 2$. Our proof of the above result also suggests a new proof to the theorem of Fournier and Fraisse in the case $k = 2$.

**Keywords:** weighted graph, (long, heavy, Hamilton) cycle, weighted degree, (weighted) degree sum.

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## 1. Terminology and Notation

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let $G = (V, E)$ be a simple graph. $G$ is called a *weighted graph* if each edge $e$ is assigned a non-negative number $w(e)$, called the *weight* of $e$. For any subgraph $H$ of $G$, $V(H)$ and $E(H)$ denote the sets of vertices and edges of $H$, respectively. The *weight* of $H$ is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

An *optimal cycle* is one with maximum weight. For each vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices in $H$ that are adjacent to $v$. We define the *weighted degree* of $v$ in $H$ by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v), d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively. An $(x, y)$-*path* is a path connecting the two vertices $x$ and $y$. The distance between two vertices $x$ and $y$, denoted by $d(x, y)$, is the length of a shortest $(x, y)$-path. If $u$ and $v$ are two vertices on a path $P$, $P[u, v]$ denotes the segment of $P$ from $u$ to $v$. The number of vertices in a maximum independent set of $G$ is denoted by $\alpha(G)$. For a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any $k$ independent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any $k$ independent vertices. Instead of $\sigma_1(G)$ and $\sigma_1^w(G)$, we use the notations $\delta(G)$ and $\delta^w(G)$, respectively.

## 2. Results

There have been many results on the existence of long cycles in graphs. The following three theorems are well-known.

**Theorem A** (Dirac [5]). *Let $G$ be a 2-connected graph such that $\delta(G) \geq r$. Then $G$ contains either a Hamilton cycle or a cycle of length at least $2r$.*

**Theorem B** (Pósa [7]). *Let $G$ be a 2-connected graph such that $\sigma_2(G) \geq s$. Then $G$ contains either a Hamilton cycle or a cycle of length at least $s$.*

**Theorem C** (Fournier and Fraisse [6]). *Let $G$ be a $k$-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then $G$ contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.*

It is easy to see that Theorem B generalizes Theorem A, and Theorem C in turn generalizes Theorem B.

An unweighted graph can be regarded as a weighted graph in which each edge $e$ is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex $v$, and the weight of a cycle is simply the length of the cycle.

Theorem A and Theorem B were generalized to weighted graphs by the following two theorems, respectively.

**Theorem 1** (Bondy and Fan [3]). *Let $G$ be 2-connected weighted graph such that $\delta^w(G) \geq r$. Then either $G$ contains a cycle of weight at least $2r$ or every optimal cycle is a Hamilton cycle.*

**Theorem 2** (Bondy et al. [2]). *Let $G$ be 2-connected weighted graph such that $\sigma_2^w(G) \geq s$. Then $G$ contains either a Hamilton cycle or a cycle of weight at least $s$.*

A natural question is whether Theorem C also admits an analogous generalization for weighted graphs. This leads to the following problem.

**Problem 1.** *Let $G$ be a $k$-connected weighted graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}^w(G) \geq m$. Is it true that $G$ contains either a Hamilton cycle or a cycle of weight at least $2m/(k+1)$?*

If the answer to the question of this problem is positive, then the result would be best possible and it would generalize Theorem C and Theorem 2.

It seems very difficult to settle this problem, even for the case $k = 2$. In the next section, we prove that the answer to the case $k = 2$ of Problem 1 is positive if we add some extra conditions. These extra conditions were motivated by a recent generalization of a theorem of Fan to weighted graphs (cf. [8]). Our result is an analogue and also a generalization of Theorem C to weighted graphs in the case $k = 2$.

**Theorem 3.** *Let $G$ be a 2-connected weighted graph which satisfies the following conditions:*

1. *The weighted degree sum of any three independent vertices is at least $m$;*

2. *$w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$;*
3. *In every triangle $T$ of $G$, either all edges of $T$ have different weights or all edges of $T$ have the same weight.*

*Then $G$ contains either a Hamilton cycle or a cycle of weight at least $2m/3$.*

## 3. Proof of Theorem 3

Let $G$ be a 2-connected weighted graph satisfying the conditions of Theorem 3. Suppose that $G$ does not contain a Hamilton cycle. Then it suffices to prove that $G$ contains a cycle of weight at least $2m/3$.

Choose a path $P = v_1v_2 \cdots v_p$ in $G$ such that

(a) $P$ is as long as possible;

(b) $w(P)$ is as large as possible, subject to (a);

(c) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a) and (b).

From the choice of $P$, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$.

**Claim 1.** *There exists no cycle of length $p$.*

***Proof.*** Suppose there exists a cycle $C$ of length $p$. Since $G$ contains no Hamilton cycle and $G$ is connected, we can find a vertex $u \in V(G)\backslash V(C)$ and a path $Q$ from $u$ to a vertex $v \in V(C)$, such that $Q$ is internally disjoint from $C$. The subgraph $C \cup Q$ of $G$ contains a path longer than $P$, contradicting the choice of $P$ in (a). ■

**Claim 2.** *$v_1v_p \notin E(G)$.*

***Proof.*** If $v_1v_p \in E(G)$, then we can find a cycle $C = v_1v_2 \cdots v_pv_1$ of length $p$, contradicting Claim 1. ■

**Claim 3.** *If $v_i \in N(v_1)$, then $v_{i-1} \notin N(v_p)$.*

***Proof.*** Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$. Then we can form a cycle $C = v_1v_iv_{i+1} \cdots v_pv_{i-1}v_{i-2} \cdots v_1$ with length $p$, again contradicting Claim 1. ■

**Claim 4.** *If $v_i \in N(v_1)$, then $w(v_{i-1}v_i) \geq w(v_1v_i)$. If $v_j \in N(v_p)$, then $w(v_jv_{j+1}) \geq w(v_jv_p)$.*

***Proof.*** If $v_i \in N(v_1)$, the path $P' = v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_p$ has the same length as $P$. So, because of (b), we must have $w(P) \geq w(P')$, hence $w(v_{i-1}v_i) \geq w(v_1v_i)$. The second assertion can be proved similarly. ■

Since $G$ is 2-connected, by Lemma 1 of [1], there is a sequence of internally disjoint paths $P_1, P_2, \ldots, P_m$ such that

(1) $P_k$ has end vertices $x_k$ and $y_k$, and $V(P_k) \cap V(P) = \{x_k, y_k\}$ for $k = 1, 2, \ldots, m$;

(2) $v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p$, where the inequalities denote the order of the vertices on $P$.
By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that

(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;

(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by $C_k$ the cycle $P_k \cup P[x_k, y_k]$ for $k = 1, 2, \ldots, m$, and let $C$ be the cycle whose edge set is the symmetric difference of the edge sets of these cycles $C_k$. By (3), (4) and Claim 3 we have for all $v_i \in N(v_1)\backslash\{y_1\}$ and $v_j \in N(v_p)\backslash\{x_m\}$ that $v_{i-1}v_i$, $v_jv_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_jv_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1y_1$ and $P_m = x_mv_p$. Using Claim 4, this shows that

$$\begin{aligned} w(C) &\geq \sum_{v_i \in N(v_1)\backslash\{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p)\backslash\{x_m\}} w(v_jv_{j+1}) \\ &\quad + w(v_1y_1) + w(x_my_p) \\ &\geq \sum_{v_i \in N(v_1)} w(v_1v_i) + \sum_{v_j \in N(v_p)} w(v_jv_p) \\ &= d^w(v_1) + d^w(v_p). \end{aligned}$$

Without loss of generality, we can assume that $d^w(v_1) \leq w(C)/2$.

Since $G$ is 2-connected, $v_1$ is adjacent to at least one vertex on $P$ other than $v_2$. Choose $v_k \in N(v_1) \cap V(P)$ such that $k$ is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p-1$.

Now we consider two cases.

*Case* 1. There exists a vertex $v_i \in V(P)$ such that $v_1v_i \in E(G)$ but $v_1v_{i-1} \notin E(G)$ for some $i$ with $3 \le i \le k$.
By Claim 3 we know that $v_{i-1}v_p \notin E(G)$, so the three vertices $v_1, v_{i-1}$ and $v_p$ are independent. From Condition 2 of the theorem and the fact $d(v_1, v_{i-1}) = 2$ we know that $v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_p$ is another longest path with the same weight as $P$. By the choice of $P$ in (c), we have $d^w(v_{i-1}) \le d^w(v_1) \le w(C)/2$. With $d^w(v_1)+d^w(v_p) \le w(C)$, we have $d^w(v_1)+d^w(v_{i-1})+d^w(v_p) \le 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle $C$ is at least $2m/3$.

*Case* 2. $v_1v_i \in E(G)$ for all $i$ with $3 \le i \le k$.

*Case* 2.1. $w(v_1v_{i-1}) = w(v_1v_i) = w(v_{i-1}v_i) = w^*$ for all $i$ with $3 \le i \le k$.
For every $i$ with $2 \le i \le k-1$, $v_i$ can not be adjacent to any vertex outside $P$. Otherwise, there will be a path of length $p$, contradicting the choice of $P$ in (a). Since $G$ is 2-connected, there must be an edge $v_jv_s \in E(G)$ with $j < k < s$. Choose $v_jv_s \in E(G)$ such that $j < k < s$ and $s$ is as large as possible. From Claim 3 we have $s < p$.

*Case* 2.1.1. $s \ge k+2$.
By the choice of $v_k$ we know that $v_1v_{s-1} \notin E(G)$. If $v_{s-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1}\cdots v_{s-1}v_p\cdots v_sv_j\cdots v_1$ of length $p$, contradicting Claim 1. So, the three vertices $v_1, v_{s-1}$ and $v_p$ are independent. By the choice of $v_k$, we have $d(v_1, v_s) = 2$. If $v_jv_{s-1} \in E(G)$, then $d(v_1, v_{s-1}) = 2$. Then it follows from Condition 2 of the theorem that $w(v_jv_{s-1}) = w(v_jv_s) = w(v_1v_j) = w^*$, and from Condition 3 of the theorem we get $w(v_{s-1}v_s) = w^*$. If $v_jv_{s-1} \notin E(G)$, then $d(v_jv_{s-1}) = 2$. This implies that $w(v_{s-1}v_s) = w(v_jv_s) = w^*$. Thus, in both cases the path $v_{s-1}v_{s-2}\cdots v_{j+1}v_1\cdots v_jv_s\cdots v_p$ is another longest path with the same weight as $P$. By the choice of $P$ in (c), we know that $d^w(v_{s-1}) \le d^w(v_1) \le w(C)/2$. With $d^w(v_1) + d^w(v_p) \le w(C)$, we have $d^w(v_1) + d^w(v_{s-1}) + d^w(v_p) \le 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle $C$ is at least $2m/3$.

*Case* 2.1.2. $s = k+1$.
By Claim 3 we may assume that $k+1 < p$. From the 2-connectedness of $G$ and the choice of $v_s$, there must be an edge $v_kv_t \in E(G)$ such that $t \ge k+2$. By the choice of $v_k$, we know that $v_1v_{t-1} \notin E(G)$. On the other hand, if $v_{t-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1}\cdots v_kv_t\cdots v_pv_{t-1}\cdots v_{k+1}$

$v_j \cdots v_1$ of length $p$, contradicting Claim 1. So, the three vertices $v_1, v_{t-1}$ and $v_p$ are independent.

If $v_k v_{t-1} \in E(G)$, then from Condition 2 of the theorem we have $w(v_k v_{t-1}) = w(v_k v_t) = w(v_1 v_k) = w^*$, and from Condition 3 of the theorem, the edge $v_{t-1}v_t$ has weight $w^*$. If $v_k v_{t-1} \notin E(G)$, then from Condition 2 of the theorem we also get $w(v_{t-1}v_t) = w^*$. Thus, in both cases the path $v_{t-1}v_{t-2} \cdots v_{k+1}v_j \cdots v_1\ v_{j+1} \cdots v_k v_t \cdots v_p$ is another longest path with the same weight as $P$. By the choice of $P$ in (c), $d(v_{t-1}) \leq d^w(v_1) \leq w(C)/2$. With $d^w(v_1) + d^w(v_p) \leq w(C)$, we have $d^w(v_1) + d^w(v_{t-1}) + d^w(v_p) \leq 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle $C$ is at least $2m/3$.

This completes the proof of Case 2.1.

*Case* 2.2. There is some vertex $v_i$ with $3 \leq i \leq k$ such that $w(v_1 v_{i-1})$, $w(v_1 v_i)$ and $w(v_{i-1}v_i)$ are all different.
In this case, choose vertex $v_j$ such that $w(v_1 v_{j-1})$, $w(v_1 v_j)$ and $w(v_{j-1}v_j)$ are all different, and $j$ is as large as possible. Denote the weight of $v_1 v_j$, $v_{j-1}v_j$ and $v_1 v_{j-1}$ by $w_1$, $w_2$ and $w_3$, respectively. It follows from Condition 3 (or Condition 2 if $j = k$) that $w(v_{j-1}v_j) = w_2 \neq w_1 = w(v_j v_{j+1})$, and from Condition 2 of the theorem that $v_{j-1}v_{j+1} \in E(G)$. If $j < k$, then the weight of the edge $v_{j-1}v_{j+1}$ is different from the weight $w_1$ of the edge $v_{j+1}v_{j+2}$ since there is a triangle $v_1 v_{j-1} v_{j+1} v_1$ and $w(v_1 v_{j-1}) = w_3 \neq w_1 = w(v_1 v_{j+1})$. With the same argument, we can prove that $v_{j-1}v_i \in E(G)$ for all $i$ with $j \leq i \leq k+1$. By the choice of $v_k$, we have that $w(v_{j-1}v_{k+1}) = w_3$.

Suppose first that $v_k v_{k+2} \in E(G)$. Then $d(v_1, v_{k+2}) = 2$. This shows that $w(v_k v_{k+2}) = w(v_1 v_k) = w_1$. From $w(v_k v_{k+1}) = w(v_k v_{k+2}) = w_1$ and Condition 3 of the theorem we know that $w(v_{k+1}v_{k+2}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ since the two edges $v_{j-1}v_{k+1}$ and $v_{k+1}v_{k+2}$ have different weights. Again, by the fact $d(v_1, v_{k+2}) = 2$, we obtain that $w(v_{j-1}v_{k+2}) = w(v_1 v_{j-1}) = w_3$. This leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ in which $w(v_{j-1}v_{k+1}) = w(v_{j-1}v_{k+2}) = w_3$ and $w(v_{k+1}v_{k+2}) = w_1$, contradicting Condition 3 of the theorem. Hence $v_k v_{k+2} \notin E(G)$. Thus $d(v_k, v_{k+2}) = 2$. This implies that $w(v_{k+1}v_{k+2}) = w(v_k v_{k+1}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ and $w(v_{j-1}v_{k+2}) = w_3$. This also leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ which is impossible by Condition 3 of the theorem.

The proof of the theorem is complete. ■

## 4. Remarks

The proof of Theorem C in [6] is very complicated. It is clear that our proof of Theorem 3 provides a simpler proof for Theorem C in the case $k = 2$. We do not know whether the extra conditions in Theorem 3 are necessary. The results in [8] indicate that for some generalizations of long cycle results to weighted graphs one cannot avoid such additional conditions. We do not believe that there is an analogous generalization of Theorem C for the case $k \neq 2$.

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