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ODD AND RESIDUE DOMINATION NUMBERS OF A GRAPH

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Abstract

Let $G = (V, E)$ be a simple, undirected graph. A set of vertices D is called an *odd dominating set* if $|N[v] \cap D| \equiv 1 (\text{mod } 2)$ for every vertex $v \in V(G)$. The minimum cardinality of an odd dominating set is called the odd domination number of G, denoted by $\gamma_1(G)$. In this paper, several algorithmic and structural results are presented on this parameter for grids, complements of powers of cycles, and other graph classes as well as for more general forms of "residue" domination.

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1 Introduction

Let $G = (V, E)$ be a simple, undirected graph. A set of vertices D is called an *odd dominating set* if for every vertex $v \in V(G)$, $|N[v] \cap D| \equiv 1(\bmod 2)$. The minimum cardinality of an odd dominating set is called the *odd domination number* of G, denoted by $\gamma_1(G)$. An *even dominating set* is analogously defined.

Sutner showed that every graph contains an odd dominating set [17] (and a different proof was given by the first author in [4]) and that the related minimization problem is NP-complete. The maximization problem (the decision problem associated with finding the largest odd dominating set) is also NP-complete [12]. However, this particular parameter has been studied very little, other than in [5]. Work on the general problem of parity domination, much of it focusing on algorithmic problems and on even dominating sets, can be found in [1, 2, 3, 8, 9, 10, 11]. For example, it is known that $\gamma_1(G)$ can be computed in polynomial time for series-parallel graphs [2]. It is also a simple exercise to adapt the perfect code algorithm from [16] to compute $\gamma_1(G)$ in linear time in circular-arc graphs. (A *perfect code* is a subset D of $V(G)$ such that every vertex in $V(G)$ contains exactly of member of D in its closed neighborhood).

In this paper we consider odd domination and a generalization called *residue domination* in complements of powers of cycles, grid graphs, and a new class called k-*exclusive* graphs. Structural results are presented as well as algorithmic results. Several open problems are presented at the conclusion of the paper.

1.1 Notation and Definitions

We use $N(u)$ to denote the open neighborhood of a vertex u and $N[u]$ to denote the closed neighborhood of u. Denote the complement of a graph G by $\overline{G}$. A graph G is called an *odd-graph* or an *even-graph* if all the degrees of it vertices are odd or even, respectively. The cycle on n vertices is denoted C_n. The domination number of a graph G is denoted by $\gamma(G)$ and $\deg(v)$ is the degree of a vertex v.

Let $A(G)$ denote the adjacency matrix of G and $N(G) = A(G) + I_n$, where G has order n and I_n is the $n \times n$ identity matrix.

2 Complements of Powers of Cycles

In [5], it was shown that $\gamma_1(\overline{C_n^k}) = n$ if n and $k(k+1)$ are relatively prime.

We now show that this condition is necessary and sufficient.

Theorem 1. *For the k^{th} power of a cycle, $\gamma_1(\overline{C_n^k}) = n$ if and only if n and $k(k+1)$ are relatively prime.*

Proof. Let $G = \overline{C_n^k}$ where n and $k(k+1)$ are relatively prime, and note that n is odd. We assume that $n > 2k+1$; otherwise, G has no edges and the theorem is obviously true. Since G is an even-graph, one odd dominating set of G consists of all vertices of G [5]. We show that G contains no other odd dominating set. To do this, we show that $N(G)$ is non-singular over $GF(2)$, the result then follows by standard linear algebra (cf. [9]).

Index the rows of $N(G)$ by $0, 1, \ldots, n-1$. The first row of $N(G)$ is $10^k1^{n-2k-1}0^k$, where a^b denotes symbol a repeated b times. The second row is the first row shifted one position to the right; by "shift" we mean a circular shift. In this proof, we consider a vector (or row) as circular, so that the n^{th} element is adjacent to the first. Note that if we can produce a row r by a linear combination of rows of $N(G)$, then we can also produce any *shift* of r by a linear combination of rows of $N(G)$. By a shift of r, we mean that the elements of r are shifted c positions in a circular fashion. For example, vector 0110^{n-3} is a shift of 110^{n-2}. We use this fact often (and, at times, implicitly) in order to facilitate the discussion.

As an overview of the proof, we shall show that there is a linear combination of rows of $N(G)$ that is equal to 110^{n-2}. Using this row (and its shifts), it is possible to generate the row 10^{n-1} (and all its shifts). This is because any row, r, from $N(G)$ has an odd number of ones, and when we add 110^{n-2} (or one of its shifts) to r, the number of ones changes by an even number each time. Therefore, we can generate a set of vectors, specifically 10^{n-1} and all its shifts, that span the set of all n-vectors. Hence the rows of $N(G)$ are linearly independent and $N(G)$ is non-singular over $GF(2)$.

We produce row 110^{n-2} as follows. Consider the sum of row 0 and row $n-k$; we call the sum z. It is equal to: $1^{n-2k}0^{3k+1-n}1^{n-2k}0^{k-1}$ if $n-2k-1 \leq k$ (case 1) or $1^{k+1}0^{n-3k-1}1^{k+1}0^{k-1}$ if $n-2k-1 > k$ (case 2). In case 1, note that sum of number of places in the first two "blocks" is $k+1$ (i.e., $n-2k+3k+1-n = k+1$). So adding the given row, z, to its shift by $k+1$ positions to the right, has the effect of moving the second block of 1's in z $k+1$ positions to the right. We can keep shifting this block by $k+1$ positions until it appears in positions 2 through $n-2k+1$, yielding a vector z'. This happens eventually because $\gcd(n, k+1) = 1$. Adding these two vectors, z and z', then cancels all but two 1's: those in in positions 1 and $n-2k+1$. Now we can repeatedly add this weight two vector to itself shifted by $n-2k$ places to the right until we get $11000\ldots$. This is because $\gcd(n,k)=1$ implies $\gcd(n,n-2k)=1$. Case 2 is similar except the weight two vector has 1's in positions 1 and $k+2$ (we can derive this vector because $\gcd(n,n-2k)=1$). Then we can add this vector to shifts of itself in order to get $11000\ldots$, because $\gcd(n,k+1)=1$.

We now prove the "only if" direction, for sake of completeness. If $\gcd(k,n)=d$ where $d>1$, then divide the n columns of the closed adjacency matrix into d classes of size n/d each by congruence modulo d. It is easy to show that in each row, the number of ones in columns of each class is the same. So the sum of all rows in same equivalence class modulo d is either all ones or all zeroes. So either these rows, or their complement, form an even dominating set, which implies the existence of an odd dominating set of size less than n. In fact, the sum of rows congruent to 0 (mod d) is all zeroes if $(n-2k)/d$ is even, and is all ones otherwise. If $\gcd(k+1,n)=d$ where $d>1$, then one of the d classes has precisely two more ones (in each row) than all the other classes. So the sum of rows congruent to $0(\bmod d)$ is all zeroes if $(n-2k-2)/d$ is even, and is all ones otherwise. ■

For reference in what follows, we re-state a result from [5].

Theorem 2. *Let $G=\overline{C_n^k}$. Then $\gamma_1(G)\geq\left\lfloor\frac{n}{k+1}\right\rfloor$.*

3 Residue Domination and More on Complements of Powers of Cycles

We introduce a more general form of parity domination called *residue domination.*

Definition. Call D a $k(\bmod m)$-*dominating set* of G if for every $v\in V(G)$, $|N[v]\cap D|\equiv k(\bmod m)$.

Theorem 3. *Let p be a prime and let G be a graph such that for any two vertices u,v the following hold:*

(1) $\deg(v)\neq -1(\bmod p)$,

(2) $|N[u]\cap N[v]|\equiv 0(\bmod p)$.

Then $N(G)$ is regular over $GF(p)$.

Proof. The (i,j) entry of $N(G)N(G)^t = N(G)^2$ is exactly the scalar product of the i^{th} and the j^{th} rows of $N(G)$, which is exactly $|N[v_i] \cap N[v_j]| \equiv 0(\text{mod } p)$ if $i \neq j$ (by Condition (2) of the theorem). The (i,i) entry is precisely $|N[v_i] \cap N[v_i]| = \deg(v_i) + 1 \not\equiv 0(\text{mod } p)$ (by Condition (1)). Hence $\det(N(G)) \neq 0(\text{mod } p)$ and $N(G)$ is regular over $GF(p)$. ■

Theorem 4. *Let p be a prime and let G be a graph on n vertices such that for any two vertices u, v the following hold:*

(1) $\deg(v) \equiv -1(\text{mod } p)$,

(2) $|N[u] \cap N[v]| \equiv 0(\text{mod } p)$.

Then $N(G)$ is singular over $GF(p)$ with rank$(N(G)) \leq n/2$.

Proof. The (i,j) entry of $N(G)N(G)^t = N(G)^2$ is exactly the scalar product of the i^{th} and the j^{th} rows of $N(G)$, which is exactly exactly $|N[v_i] \cap N[v_j]| \equiv 0(\text{mod } p)$ (by Conditions (1) and (2) of the theorem). Hence $N(G)^2 = 0_n$ (the zero matrix of order n). But using the inequality for multiplication of matrices [14] we get from the fact that $N(G)N(G)^t = 0_n$ that $\text{rank}(G) + \text{rank}(N(G)^t) - n \leq \text{rank}(0_n)$. Hence $2 \cdot \text{rank}(N(G)) - n \leq 0$ which gives $\text{rank}(N(G)) \leq n/2$. ■

The following is related to Theorem 1.

Theorem 5. *Let p be a prime and let G be a graph on n vertices such that for any two vertices u, v the following hold:*

(1) $\deg(v) \equiv b(\text{mod } p)$, *where* $b \neq -1$,

(2) $|N[u] \cap N[v]| \equiv 0(\text{mod } p)$.

Then $V(G)$ is the unique set of vertices D of G having the property that for every vertex v of G $|N[v] \cap D| \equiv b + 1(\text{mod } p)$.

Proof. As $\deg(v) \neq -1(\text{mod } p)$ we have the conditions of Theorem 3 and $N(G)$ is regular over $GF(p)$. Hence $N(G)X = J$ has a unique solution, where J is the all 1's vector. But as $X = V(G)$ ($xv = 1$ for every v in G) is a solution it is, in fact, the only one. ■

Theorem 6. *Let G be any graph with all degrees congruent to $b(\text{mod } m)$. Then for any $k(\text{mod } m)$ dominating set D of G, the following congruence holds: $|D|(b+1) \equiv k|V|(\text{mod } m)$.*

Proof. Consider any $k(\text{mod } m)$ dominating set D of G, and set $A = V \setminus D$. Now consider the subgraph H induced by the edges incident with the vertices of D. Clearly for a vertex $v \in D$, $\deg(H, v) = \deg(G, v)$ since v loses no edges. For a vertex $v \notin D$, $\deg(H, v) \equiv k(\text{mod } m)$ as D is $k(\text{mod } m)$ dominating. Now the number of edges between A and D is equal to $|A|k(\text{mod } m)$ because we add $|A|$ numbers each congruent to $k(\text{mod } m)$. On the other hand, every vertex in D is adjacent to $k - 1(\text{mod } m)$ vertices in D (and then with itself has the $k(\text{mod } m)$ intersection property). But since the degree in H is the same as in the original graph, it follows that every vertex of D is adjacent to $b - k + 1(\text{mod } m)$ vertices of A. Hence the number of edges between A and D is $|D|(b - k + 1)(\text{mod } m)$ if we count from D's side. Therefore $|D|(b - k + 1) \equiv k|V \setminus D|(\text{mod } m)$ which gives $|D|(b + 1) \equiv k|V|(\text{mod } m)$. ■

The next two corollaries follow directly from Theorem 6.

Corollary 7. *Let G be a graph with an even number of vertices and all vertex degrees even. Then any odd dominating set, as well as any even dominating set, of G has even cardinality.*

Corollary 8. *Let G be any graph on $n \equiv 1(\text{mod } 2)$ vertices, with all degrees even. Then any odd* (*even*) *dominating set D of G must contain an odd* (*even*) *number of vertices.*

Note that if n is even, then it follows from [5] that $\gamma_1(\overline{C_n^k}) \leq \frac{n}{2}$, since all vertex degrees are odd.

One may generalize Theorem 1 to certain other graphs of odd order whose closed adjacency matrices are circulants. If this matrix is singular, then there is a unique odd dominating set (cf. [9]). Furthermore, if G is an even-graph, then the odd domination number will be equal to n [5]. For example, the projective plane incidence matrix given by 1101000 for the first row (and then subsequent rows generated in a circulant fashion) is singular because 0010111 is in its nullspace.

Using perfect difference sets, one can produce prime-order odd row-sum singular circulants: i.e., a family of even-graphs each on an odd number of vertices. Let D be a perfect difference set modulo n, that is, the size of D is k where $C(k, 2) = (n - 1)/2$, and the $n - 1$ differences $i - j$ are distinct (modulo n), where i, j are in D. For example, $\{0, 1, 3\}$ is a perfect difference set with $k = 3, n = 7$. This particular example produces the circulant with first row 1101000 mentioned above. As long as k is odd and

n is prime, a circulant with the desired property is produced. For example, $k = 9, n = 73$ produces perfect difference set $\{0, 1, 3, 7, 15, 31, 36, 54, 63\}$ and if you generate the order 73 circulant with ones in these positions, each two rows will have exactly one column where both have ones. So the sum of the 64 rows whose numbers are not in the difference set, will be all zeroes, i.e., the matrix is singular.

It was shown above that $\gamma_1(\overline{C_n^k})$ is odd when n is odd. We can go further showing that $\gamma_1(\overline{C_n^k})$ has an odd dominating set of the form "every i^{th} vertex" when n is odd.

Proposition 9. *Let $G = \overline{C_n^k}$ and let d be the largest integer less than or equal to $k+1$ such that $d|\gcd(n, k(k+1))$ and $n/d \equiv 1(\bmod\ 2)$. Then $n/d \geq \gamma_1(\overline{C_n^k}) \geq \left\lfloor \frac{n}{k+1} \right\rfloor$. In particular, if $(k+1)|n$ and n is odd, then $\gamma_1(\overline{C_n^k}) = \frac{n}{k+1}$.*

Proof. In general, we know that $\gamma_1(G) \geq \left\lfloor \frac{n}{k+1} \right\rfloor$ [5]. Label the n vertices in clockwise order as $v_0, \ldots, v_{n-1}$. For the upper bound, we show that G contains an odd dominating set D, consisting of the vertices whose labels belong to the set $\{jd | j = 0, 1, \ldots, n/d - 1\}$. If $d = 1$, then clearly n is odd and we are finished as $D = V(G)$ and $|D| = n \equiv 1(\bmod\ 2)$ and for each vertex v we have that $|N[v] \cap D| = n - 2k \equiv 1(\bmod\ 2)$. So assume that $1 < d \leq k+1$.

If $d|k$, then for every vertex $v \in V(G)$ we have $|N[v] \cap D| = n/d - 2k/d \equiv 1(\bmod\ 2)$ and we are finished. If $d|(k+1)$, then for every vertex $v \in D$ we have $|N[v] \cap D| = n/d - 2(k+1)/d \equiv 1(\bmod\ 2)$ and for every vertex $v \notin D$ we have $|N[v] \cap D| = n/d - 2(k+1)/d - 2 \equiv 1(\bmod\ 2)$, thereby completing the proof. ∎

Conjecture 1. *Let $G = \overline{C_n^k}$ where n is odd. Then*
$\gamma_1(G) \geq \max\{\left\lfloor \frac{n}{k+1} \right\rfloor, \frac{n}{\gcd(n, k(k+1))}\}$.

A partial solution to the conjecture is given next.

Proposition 10. *Let $G = \overline{C_n^k}$ and $\gcd(n, k(k+1)) = k$. If n is odd, then $\gamma_1(G) = \frac{n}{k}$.*

Proof. By Proposition 9, G contains an odd dominating set of size $\frac{n}{k}$. So we have to show no smaller odd dominating set exists. Number the vertices in a clockwise fashion from v_1 to v_n.

Let D be an odd dominating set. Set $q(v) = 1$ if $v \in D$, otherwise $q(v) = 0$. For vertex v_1 we have $q(v_1) + \sum_{j=k+2}^{n-k} q(v_j) \equiv 1(\bmod 2)$. Hence (working in GF(2)) and equating the formulae for v_1 and v_2 we get:
$q(v_1) + q(v_{k+2}) = q(v_2) + q(v_{n-k+1})$. Doing the same for v_2 and v_3 we get:
$q(v_2) + q(v_{k+3}) = q(v_3) + q(v_{n-k+2})$. Continuing we get:
$q(v_3) + q(v_{k+4}) = q(v_4) + q(v_{n-k+3})$,
$q(v_4) + q(v_{k+5}) = q(v_5) + q(v_{n-k+4})$,
$\ldots$
$q(v_{k+1}) + q(v_{2k+2}) = q(v_{k+2}) + q(v_1)$.

Summing up and dropping equal terms we get:
$q(v_{k+3}) + q(v_{k+4}) + \ldots + q(v_{2k+2}) = q(v_{n-k+1}) + q(v_{n-k+2}) + \ldots + q(v_n)$. Writing $Q(j) = \sum_{r=j}^{j+k-1} q(v_r)$, we have that $Q(j) = Q(j + 2(k + 1))$ holds for every vertex v_j. But as $\gcd(n, k(k + 1)) = k$ and n is odd, we infer that $\gcd(n, 2(k + 1)) = 1$. Hence by shifting cyclically the equation $Q(j) = Q(j+2(k+1))$, we get $Q(j) = Q(j+1)$ for every j. But this gives immediately $q(v_j) = q(v_{j+k})$ for every j. Since we may assume without loss of generality that $v_0 \in D$, we immediately get that $v_j \in D$ for every $j \equiv 0(\bmod\ k)$, hence proving the other side of the inequality. ■

For $k = 3$ we can give precise values of $\gamma_1(\overline{C_n^k})$. The $k = 1, k = 2$ cases were resolved in [5].

Fact 11. *Let* $G{=}\gamma_1(\overline{C_n^3})$.

(i) *If* $n \leq 7$, *then* $\gamma_1(G) = n$.
(ii) *If* $n \equiv \{1, 3, 5, 7, 11, 13, 17, 19, 23\}(\bmod 24)$, *then* $\gamma_1(G) = n$.
(iii) *If* $n \equiv \{3, 9, 15, 21\}(\bmod 24)$, *then* $\gamma_1(G) = \frac{n}{3}$.
(iv) *If* $n \equiv \{4, 12, 20\}(\bmod 24)$, *then* $\gamma_1(G) = \frac{n}{4}$.
(v) *If* $n \equiv \{2, 6, 8, 10, 14, 16, 18, 22\}(\bmod 24)$, *then* $\gamma_1(G) = \frac{n}{2}$.

Proof. The odd n cases follow from Theorem 1, Propositions 9 and 10 and a simple "shifting" argument as used in Proposition 10. The case when $n = 8m + 4$, for any positive integer m, is a direct application of Proposition 9. The remaining cases were verified using a computer and the fact that a "periodic" minimum odd dominating set exists for n sufficiently large [5]. ■

Theorem 12. *Let* $n = 2^m$ *for an integer* $m \geq 1$ *and let* $n \geq 2k + 2$. *Then* $\gamma_1(\overline{C_n^k}) = \frac{n}{2}$.

Proof. The case for $k = 2$ was proved in [5]. One may note that $\gamma_1(\overline{C_n^{n/2-1}}) = \frac{n}{2}$, as $\overline{C_n^{n/2-1}}$ is a perfect "matching" (i.e., a one-factor). Since n is even, we know that $\gamma_1(\overline{C_n^k}) \leq \frac{n}{2}$. One may also observe that $G = \overline{C_n^k}$ cannot contain an odd dominating set of odd order, since the all-ones vector is in the nullspace of the closed adjacency matrix of G (and the nullspace and rangespace of this matrix are orthogonal complements). Denote the all-ones vector of length n as J_n (the subscript n will be omitted when clear from the context).

Let w be the first row of the closed neighborhood matrix of G, which we denote as matrix A. So w is a length n ($n = 2^m$) 0-1 vector. Vector w can be described as follows: first a single 1, then k 0's, then $n - 1 - 2k$ 1's, then k 0's. Each subsequent row of A is this row shifted one place to the right, i.e., A is a circulant. We want to find vectors Y such that $AY = J$, and show that entries of any such Y contains precisely $\frac{n}{2}$ 1's. We use polynomials to do this. If entries of w are $a, b, c, d, \ldots$, we represent w by the polynomial $a + bx + cx^2 + dx^3 + \ldots$, a polynomial of degree at most $n - 1$ (and we shall work with these polynomials *modulo* $(x^n + 1)$). The successive shifts of the vector w are products of the polynomial representing w times x^j for various integers j. A sum of rows of A which equal the all 1's vector corresponds to finding a polynomial q such that q times w equals the polynomial corresponding to the all 1's vector, J. (Abusing notation, we are using w for the polynomial representing vector w). Using the fact that $n = 2^m$, the polynomial representing J is $(1 + x)^{(n-1)}$. Hence, the sum of the exponents of $(1 + x)$ in the prime factorizations of q and w (in the ring of polynomials *modulo* $(x^n + 1)$) must be equal to $n - 1$. It turns out that the possible prime factorizations of w have a special form. The exponent of $(x + 1)$ is always a power of two. Furthermore, if $j = (n - 1) - 2^i$ for some positive integer i, then $(x + 1)^j$ has precisely $\frac{n}{2}$ non-zero terms. This yields which rows of A to add to get the all 1's vector, i.e., an odd dominating set of G. ∎

An example is in order. Consider the case when $n = 32$ and $k = 4$. Then w has a 1, then four 0's, then twenty-three 1's, then four 0's. Written in polynomial form, $w = (1+x)^{16} + x^5(1+x+x^2+x^3+\ldots+x^{10})(1+x+x^2+\ldots+x^{11})(1+x)$. The polynomial $(1+x)$ does not divide $(1+x+\ldots+x^{10})$. The exponent of $(1+x)$ in the prime factorization of $(1+x+\ldots+x^{11})(1+x)$ is $\gcd(12, 32) = 4$. Thus $(1 + x)^{27}$ yields the desired odd dominating set: 11110000111100001111000011110000.

Since A is a symmetric matrix, w is a self-reciprocal polynomial. We suspect this fact may turn out to be significant, because, for example, this fact is important for binary cyclic codes, which have some of the same algebraic structure as above. But, in general, an arbitrary self-reciprocal w may not have useful property that the exponent of $(x+1)$ in the prime factorization of w is a power of two, which was needed to show that the size of any odd dominating set is $\frac{n}{2}$.

4 Other Structural Results

4.1 Grids

In [5], the odd domination number for $n \times m$ grids were given for $1 \le n \le 4$, and it was shown that $\gamma_1(G) \le \frac{20}{7}\gamma(G)$ where G is a sufficiently large grid graph. Those grids with unique odd dominating sets were characterized in [9]. In [11], it was shown that if G is an $n \times n$ grid graph, with $n = 2^m$ or $n = 2^m - 2$ and $m \ge 4$, then G contains more than one odd dominating set. Table 1 shows the odd domination numbers of some small square grids, which were determined using a computer program that performed an exhaustive search. An "x" in the table indicates that the $n \times n$ grid has a unique odd dominating set. The domination number of the $n \times n$ grid is at most $\frac{n^2+4n-c}{5}$ for $16 \le c \le 20$ [6]. The 6×6 grid has $\gamma_1(G)\Box\frac{7}{2}\gamma(G)$, and the 13×13 and 21×21 grids have $\gamma_1(G) > \frac{21}{9}\gamma(G)$. An infinite family of grids was shown in [5] to have $\frac{\gamma_1(G)}{\gamma(G)} \approx \frac{20}{9}$.

4.2 Cubes

Theorem 13. *The odd domination number of the even cube Q_{2n} is precisely $|Q_{2n}| = 2^{2n}$.*

Proof. The cube Q_{2n} satisfies Theorem 5 with $p = 2$. ■

If $n = 2^m - 1$, there exists a perfect error-correcting code of distance one (a Hamming code) where every word (node of Q_n) is at distance one from exactly one word of the code. (Such a set of vertices in a graph is, in general, called a *perfect code*, cf. [13, 16]). This gives that in these cases $\gamma_1(Q_n) = \frac{2^n}{\deg(v)+1} = \frac{2^n}{2^m-1+1} = 2^{n-m}$. For other values of n, we have the inequality $\gamma_1(Q_{n+2}) \le 4\gamma_1(Q_n)$ (which has meaning only for odd n). This comes by observing that if D is an odd dominating set for Q_n, then the corresponding four "copies" of D in Q_{n+2} form an odd dominating set for Q_{n+2}. So, $\gamma_1(Q_n)$ is approximately $\frac{|V(Q_n)|}{\log|V(Q_n)|}$ for n odd.

Table 1. Odd Domination Numbers for Square Grids

n	$\gamma_1(n \times n \text{ grid})$	Unique Odd Dom Set?
1	1	x
2	4	x
3	5	x
4	4	
5	15	
6	28	x
7	33	x
8	40	x
9	25	
10	44	x
11	55	
12	72	x
13	105	x
14	56	
15	117	x
16	104	
17	147	
18	188	x
19	141	
20	224	x
21	245	x
22	276	x
23	231	
24	270	
30	≤ 414	
32	≤ 458	

5 Algorithmic Results

5.1 Residue Domination

We describe a family of graphs in which one can find optimal residue domination sets in polynomial time, i.e., a minimum sized set satisfying a residue domination constraint.

Definition. A graph is called *k-exclusive* if its vertices can be ordered $v_1, \ldots, v_n$ such that for every $j > k$, v_j is the unique neighbor in $\{v_j, \ldots, v_n\}$ of at least one vertex in $\{v_1, \ldots, v_{j-1}\}$.

Note that this class contains several well-known classes of graphs including the k^{th} power of paths and cycles and grids of dimension $k \times m$.

Proposition 14. *Let G be a k-exclusive graph with vertex ordering $v_1, \ldots, v_n$ realizing the k-exclusiveness. Then for every $j \geq 1$, the following hold:*

(1) *There are at most k vertices in $\{v_1, \ldots, v_j\}$ having neighbors in $\{v_{j+1}, \ldots v_n\}$,*
(2) *Vertex v_j is adjacent to at most k vertices in $\{v_1, \ldots, v_{j-1}\}$.*

Proof. The proof is by induction on j. The proposition holds trivially for $1 \leq j \leq k$. Assume the proposition holds for v_j and we show it holds for v_{j+1}. By definition, v_{j+1} is the unique neighbor in $\{v_{j+1}, \ldots, v_n\}$ of at least one vertex in $\{v_1, \ldots, v_j\}$. By the inductive hypothesis, at most k vertices in the set $\{v_1, \ldots, v_j\}$ have neighbors in $\{v_{j+1}, \ldots v_n\}$. Thus v_{j+1} can have at most k neighbors in $\{v_1, \ldots, v_j\}$. Furthermore, there are at most k vertices in $\{v_1, \ldots, v_{j+1}\}$ having neighbors in $\{v_{j+2}, \ldots, v_n\}$ as we gain at most one vertex having a neighbor in $\{v_{j+2}, \ldots, v_n\}$ (i.e., v_{j+1}), but we lose at least one vertex: the vertex having v_{j+1} as its unique neighbor in $\{v_{j+1}, \ldots, v_n\}$, thereby completing the induction step. ■

Proposition 15. *Let G be a k-exclusive graph. Then G is k-degenerate and is $k+1$-colorable.*

Proof. Let $\beta = v_1, \ldots, v_n$ be an ordering of the vertices realizing the k-exclusiveness of G. We must show that every subgraph H of G has a vertex with degree in H of at most k. However, this is clear as the vertex in H with highest index in β is adjacent to at most k vertices of lower indices and we are done. Since it is well-known that k-degenerate graphs are $k+1$ colorable (the greedy coloring algorithm with vertex ordering β will produce such a coloring) the second claim is proved. ■

Theorem 16. *Let G be a k-exclusive graph and let H be any graph. Then $G \times H$ is $k|H|$-exclusive.*

Proof. Recall that $G \times H$ is defined by:
$V(G \times H) = \{(u, v)|u \in V(G), v \in V(H)\}$;
$E(G \times H = \{((x, y), (u, v))|$ either $(x = u)$ and $(y, v) \in E(H)$ or $(y = v)$ and $(x, u) \in E(G)\}$.
As G is k-exclusive, let $\beta = u_1, \ldots, u_n$ be the vertex ordering of G realizing its k-exclusiveness. Let $V(H) = \{v_1, \ldots, v_m\}$. The following ordering, β, of the vertices of $G \times H$ realizing its $k|H|$-exclusiveness: $(u_1, v_1), (u_1, v_2), \ldots,$ $(u_1, v_m), (u_2, v_1), (u_2, v_2), \ldots,$ $(u_2, v_m), \ldots, (u_n, v_1), \ldots, (u_n, v_m)$. This is true because every vertex in $G \times H$ of the form (u_j, v_t) (u_j, v_t) with $j > k$ is the unique neighbor, in β, of vertex (u_{j-1}, v_t), and by definition (u_{j-1}, v_t) has lower index in β than (u_j, v_t). ■

We are now ready to describe the residue domination algorithm.

Proposition 17. *Let G be a k-exclusive graph with its k-exclusive vertex ordering given. Then a minimum $b(\text{mod } m)$ dominating set can be found or the non-existence of such a residue dominating set can be decided in time $O(2^k k^3 n)$.*

Proof. Let $v_1, \ldots, v_n$ be a vertex order realizing the k-exclusiveness of G. Using an "exhaustive-search" strategy, we shall construct a candidate residue dominating set, D, by considering all possible 2^k combinations of vertices $\{v_1, \ldots, v_k\}$. Let f denote the characteristic function of D. For each combination of these k vertices, we verify that for each vertex v in $\{v_1, \ldots, v_k\}$ that has no neighbor in $\{v_{k+1}, \ldots, v_n\}$ satisfies $|N[v] \cap D| \equiv b(\text{mod } m)$. If this congruence is satisfied for all such vertices in $\{v_1, \ldots, v_k\}$, we can continue. Otherwise, the initial combination is illegal and the next combination is tested. If the initial combination is legal, we consider (in increasing order of index) vertex v_j, which is the unique neighbor in $\{v_j, \ldots, v_n\}$ of at least one vertex in $\{v_1, \ldots, v_{j-1}\}$ and, by Proposition 14, of at most k vertices, in $\{v_1, \ldots, v_{j-1}\}$. The possible value, 0 or 1, of $f(v_j)$ is completely determined by those vertices in $\{v_1, \ldots, v_{j-1}\}$ for which v_j is the unique neighbor in the set $\{v_j, \ldots, v_n\}$. Add v_j to D (or not) if its addition (omission) properly satisfies the residue constraints for these vertices. If so, proceed to v_{j+1}, otherwise consider the next initial combination. If this algorithm does not terminate with a successful construction of D, then we infer no $b(\text{mod } m)$ dominating set exists. Otherwise, output the smallest such dominating set discovered in this process, which must be a minimum sized $b(\text{mod } m)$ dominating set.

Testing each initial combination takes $O(k^2)$ time. As we can maintain during the algorithm the value $p(v) \equiv |N[v] \cap D| (\bmod m)$ for every vertex v already visited, we can decide "legality" of the value of $f(v_j)$ in $O(k)$ time: updating the $p(v)$ values that may be changed can be done in $O(k)$ time as v_j has at most k neighbors. Thus the running time of the algorithm is as claimed. ■

5.2 NP-Completeness Results

Theorem 18. *Deciding if a planar graph G with maximum degree six contains an odd dominating set of size at most k is NP-complete.*

Proof. The problem is clearly in NP. We perform a reduction from **Planar 3-SAT** using the special case in which each variable appears in at most five clauses [7].

Let F be an instance of Planar 3-SAT with clause set C and variable set U. Assume without loss of generality that no clause contains a variable and its negation. Construct graph G_0 as follows. Create a *clause* vertex c in G for each clause c in C and a *variable* vertex v in G for each variable v in U (using the same names for vertices and elements from C, U when the context is clear). Add an edge from c to v if v appears in clause c (in negated or non-negated form). This graph is planar [7]. Observe that the following operations preserve planarity: inserting an edge (u, v) parallel to existing edge (u, v); subdividing edge (u, v) into path uwv (i.e., inserting a new vertex w); deleting an edge. Graph G constructed from G_0 in what follows will be shown to be planar.

Graph G is constructed from G_0 by making a parallel copy of each edge in G_0 and partitioning one of each parallel pair into a path of length two. The newly added vertices (denoted as $\overline{v}$) will represent the negation of variables. If variable v appears in c in negated form, delete edge (v, c) (leaving edge $(\overline{v}, c)$) and vice versa if v appears in c in non-negated form. Attach a pendant path with two (new) vertices c_{p_1}, c_{p_2} (the latter will have degree one) to each clause vertex.

The resulting graph G is planar and has maximum degree six, and has $3|C| + 2|U| + 2|C|$ vertices. We claim F is satisfiable if and only if G has an odd dominating set of size at most $|C| + |U|$.

Suppose F is satisfiable. Include in set D each vertex v if $v =$ true and each vertex of the form $\overline{v}$ if $v =$ false. Add to D each c_{p_2} vertex if c has an odd number of "witnesses" ("true" literals) and c_{p_1} otherwise. It is easy to see that D is an odd dominating set of size $|C| + |U|$.

Suppose there is an odd dominating set D, $|D| \leq |C| + |U|$. It is easy to see that no clause vertex c can be in D, else $N[c_{p_i}] \cap D \equiv 0 (\mathrm{mod}\ 2)$ for some $1 \leq i \leq 2$. Thus exactly one of these c_{p_i} vertices must be in any odd dominating set and further, at least one of each $v, \overline{v}$ pair must be in any odd dominating set. Thus D contains exactly one vertex from each $v, \overline{v}$ pair and each clause vertex must be adjacent to at least one "witness" vertex. Hence the proof. ■

Theorem 19. *Deciding if a bipartite planar graph G contains an odd dominating set of size at most k is NP-complete.*

Proof. The problem is clearly in NP. We perform a reduction from **Planar 3-SAT** [7].

Let F be an instance of Planar 3-SAT with clause set C and variable set U. Assume without loss of generality that no clause contains a variable and its negation and that $|C| > 3$ (else F is trivially satisfiable). Construct graph G_0 as follows. Create a *clause* vertex c in G for each clause c in C and a *variable* vertex v in G for each variable v in U (using the same names for vertices and elements from C, U when the context is clear). Add an edge from c to v if v appears in clause c (in negated or non-negated form). This graph is planar [7]. Observe that the following operations preserve planarity: inserting an edge (u, v) parallel to existing edge (u, v); subdividing edge (u, v) into path uwv (i.e., inserting a new vertex w); deleting an edge. Graph G constructed from G_0 in what follows will be seen to be planar.

Graph G is constructed G_0 by making a parallel copy of each edge in G_0 and partitioning one of each parallel pair into a path of length two. The newly added vertices (denoted as $\overline{v}$) will represent the negation of variables. If variable v appears in c in negated form, delete edge (v, c) (leaving edge $(\overline{v}, c)$) and vice versa if v appears in c in non-negated form. Subdivide each edge $(v, \overline{v})$ by adding a new vertex v_0. Attach to each v_0 and to each clause vertex c a new copy of the subgraph H shown in Figure 1, by making v_0 (c) adjacent to vertices $a_1, a_2, \ldots a_{|C|*|U|}$, i.e., v_0 (c) becomes the bottom (unlabeled) vertex in Figure 1.

The resulting graph G is planar, bipartite, and has $|C|+3|U|+|U|(2|C| * |U| + 1) + |C|(2|C| * |U| + 1)$ vertices. We claim F is satisfiable if and only if G has an odd dominating set of size at most $|C| + 2|U|$.

Suppose F is satisfiable. Include in set D each vertex v if $v =$ true and each vertex of the form $\overline{v}$ if $v =$ false. Add to D each h_2 vertex (of which there are $|C| + |U|$). It is easy to see that D is an odd dominating set of size $|C| + 2|U|$.

Suppose there is an odd dominating set D, $|D| \leq |C| + 2|U|$. It is easy to see that no clause vertex c nor any vertex of the form v_0 can be in D, else we would have to include more than $|C| + 2|U|$ vertices from a single H subgraph in order that D be an odd dominating set. Thus D contains exactly one vertex from each $v, \overline{v}$ pair, D contains each h_2 vertex, and each clause vertex must be adjacent to at least one "witness" vertex. Hence the proof. ∎

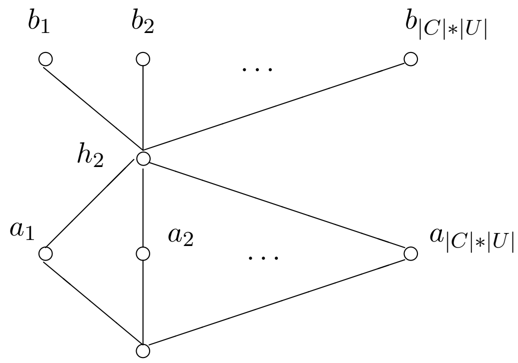

Figure 1. Subgraph H

6 Future Directions

We list several open problems.

Problem 1. *Characterize the odd domination number for complements of powers of cycles. In particular, we leave open the case when* $\gcd(n, k(k+1)) < k$.

Problem 2. *Find a tighter upper bound than* $\frac{20}{7}$ *on* $\frac{\gamma_1(G)}{\gamma(G)}$ *for all sufficiently large grid graphs* G.

Problem 3. *Can the odd domination number be computed in polynomial time in graphs of bounded tree-width? In chordal bipartite graphs? In grids?*

Problem 4. *Does knowing that a graph has a perfect code help one to compute its odd domination number* (*in certain classes of graphs*)?

Problem 5. *Prove* (*or disprove*) *that the minimum odd dominating set cannot be approximated to within any constant multiplicative factor unless* $P = NP$.

Problem 6. *Can k-exclusive graphs be recognized in polynomial time, for fixed values of k? If G has maximum degree three, can it be determined in polynomial time if G is k-exclusive*?

The next problem is re-stated from [5].

Problem 7. *Find an infinite family of even-order graphs such that each graph in the class has* $\gamma_1(G) + \gamma_1(\overline{G}) = 2n - 4$ (*or* $2n - 3$). *Likewise for graphs with* $n \equiv 3 \pmod 6$ *vertices.*

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