MINIMAL FORBIDDEN SUBGRAPHS
OF REDUCIBLE GRAPH PROPERTIES

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Abstract

A property of graphs is any class of graphs closed under isomorphism. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A graph G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if the vertex set $V(G)$ can be partitioned into n sets, $\{V_1, V_2, \ldots, V_n\}$, such that for each $i = 1, 2, \ldots, n$, the graph $G[V_i] \in \mathcal{P}_i$. We write $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ for the property of all graphs which have a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. An additive induced-hereditary property $\mathcal{R}$ is called reducible if there exist additive induced-hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$. Otherwise $\mathcal{R}$ is called irreducible. An additive induced-hereditary property $\mathcal{P}$ can be defined by its minimal forbidden induced subgraphs: those graphs which are not in $\mathcal{P}$ but which satisfy that every proper induced subgraph is in $\mathcal{P}$. We show that every reducible additive induced-hereditary property has infinitely many minimal forbidden induced subgraphs. This result is also seen to be true for reducible additive hereditary properties.

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1   Introduction

An *additive induced-hereditary property* of graphs is any class of simple graphs which is closed under isomorphisms, disjoint unions and induced

subgraphs. Similarly an *additive hereditary property* is closed under isomorphisms, disjoint unions and subgraphs. The set of all additive induced-hereditary properties of graphs, partially ordered by set inclusion, is a lattice. We will use the notation $\mathbb{M}^a$ to denote this lattice of properties. The set of all additive hereditary properties also forms a lattice, and we will denote this lattice by $\mathbb{L}^a$.

For any property $\mathcal{P}$ in $\mathbb{M}^a$, the set $\boldsymbol{C}(\mathcal{P})$ of *minimal forbidden induced subgraphs* of $\mathcal{P}$ is defined by $\boldsymbol{C}(\mathcal{P}) = \{G : G \notin \mathcal{P}$ but every proper induced subgraph of G is in $\mathcal{P}\}$. It is easy to see that, for an additive property of graphs, the set $\boldsymbol{C}(\mathcal{P})$ contains only connected graphs. The set $\boldsymbol{C}(\mathcal{P})$ characterises $\mathcal{P}$ in the sense that a graph is in $\mathcal{P}$ if and only if it contains no graph from $\boldsymbol{C}(\mathcal{P})$ as an induced subgraph.

Let n be a positive integer with $n \geq 2$ and consider properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ in $\mathbb{M}^a$. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition of a graph* G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$ the subgraph $G[V_i]$ induced by V_i has property $\mathcal{P}_i$. We allow V_i to be empty. $G[\emptyset]$ is in every property. The *product* $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is now defined as the set of all graphs having a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Such a product is easily seen to be induced hereditary if each $\mathcal{P}_i$ is induced hereditary and additive if each $\mathcal{P}_i$ is additive. A *reducible property* $\mathcal{R}$ is a property in $\mathbb{M}^a$ which can be written as a product of non-trivial properties from $\mathbb{M}^a$; if this is not possible we call $\mathcal{R}$ an *irreducible* property. If $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, we call $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ a *factorization* of $\mathcal{R}$.

Analogous definitions and results hold for additive hereditary properties. More basic results concerning hereditary graph properties can be found in the survey paper [1].

In Section 2 we show that every reducible property in $\mathbb{M}^a$ has an infinite set of minimal forbidden induced subgraphs. The analogous result for a reducible property in $\mathbb{L}^a$ follows easily and is given in Section 3

2   Additive Induced-Hereditary Properties

Let $\mathcal{P} \in \mathbb{M}^a$ be a reducible property. We will show that $\boldsymbol{C}(\mathcal{P})$ is infinite by showing that the set of all cyclic blocks making up the graphs in $\boldsymbol{C}(\mathcal{P})$ is infinite. In order to do this we will use the following three lemmas, and Theorem 25 which is an extension of a result of Nešetril and Rödl ([3]). The main result is then presented in Theorem 26.

Lemma 21. *Let G be a graph with at least one cyclic block. If B is a maximal cyclic block of G (under containment as a subgraph), then there exists an independent set $I \subseteq V(G)$ such that $G - I$ has no subgraph isomorphic to B.*

Proof. We work by induction on the number of cyclic blocks of G. If G has one cyclic block, this must be B and we can let I be the set containing any single vertex from this block.

Suppose the result is true if G has k cyclic blocks, $k > 0$. Now suppose that G has $k + 1$ cyclic blocks, and that B is a maximal cyclic block of G. Delete vertices of degree one from G until the result is a graph all of whose endblocks are cyclic. Call this graph G'. Let D be an endblock of G', with cut-vertex v. If $G' - (D - v)$ contains no blocks isomorphic to B we can take I to be the set containing any single vertex from D. Otherwise by the induction assumption, $V(G' - (D - v))$ has an independent set I' such that $(G' - (D - v)) - I'$ has no subgraphs isomorphic to B. If $D \neq B$, we can take $I = I'$. If $D = B$, and $v \in I'$, again let $I = I'$. If $D = B$ and $v \notin I'$ let $I = I' \cup \{w\}$ where w is any vertex of $D - v$. ∎

Lemma 22. *Let $F_1, F_2, \ldots, F_n$ be graphs each with at least one cyclic block. Then there exists a graph G of the form $F_i - I$, with $I \subseteq V(F_i)$ independent, such that for each $j \in \{1, 2, \ldots, n\}$, there is a cyclic block of F_j not contained in G.*

Proof. We work by induction on n. If $n = 1$, the result follows by the previous lemma. Suppose then that the result is true if we start with k graphs, $k > 0$.

Now suppose we are given $F_1, F_2, \ldots, F_{k+1}$ each with at least one cyclic block. By the induction assumption, there exists a graph $G' = F_i - I$ with $i \in \{1, 2, \ldots, k\}$ such that for each $j \in \{1, 2, \ldots, k\}$, there is a cyclic block of F_j which is not a subgraph of G'. If there is a cyclic block of F_{k+1} that is not a subgraph of G', then we can take G equal to G' and we are done.

Suppose that given any cyclic block of F_{k+1} there exists a block of G' that contains it. Let B be a maximal cyclic block of F_{k+1}. By the previous lemma, there exists an independent set J of vertices of F_{k+1} such that $F_{k+1} - J$ does not contain B. Let $G = F_{k+1} - J$. Note that every cyclic block of G is contained in a block of G'. If every cyclic block of F_1 is contained in G, then every cyclic block of F_1 is contained in G', which we know is not true. Hence there must be a cyclic block of F_1 not contained in G. Similarly for $F_2, F_3, \ldots, F_k$. ∎

Lemma 23. *Let $F_1, F_2, \ldots$ be graphs each with at least one cyclic block, such that the set of all the cyclic blocks making up $F_1, F_2, \ldots$ is finite. Then there exists a graph G of the form $F_i - I$, with $I \subseteq V(F_i)$ independent, such that for each $j \geq 1$, there is a cyclic block of F_j not contained in G.*

Proof. Let $\mathcal{S}$ denote the set of all the cyclic blocks making up the graphs $F_1, F_2, \ldots$. Then $\mathcal{S}$ is finite by assumption and for any F_i, the set of cyclic blocks of F_i is a subset of $\mathcal{S}$. Call a subset of $\mathcal{S}$ *determined* if it is the set of cyclic blocks of some F_i. Without loss of generality, suppose that every determined subset of $\mathcal{S}$ arises from exactly one of $F_1, F_2, \ldots, F_m$.

By the previous lemma, there exists a graph G of the form $F_i - I$, with $1 \leq i \leq m$ and with $I \subseteq V(F_i)$ independent, such that for each $j = 1, 2, \ldots, m$, there is a cyclic block of F_j not contained in G. Given any F_k with $k \geq 1$, F_k has the same determined subset of $\mathcal{S}$ as some F_j with $1 \leq j \leq m$, so there is a cyclic block of F_k not contained in G. ∎

Theorem 25 that follows is a generalisation of the result in Theorem 1 of the paper [3] by Nešetřil and Rödl, and the proof of Theorem 25 mimics theirs. We first copy some of their results here for completeness, with some changes in notation.

Nešetřil and Rödl used the 1966 Erdös and Hajnal ([2]) result which proved: For all positive integers $k \geq 2, l \geq 2, n \geq 1$ there exists a hypergraph $\Upsilon = \Upsilon(k, l, n) = (X, \mathcal{M})$ with the following properties:

(a) Υ is a k-uniform hypergraph.

(b) Υ does not contain cycles of length smaller than l.

(c) $\chi(\Upsilon) > n$.

If G and H are graphs and $r \geq 2$ is an integer, we write $H \xrightarrow{r} G$ if any partition of $V(H)$ into r parts has an induced copy of G in the subgraph of H induced by one of the parts. With our notation, Theorem 1 of [3] now reads as follows:

Theorem 24 [3]. *Let $\mathcal{P} \in \mathbb{M}^a$ so that $C(\mathcal{P})$ is a finite set of two connected graphs, and let $r \geq 2$ be an integer. Then for any $G \in \mathcal{P}$ there exists a graph $H \in \mathcal{P}$ such that $H \xrightarrow{r} G$.*

We can now generalise this theorem as follows:

Theorem 25. *Let $r \geq 2$ be an integer. Let $\mathcal{P} \in \mathbb{M}^a$, and suppose $C(\mathcal{P})$ is a set of graphs each with at least one cyclic block, such that the set of all*

the cyclic blocks forming the graphs in $\boldsymbol{C}(\mathcal{P})$ is finite. Then for each graph $G \in \mathcal{P}$ satisfying that for every $F \in \boldsymbol{C}(\mathcal{P})$ there exists a cyclic block of F which is not an induced subgraph of G, there exists a graph $H \in \mathcal{P}$ such that $H \xrightarrow{r} G$.

Proof. Suppose that $G = (V, E)$ satisfies the condition. The result follows easily if $|V| = 1$, so assume that $|V| > 1$. Let $l = \max\{|V(B)| + 1 : B$ is a block of a graph in $\boldsymbol{C}(\mathcal{P})\}$, and let $|V| = k$. Note that $l > 2$. Choose $\Upsilon(k, l, r) = (X, \mathcal{M})$ as in the Erdös-Hajnal result. For each $M \in \mathcal{M}$, let $f_M : V \to M$ be a fixed bijection. Define the graph $H = (X, K)$ such that $\{x, y\} \in K$ iff there exists $M \in \mathcal{M}$ and $\{a, b\} \in E$ such that $\{f_M(a), f_M(b)\} = \{x, y\}$. As $l > 2$, we have $|M \cap N| \leq 1$ whenever $M \neq N, M, N \in \mathcal{M}$, so that each $M \in \mathcal{M}$ is isomorphic to G.

We must show that this graph H is in $\mathcal{P}$. Let $F \in \boldsymbol{C}(\mathcal{P})$, and suppose that H contains an induced copy of F. Let B be a cyclic block of F which is not an induced subgraph of G. Since $(X, \mathcal{M})$ does not contain a cycle of length less than $|V(B)| + 1$, B must be contained in (an induced subgraph of) some $M \in \mathcal{M}$. But each M is isomorphic to G, and so we have a contradiction to the condition satisfied by G. Hence H cannot contain any graph from $\boldsymbol{C}(\mathcal{P})$ as an induced subgraph and so H must be in $\mathcal{P}$.

Finally $H \xrightarrow{r} G$ follows immediately from $\chi(\Upsilon) > r$. ∎

With this last theorem, we can now prove the main result:

Theorem 26. *If $\mathcal{P}$ is a reducible property in $\mathbb{M}^a$, then the set of cyclic blocks making up the graphs in $\boldsymbol{C}(\mathcal{P})$ is infinite and hence $\boldsymbol{C}(\mathcal{P})$ is infinite.*

Proof. Let $\mathcal{P} = \mathcal{R} \circ \mathcal{S}$ be a factorisation of $\mathcal{P}$, and suppose to the contrary that the set of cyclic blocks making up the graphs in $\boldsymbol{C}(\mathcal{P})$ is finite. Let $\boldsymbol{C}(\mathcal{P}) = \{F_1, F_2, \ldots\}$ (finite or infinite). Since every bipartite graph is in $\mathcal{P} = \mathcal{R} \circ \mathcal{S}$, none of the graphs in $\boldsymbol{C}(\mathcal{P})$ is bipartite. Each F_i thus contains an odd cycle and therefore has at least one cyclic block.

By Lemma 23 there exists a graph G of the form $F_i - I$, with $I \subseteq V(F_i)$ independent, such that for each $j, 1 \leq j \leq n$, there is a cyclic block of F_j not contained in G. Since G is an induced subgraph of F_i, G is in $\mathcal{P}$.

Now by Theorem 25, there exists $H \in \mathcal{P}$ such that $H \xrightarrow{2} G$. H has an $(\mathcal{R}, \mathcal{S})$ partition (V_R, V_S) in which neither part is empty. Since $H \xrightarrow{2} G$ we have $G \in \mathcal{R}$ or $G \in \mathcal{S}$. Suppose that $G \in \mathcal{R}$. $\bar{K}_{|I|} \in \mathcal{S}$, so $G + \bar{K}_{|I|} \in \mathcal{R} \circ \mathcal{S} = \mathcal{P}$. But this is impossible, since $F_i \subseteq G + \bar{K}_{|I|}$ and so F_i has an

$(\mathcal{R}, \mathcal{S})$-partition, a contradiction since F_i is not in $\mathcal{P} = \mathcal{R} \circ \mathcal{S}$. Hence the graphs in $\boldsymbol{C}(\mathcal{P})$ are formed from infinitely many cyclic blocks.   ■

We remark that the converse of this result is not true , for example, if $\boldsymbol{C}(\mathcal{P})$ is the set of all cycles, then $\mathcal{P}$ is the class of all forests, which is irreducible (since it does not contain all the bipartite graphs). It seems to be a very difficult problem to decide from an infinite $\boldsymbol{C}(\mathcal{P})$ whether $\mathcal{P}$ is reducible or irreducible.

3   Additive Hereditary Properties

A result analogous to Theorem 26 holds for any property $\mathcal{P}$ in $\mathbb{L}^a$ which is reducible in $\mathbb{L}^a$. We will use the notation $\boldsymbol{F}(\mathcal{P})$ to denote the set of minimal forbidden subgraphs of $\mathcal{P}$. We state the analogues of Theorems 25 and 26 here. Their proofs follow exactly as in the case of induced hereditary properties, replacing $\boldsymbol{C}(\mathcal{P})$ with $\boldsymbol{F}(\mathcal{P})$ and 'induced subgraph' with 'subgraph'.

Theorem 31. *Let $r \geq 2$ be an integer. Let $\mathcal{P} \in \mathbb{L}^a$, and suppose $\boldsymbol{F}(\mathcal{P})$ is a set of graphs each with at least one cyclic block, such that the set of all the cyclic blocks forming the graphs in $\boldsymbol{F}(\mathcal{P})$ is finite. Then for each graph $G \in \mathcal{P}$ satisfying that for every $F \in \boldsymbol{F}(\mathcal{P})$ there exists a cyclic block of F which is not a subgraph of G, there exists a graph $H \in \mathcal{P}$ such that $H \xrightarrow{r} G$.*

Theorem 32. *If $\mathcal{P}$ is a reducible property in $\mathbb{L}^a$, then the set of cyclic blocks making up the graphs in $\boldsymbol{F}(\mathcal{P})$ is infinite and hence $\boldsymbol{F}(\mathcal{P})$ is infinite.*

Note that if $\mathcal{P}$ is a reducible property in $\mathbb{L}^a$, then $\mathcal{P}$ is also a reducible property in $\mathbb{M}^a$ and so $\boldsymbol{C}(\mathcal{P})$ is infinite, by Theorem 26. From this fact we can immediately conclude that $\boldsymbol{F}(\mathcal{P})$ is infinite, since every graph in $\boldsymbol{C}(\mathcal{P})$ must contain an element of $\boldsymbol{F}(\mathcal{P})$ with the same order as a subgraph.

However if $\mathcal{P}$ is a property in $\mathbb{L}^a$ such that the set of cyclic blocks making up the graphs in $\boldsymbol{C}(\mathcal{P})$ is infinite, then it is not always true that the graphs in $\boldsymbol{F}(\mathcal{P})$ are made up of infinitely many cyclic blocks. For example, let $\mathcal{P}$ be the property defined by $\boldsymbol{F}(\mathcal{P}) = \{F_1, F_2, F_3, \ldots\}$, where F_i is the graph formed by attaching a copy of K_3 to a degree two vertex of a copy of $K_{2,1,1}$ by a path with i edges. Then only two cyclic blocks are found in the graphs of $\boldsymbol{F}(\mathcal{P})$. However for each $i \geq 1$, $\boldsymbol{C}(\mathcal{P})$ contains the graph formed from F_i by attaching each remaining degree two vertex of K_3 to a different one of the original degree three vertices of $K_{2,1,1}$ by an edge, and hence infinitely many cyclic blocks are found in the graphs of $\boldsymbol{C}(\mathcal{P})$.

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