

## ON $(k, l)$ -KERNELS OF SPECIAL SUPERDIGRAPHS OF $P_m$ AND $C_m$

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### Abstract

The concept of  $(k, l)$ -kernels of digraphs was introduced in [2]. Next, H. Galeana-Sanchez [?] proved a sufficient condition for a digraph to have a  $(k, l)$ -kernel. The result generalizes the well-known theorem of P. Duchet and it is formulated in terms of symmetric pairs of arcs. Our aim is to give necessary and sufficient conditions for digraphs without symmetric pairs of arcs to have a  $(k, l)$ -kernel. We restrict our attention to special superdigraphs of digraphs  $P_m$  and  $C_m$ .

**Keywords:** kernel, semikernel,  $(k, l)$ -kernel.

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## 1. Introduction

For general concepts we refer the reader to [?]. Let  $D$  denote a finite, directed graph without loops and multiple arcs (for short: a *digraph*), where  $V(D)$  is the set of vertices of  $D$  and  $A(D)$  is the set of arcs of  $D$ . We restrict our considerations to digraphs not having symmetric pairs of arcs. A *path* is a digraph  $P_m$  with  $V(P_m) = \{x_1, x_2, \dots, x_m\}$  and  $A(P_m) = \{x_i x_{i+1} : i = 1, \dots, m-1\}$  for  $m \geq 2$ . A circuit  $C_m$  is a digraph with  $V(C_m) = V(P_m)$  and  $A(C_m) = A(P_m) \cup \{x_m x_1\}$ , for  $m \geq 3$ . For simplicity,  $x_{m+i} = x_i$ , with  $1 \leq i \leq m$ . The cardinality of  $A(P_m)$  and  $A(C_m)$  we call the *length* of  $P_m$  and  $C_m$ , respectively. We denote by  $d_D(x, y)$

the length of the shortest path from  $x$  to  $y$  in  $D$ . This path is meant as a subdigraph of  $D$  isomorphic to  $P_m$ , where  $x_1 = x$  and  $x_m = y$ . For any  $X \subseteq V(D)$  and  $x \in V(D) \setminus X$  we put  $d_D(x, X) = \min_{y \in X} d_D(x, y)$ ,  $d_D(X, x) = \min_{y \in X} d_D(y, x)$  and  $N_D^l(X) = \{x \in V(D) \setminus X : d_D(x, X) > l\}$ . For the sake of clarity, we introduce the following notations. A *spanning superdigraph* of  $D$  is a digraph  $H$  such that  $V(H) = V(D)$  and  $A(H) \supset A(D)$ . If  $H$  is a spanning superdigraph of  $P_m$  ( $C_m$ ), then an arc  $a \in A(H) \setminus A(C_m)$  ( $a \in A(H) \setminus A(C_m)$ ) will be called a *chord* of  $D$  and a chord  $x_i x_{i+2}$  we will call a *short chord* of  $D$ . Two vertices  $x_i, x_j \in X \subset V(P_m) = V(C_m)$  with  $i < j$  are called *consecutive in  $X$*  if for every integer  $t$  with  $i < t < j$ , we have that  $x_t \in V(P_m) \setminus X$ . If  $i > j$ , then instead of  $j$  we take  $j + m$  and we define that  $x_i, x_j$  are consecutive in  $X$  as the above. Let  $k, l$  be fixed positive integers,  $k \geq 2$  and  $l \geq 1$ . A subset  $J \subseteq V(D)$  is called a  $(k, l)$ -kernel of  $D$  if

- (1) for each  $x, y \in J$  and  $x \neq y$ ,  $d_D(x, y) \geq k$  and
- (2) for each  $x \in V(D) \setminus J$  there exists  $y \in J$  that  $d_D(x, y) \leq l$ .

The concept of a  $(k, l)$ -kernel of a digraph was introduced in [?] and considered in [?] and [?]. It may be noted that for  $k = 2$  and  $l = 1$  we obtain the definition of a kernel of  $D$  in the sense of Berge [?]. If  $J$  satisfies the condition (1), then we say that  $J$  is *k-stable in  $D$* . Moreover, we assume that the subset including exactly one vertex is also *k-stable in  $D$* . We say that the vertex  $x$  is *l-dominated by  $J$  in  $D$*  or  $J$  *l-dominates  $x$  in  $D$*  or  $J$  is *l-dominating in  $D$* , when the condition (2) is fulfilled. A subset  $J \subseteq V(D)$  is a *strong  $(k, l)$ -kernel of  $D$*  if  $J$  is a  $(k, l)$ -kernel of  $D$  and

- (3) there exist  $x, y \in J$ ,  $x \neq y$  that  $d_D(x, y) = k$  and
- (4) there exists  $x \in V(D) \setminus J$  that  $d_D(x, J) = l$ .

Notice that a  $(k, l)$ -kernel consisting of exactly one vertex cannot be a strong  $(k, l)$ -kernel. A subset  $J \subset V(D)$  is a  $(k, l)$ -semikernel of  $D$  if  $J$  is *k-stable in  $D$*  and

- (5) for each  $x \in V(D) \setminus J$  for which  $d_D(J, x) \leq l$ , there must be  $d_D(x, J) \leq l$ .

It is clear that if  $J$  is a  $(k, l)$ -kernel of  $D$ , then  $J$  is a  $(k, l)$ -semikernel of  $D$ . For  $k = 2$  and  $l = 1$  we obtain the definition of semikernel [?].

All definitions are similar for undirected graphs, which are also considered.

## 2. The Existence of $(k, l)$ -Kernels in $P_m$ and its Spanning Superdigraph

For a fixed  $k \geq 2$  we can write an arbitrary positive integer number  $m \geq 2$  in the form  $m = nk + r$ , where  $n \geq 0$  and  $0 \leq r < k$ . By the way, if  $J$  is a  $(k, l)$ -kernel of  $P_m$ , then  $|J| \leq n + 1$ .

First, we give a necessary and sufficient condition for a digraph  $P_m$  to have a  $(k, l)$ -kernel. If  $n = 0$ , then  $P_m$  has a  $(k, l)$ -kernel if and only if  $r \leq l + 1$ . For  $n \geq 1$  we have the following result.

**Theorem 2.1.** *Let  $P_m$  be a digraph of order  $m = nk + r$  and  $n \geq 1$ . Then  $P_m$  has a  $(k, l)$ -kernel if and only if  $k \leq l + 1$ .*

**Proof.** Let  $k \leq l + 1$ . It is not difficult to observe that  $J = \{x_r, x_{r+k}, x_{r+2k}, \dots, x_{r+(n-1)k}, x_{r+nk=m}\}$  is a  $(k, l)$ -kernel of  $P_m$ . Indeed,  $J$  is  $k$ -stable and for every  $x \in V(P_m) \setminus J$  we have  $d_{P_m}(x, J) \leq k - 1 \leq l$ .

Now suppose on the contrary that  $P_m$  has a  $(k, l)$ -kernel  $J$ , but  $k > l + 1$ . Then for every two consecutive vertices  $x_i, x_j \in J$ ,  $d_{P_m}(x_i, x_j) \geq k > l + 1$ . Moreover,  $d_{P_m}(x_{i+1}, J) = d_{P_m}(x_{i+1}, x_j) \geq k - 1 > l$  and this means that  $x_{i+1}$  is not  $l$ -dominated by  $J$ . This contradicts the assumption that  $J$  is a  $(k, l)$ -kernel of  $P_m$  and completes the proof. ■

It is natural to ask whether adding a new arc (the opposite arcs are not taken into consideration) to  $P_m$  guarantees the existence of a  $(k, l)$ -kernel in an obtained spanning superdigraph, for  $k > l + 1$ . We shall calculate the smallest number of chords of a spanning superdigraph of  $P_m$  having a  $(k, l)$ -kernel for the case, when  $k > l + 1$ . In order to do it, we start with a simple assertion noting that throughout all sections we assume  $m = nk + r$ ,  $n \geq 1$  and  $0 \leq r < k$ .

**Lemma 2.2.** *Let  $D$  be a spanning superdigraph of  $P_m$  such that  $|A(D) \setminus A(P_m)| = 1$ . Then, for any  $X \subset V(D)$ ,  $|N_{P_m}^l(X) \setminus N_D^l(X)| \leq l$ .*

**Proof.** Let  $D$  be a spanning superdigraph of  $P_m$  having exactly one additional arc from  $A(D) \setminus A(P_m)$ . We extend the numbering of the vertices in the natural fashion assuming that the sequence  $(x_1, x_2, \dots, x_m)$  constitutes the path  $P_m$ . Suppose for an indirect proof that there exists a subset  $X \subset V(D)$  such that  $|N_{P_m}^l(X) \setminus N_D^l(X)| \geq l + 1$ . Certainly,  $N_D^l(X) \subset N_{P_m}^l(X)$ . For convenience, we put  $\eta = |N_{P_m}^l(X) \setminus N_D^l(X)|$ .

Further, let  $x_s x_t$  denote a unique arc belonging to the set  $A(D) \setminus A(P_m)$  with  $|s - t| \geq 2$ . Notice that  $x_s \in V(D) \setminus X$ . Otherwise, it would be  $N_{P_m}^l(X) = N_D^l(X)$ . Hence  $\eta = 0$  but this is a contradiction to the assumption that  $\eta \geq l + 1$ . Choose a vertex  $x_{u_0} \in N_{P_m}^l(X) \setminus N_D^l(X)$  such that  $d_{P_m}(x_{u_0}, X) = \max_{x_u \in N_{P_m}^l(X) \setminus N_D^l(X)} d_{P_m}(x_u, X)$ . It follows from the choice of  $x_{u_0}$  that if  $x_u \in N_{P_m}^l(X) \setminus N_D^l(X)$ , then  $u_0 \leq u \leq u_0 + \eta - 1$  and  $d_{P_m}(x_{u_0}, x_s) \geq \eta - 1$ . As it was noted  $x_{u_0} \in N_{P_m}^l(X) \setminus N_D^l(X)$ , so  $d_{P_m}(x_{u_0}, X) > l$  and  $d_D(x_{u_0}, X) \leq l$ . This means that the shortest path from  $x_{u_0}$  to the set  $X$  includes the arc  $x_s x_t$ . Therefore, we can conclude that  $d_D(x_{u_0}, X) = d_D(x_{u_0}, x_s) + d_D(x_s, x_t) + d_D(x_t, X) = d_{P_m}(x_{u_0}, x_s) + 1 + d_{P_m}(x_t, X) \geq (\eta - 1) + 1 + d_{P_m}(x_t, X) \geq \eta \geq l + 1$ . Finally we obtain that  $d_D(x_{u_0}, X) \leq l + 1$ , a contradiction.

Note that the Lemma ?? shows that adding exactly one arc to  $P_m$  creates superdigraph  $D$  such that the number  $s$  of  $l$ -dominated vertices by a fixed subset  $X \subset V(P_m)$  in  $D$  is more than the number  $p$  of  $l$ -dominated vertices by  $X$  in  $P_m$ . Moreover,  $s - p \leq l$ . This leads to the following corollary.

**Corollary 2.3.** *Let  $X \subseteq V(P_m)$ , such that  $|N_{P_m}^l(X)| = \eta > 0$ . Then every spanning superdigraph  $D$  of  $P_m$ , in which  $X$  is  $l$ -dominating, has to possess at least  $\lceil \frac{\eta}{l} \rceil$  additional arcs (i.e.,  $|A(D) \setminus A(P_m)| \geq \lceil \frac{\eta}{l} \rceil$ ), where  $\lceil p \rceil$  denotes the smallest integer greater than or equal to  $p$ .*

It may be noted that if  $X \subset V(P_m)$  and  $|X| = 1$ , then  $X$  can  $l$ -dominate at most  $l$  vertices of  $P_m$ . Moreover, if  $|X| = s$ , then  $X$  can  $l$ -dominate at most  $s \cdot l$  vertices of  $P_m$ . Now we discuss the case when  $k > l + 1$  with respect to the existence of a  $(k, l)$ -kernel in spanning superdigraph  $D$  of  $P_m$ . More precisely, we estimate a number of additional arcs which are needed for a superdigraph  $D$  having a  $(k, l)$ -kernel with  $k > l + 1$ .

**Theorem 2.4.** *Let  $D$  be a spanning superdigraph of  $P_m$ . If  $k > l + 1$  and  $D$  has a  $(k, l)$ -kernel, then  $|A(D) \setminus A(P_m)| \geq \lceil \frac{m-n-r}{l} \rceil - n$  for  $r \leq l + 1$  and  $|A(D) \setminus A(P_m)| \geq \lceil \frac{m-n-1}{l} \rceil - n - 1$  for  $r > l + 1$ .*

**Proof.** Let  $J$  be a  $(k, l)$ -kernel of  $D$ . Since  $J$  is  $k$ -stable in  $D$ , then it is  $k$ -stable in  $P_m$ , too. Moreover, from the assumption that  $k > l + 1$  we have that  $J$  is not a  $(k, l)$ -kernel of  $P_m$  (see Theorem ??). Thus  $J$  is not  $l$ -dominating in  $P_m$ . Then  $N_{P_m}^l(J) \neq \emptyset$ . We can present the set of vertices as a sum of disjoint subsets, namely  $V(P_m) = J \cup \{y \in V(P_m) \setminus J : d_{P_m}(y, J) \leq l\} \cup$

$N_{P_m}^l(J)$ . Hence if we take the cardinalities of these sets into consideration, we have the following equality:  $m = |J| + |\{y \in V(P_m) \setminus J : d_{P_m}(y, J) \leq l\}| + |N_{P_m}^l(J)|$ . Moreover,  $|\{y \in V(P_m) \setminus J : d_{P_m}(y, J) \leq l\}| \leq l|J|$ . Then  $m \leq |J| + l|J| + |N_{P_m}^l(J)|$ . As it was mentioned earlier,  $|J| \leq n + 1$ . This means that  $m \leq (n + 1)(l + 1) + |N_{P_m}^l(J)|$  i.e.,  $|N_{P_m}^l(J)| \geq m - (n + 1)(l + 1)$ . As a consequence  $|A(D) \setminus A(P_m)| \geq \left\lceil \frac{m - (n + 1)(l + 1)}{l} \right\rceil = \left\lceil \frac{m - n - 1}{l} \right\rceil - n - 1$  in view of Corollary ???. If  $r \leq l + 1$  we can give a better estimate. We shall show that in this case  $|N_{P_m}^l(J)| \geq m - n(l + 1) - r$ . Assume that  $|N_{P_m}^l(J)| < m - n(l + 1) - r$ . Combining the upper bound of  $m$  (given above) and the last inequality we deduce that  $m < (n + 1)(l + 1) + m - n(l + 1) - r = l + 1 - n(l + 1) = (1 - n)(l + 1)$ . If  $n = 0$ , then  $m = r < l + 1$  i.e.,  $J$  is a  $(k, l)$ -kernel of  $P_m$ , which contradicts the assumption. If  $n \geq 1$ , then  $m < 0$ , the next contradiction. Thus we conclude that  $|N_{P_m}^l(J)| \geq m - n(l + 1) - r$ . This means that  $|A(D) \setminus A(P_m)| \geq \left\lceil \frac{m - n(l + 1) - r}{l} \right\rceil = \left\lceil \frac{m - n - r}{l} \right\rceil - n$  in view of Corollary ?? and completes the proof. ■

### 3. Special Kinds of $(k, l)$ -Kernels in $C_m$ and its Superdigraphs

At the beginning, we prove the relationship between the existence of  $(k, l)$ -kernel and  $(k, l)$ -semikernel in  $C_m$ . We extend the numbering of the vertices in the natural fashion around the circuit  $C_m$  i.e., the sequence  $(x_1, x_2, \dots, x_m)$  constitutes the digraph  $C_m$ .

**Theorem 3.1.** *Let  $m \geq 3$ . Then  $C_m$  has a  $(k, l)$ -semikernel if and only if it has a  $(k, l)$ -kernel.*

**Proof.** Let  $J$  be a  $(k, l)$ -semikernel of  $C_m$ . To prove that  $J$  is a  $(k, l)$ -kernel of  $C_m$  it is enough to show that  $J$  is  $l$ -dominating in  $C_m$ . Let  $x_i, x_j \in J$  be any consecutive vertices in  $J$ . If  $i > j$ , then instead of  $j$  we take  $j + m$ . Since  $d_{C_m}(x_i, x_{i+1}) = 1 \leq l$ , then we have that  $d_{C_m}(x_{i+1}, x_j) \leq l$ . Hence  $J$  is  $l$ -dominating in  $C_m$ . As it was remarked in Introduction, each  $(k, l)$ -kernel of a digraph is a  $(k, l)$ -semikernel of the digraph which completes the proof. ■

Recall that  $m = nk + r$ ,  $n \geq 0$  and  $0 \leq r < k$ . It is not difficult to see that if  $J$  is a  $(k, l)$ -kernel of  $C_m$ , then  $|J| \leq n$ , for  $n \geq 1$  or  $|J| = 1$ , for  $n = 0$ .

Moreover, if  $n = 0$ , then  $C_m$  has a  $(k, l)$ -kernel  $J$  iff  $r \leq l + 2$ . If  $n \geq 1$ , then we have the following theorem.

**Theorem 3.2.** *Let  $C_m$  be given with  $m = nk + r$ ,  $n \geq 1$ . Then  $C_m$  has a  $(k, l)$ -kernel if and only if  $k \leq l + 1$  and  $r \leq n(l - k + 1)$ .*

**Proof.** I. Let  $k \leq l + 1$  and  $r \leq n(l - k + 1)$ . It is easy to observe that if  $r = 0$  (i.e.,  $m = nk$ ), then the subset  $J = \{x_1, x_{1+k}, x_{1+2k}, \dots, x_{1+(n-1)k}\}$  is a  $(k, l)$ -kernel of  $C_m$ .

Assuming that  $r > 0$  we shall prove that there exists an integer  $s$  such that  $0 \leq s \leq l - k + 1$  and  $m = n(k + s) + r_s$ , where  $0 \leq r_s < n$ . Assume that this is not true, or in other words for every  $s$  with  $0 \leq s \leq l - k + 1$  we have  $r_s > n$ . Taking  $s = l - k$  we have  $m = n(k + s) + r_s = nl + r_{l-k}$ . Since  $r_{l-k} > n$ , so  $m > n(l + 1)$ . But at the same time we have  $m = nk + r \leq nk + n(l - k + 1) = n(l + 1)$ , a contradiction.

Now, we shall show that the existence of a  $(k, l)$ -kernel in  $C_m$  is assured. For  $r_s = 0$  the set  $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \dots, x_{1+(n-1)(k+s)}\}$  is a  $(k, l)$ -kernel of  $C_m$ . For  $r_s > 0$ , we put  $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \dots, x_{1+(n-r_s)(k+s)}, x_{1+(n-r_s+1)(k+s)+1}, x_{1+(n-r_s+2)(k+s)+2}, \dots, x_{1+(n-r_s+(r_s-2))(k+s)+r_s-2}, x_{1+(n-1)(k+s)+r_s-1}\}$ . In order to show that  $J$  is  $k$ -stable in  $C_m$  it suffices to observe that  $d_{C_m}(x_{1+(n-1)(k+s)}, x_1) = m + 1 - [1 + (n - 1)(k + s) + r_s - 1] = k + s + 1 > k$ . We have also for every  $x \in V(C_m) \setminus J$  that  $d_{C_m}(x, J) \leq k + s < l + 1$ , what proves that  $J$  is  $l$ -dominating in  $C_m$ . Consequently,  $J$  is a  $(k, l)$ -kernel of  $C_m$  and the first part of the theorem is proved.

II. Assume that  $J$  is a  $(k, l)$ -kernel of  $C_m$ , but  $k > l + 1$  or  $r > n(l - k + 1)$ . If  $|J| = 1$ , then it can be verified that  $n = 1$  and  $J = \{x_i\}$ , where  $1 \leq i \leq m$ . As a consequence  $d_{C_m}(x_{i+1}, J) = d_{C_m}(x_{i+1}, x_i) = m - 1 = k + r - 1$ . Further, from the assumption that  $k > l + 1$  or  $r > n(l - k + 1)$  it follows that  $k + r - 1 > l$ . This means that the vertex  $x_{i+1}$  is not  $l$ -dominated by  $J$  and contradicts our assumption that  $J$  is a  $(k, l)$ -kernel of  $C_m$ . Now we consider the case when  $|J| \geq 2$ . Let  $x_i, x_j \in J$  be two consecutive vertices in  $J$ . If  $k > l + 1$ , then  $d_{C_m}(x_{i+1}, x_j) = d_{C_m}(x_i, x_j) - 1 \geq k - 1 > l$ . This means that  $x_{i+1}$  is not  $l$ -dominated by  $J$ , a contradiction to the assumption that  $J$  is a  $(k, l)$ -kernel of  $C_m$ . If  $r > n(l - k + 1)$ , then  $m = nk + r > n(l + 1)$ . From this and in fact that  $|J| \leq n$ , the existence of two consecutive vertices in  $J$ , say  $x_i, x_j$  such that  $d_{C_m}(x_i, x_j) > l + 1$  is assured. Hence  $d_{C_m}(x_{i+1}, x_j) = d_{C_m}(x_i, x_j) - 1 > l$ . This means that  $x_{i+1}$  is not  $l$ -dominated by  $J$  i.e.,  $J$  is not a  $(k, l)$ -kernel of  $C_m$ . This contradiction completes the proof of the theorem.  $\blacksquare$

Certainly, if  $n \leq 1$ , then each  $k$ -stable set of  $C_m$  contains exactly one vertex. Therefore, we conclude that  $C_m$  does not have a strong  $(k, l)$ -kernel, since the condition (3) is not satisfied. Now, we give a necessary and sufficient condition for  $C_m$  to have a strong  $(k, l)$ -kernel.

**Theorem 3.3.** *The digraph  $C_m$  possesses a strong  $(k, l)$ -kernel if and only if:*

- (6)  $m - k - l - 1 = 0$  or
- (7)  $m - k - l - 1 \geq k$  and  $C_{m-k-l-1}$  has a  $(k, l)$ -kernel.

**Proof.** I. Let  $J$  be a strong  $(k, l)$ -kernel of  $C_m$ . This implies that it must be  $k \leq l + 1$ , by Theorem ???. By the way, it is easy to observe that  $m - k - l - 1 \geq 0$ . Suppose on the contrary that  $m - k - l - 1 < 0$ . This is equivalent to  $m < k + l + 1 \leq 2k$ , since  $k \leq l + 1$ . In conclusion there must be  $|J| = 1$ , which is impossible by the assumption that  $J$  is a strong  $(k, l)$ -kernel of  $C_m$ . Finally, we state  $m - k - l - 1 \geq 0$ . Next, assume on the contrary that both conditions (6) and (7) do not hold simultaneously. In other words (by the condition  $m - k - l - 1 \geq 0$ ) there must hold: (a)  $0 < m - k - l - 1 < k$  or (b)  $m - k - l - 1 > 0$  and  $C_{m-k-l-1}$  has no  $(k, l)$ -kernel. Suppose that the condition (a) holds. Since  $J$  is a strong  $(k, l)$ -kernel of  $C_m$ , then there exist  $x_q, x_p \in J$  and  $x_s \in V(C_m) \setminus J$  such that  $d_{C_m}(x_q, x_p) = k$  and  $d_{C_m}(x_s, J) = l$ . Without loss of generality, let  $q < p$  (if  $q > p$ , then take  $p+m$  instead of  $p$ ). If  $q < s < p$ , then  $s = q+1$  and  $d_{C_m}(x_q, x_p) = l+1$ , hence  $k = l + 1$ . In conclusion, the condition (a) is equivalent to the expression  $0 < m - 2k < k$ . This means that  $m = 2k + r$ , where  $r > 0$ . On the other hand, since  $C_m$  has a  $(k, l)$ -kernel, then  $r \leq n(l - k + 1)$  in view of Theorem ??. Therefore, putting  $k = l + 1$  we have  $r \leq 0$ , contrary to the conclusion that  $r > 0$ . If  $s < q$  or  $s > p$ , there exists a vertex  $x_t \in J$ , such that  $t \neq q$  and  $t \neq p$ . Figure 1 illustrates the positions of the vertex  $x_t$  with respect to the vertex  $x_s$ .

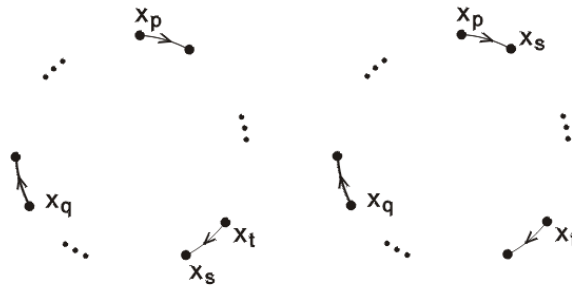


Figure 1

Otherwise (i.e.,  $J = \{x_q, x_p\}$ ), we would have  $m = k + l + 1$  or equivalently  $m - k - l - 1 = 0$ , which is impossible by (a). Thus,  $t \neq q$  and  $t \neq p$ . This means that  $d_{C_m}(x_p, x_q) = d_{C_m}(x_p, x_t) + d_{C_m}(x_t, x_s) + d_{C_m}(x_s, x_q)$  or  $d_{C_m}(x_p, x_q) = d_{C_m}(x_p, x_s) + d_{C_m}(x_s, x_t) + d_{C_m}(x_t, x_q)$  (see Figure 1). As it was noted  $x_q, x_p, x_t \in J$ , where  $J$  is  $k$ -stable and  $d_{C_m}(x_s, J) = l$ , then  $d_{C_m}(x_p, x_q) \geq k + l + 1$ . Using the last inequality we can write that  $m = d_{C_m}(x_p, x_q) + d_{C_m}(x_q, x_p) \geq k + (k + l + 1) = 2k + l + 1$ . Thus  $m - k - l - 1 \geq k$ , which is a contradiction to (a). Assume that the condition (b) holds. Since  $m > k + l + 1$ , then  $|J| \geq 3$ . Otherwise, (i.e.,  $|J| \leq 2$ ) the subset  $J$  could not be a strong  $(k, l)$ -kernel of  $C_m$ . Finally  $|J| \geq 3$ . Therefore, we may assume without loss of generality that  $x_{m-k-l-1}, x_{m-k}, x_m \in J$  (see Figure 2).

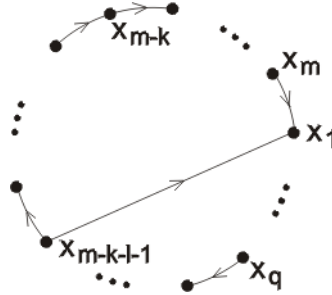


Figure 2

Create a spanning superdigraph  $D$  of  $C_m$  adding a new arc  $x_{m-k-l-1}x_1$  to  $C_m$ . Thus a subdigraph  $H$  of  $D$  induced by the set  $\{x_1, x_2, \dots, x_{m-k-l-1}\}$  is isomorphic to  $C_{m-k-l-1}$ . Then  $H$  has no  $(k, l)$ -kernel either. We define  $J_0 = J \setminus \{x_{m-k}, x_m\}$ . Since  $J$  is a  $(k, l)$ -kernel of  $C_m$ , then for  $1 \leq s \leq m - k - l - 1$  we have  $d_{C_m}(x_s, J) \leq l$ . Moreover,  $d_{C_m}(x_s, \{x_{m-k}, x_m\}) > l$ . This means that  $d_H(x_s, J_0) = d_{C_m}(x_s, J) \leq l$ . Hence  $J_0$  is  $l$ -dominating in  $H$ . Now we show that  $J_0$  is  $k$ -stable in  $H$ . Choose a vertex  $x_q \in J$  that  $x_m, x_q$  are consecutive in  $J$  (of course  $q < m$ ). In order to show that  $J_0$  is  $k$ -stable in  $H$ , it is enough to observe that  $d_H(x_{m-k-l-1}, x_q) \geq k$ . Indeed,  $d_H(x_{m-k-l-1}, x_q) = q = d_{C_m}(x_m, x_q) \geq k$ , since  $x_m, x_q \in J$ . Thus  $J_0$  is a  $(k, l)$ -kernel of  $H$ , what is a required contradiction and proves the first part of the theorem.

II. Let  $m - k - l - 1 = 0$ . Thus  $J = \{x_1, x_{1+k}\}$  is a strong  $(k, l)$ -kernel of  $C_m$ . Indeed,  $d_{C_m}(x_1, x_{1+k}) = k$  and  $d_{C_m}(x_{1+k}, x_1) = m + 1 - (1 + k) = m - k = l + 1$ , hence  $d_{C_m}(x_{2+k}, x_1) = l$ . Now let  $m - k - l - 1 \geq k$  and



$C_{m-k-l-1}$  has a  $(k, l)$ -kernel. Let a subdigraph  $H$  be defined in the same way as in Part I of the proof. Then  $H$  has a  $(k, l)$ -kernel, too. We denote it by  $J_0$  and assume without loss of generality that  $x_1 \in J_0$ . We show that  $J = J_0 \cup \{x_{m-k-l}, x_{m-l}\}$  is a strong  $(k, l)$ -kernel of  $C_m$ . Observe that because of the structure of  $C_m$  we have  $d_{C_m}(x_{m-k-l}, x_{m-l}) = k$  and  $d_{C_m}(x_{m-l}, x_1) = l + 1$ . Thus  $d_{C_m}(x_{m-l+1}, x_1) = l$ . This means that if  $J$  is a  $(k, l)$ -kernel of  $C_m$ , then it also is a strong  $(k, l)$ -kernel of  $C_m$ . If  $|J_0| = 1$  (i.e.,  $J_0 = \{x_1\}$ ), then  $d_{C_m}(x_1, x_{m-k-l}) = m - k - l - 1 \geq k$  and  $J = \{x_1, x_{m-k-l}, x_{m-l}\}$  is a  $(k, l)$ -kernel of  $C_m$ . If  $|J_0| > 1$ , then there exists  $x_q \in J_0$ , such that  $x_q, x_1$  are consecutive in  $J_0$ . Since  $d_H(x_q, x_1) = m - k - l - q \geq k$ , then  $d_{C_m}(x_q, x_{m-k-l}) = m - k - l - q \geq k$ . Thus  $J$  is  $k$ -stable and  $l$ -dominating in  $C_m$  i.e.,  $J$  is a  $(k, l)$ -kernel of  $C_m$  and this completes the proof of the theorem. ■

Proceeding by the same argument as for  $P_m$  in the proof of Lemma ?? and Corollary ?? we state two assertions with respect to  $C_m$ .

**Theorem 3.4.** *Let  $D$  be a spanning superdigraph of  $C_m$  including only one chord and  $X \subset V(D)$ . Then  $|N_{C_m}^l(X) \setminus N_D^l(X)| \leq l$ .*

**Corollary 3.5.** *Let  $X \subset V(C_m)$ , where  $|N_{C_m}^l(X)| = \eta > 0$ . Then every spanning superdigraph  $D$  of  $C_m$ , in which  $X$  is  $l$ -dominating, has at least  $\lceil \frac{\eta}{l} \rceil$  additional arcs (i.e.,  $|A(D) \setminus A(C_m)| \geq \lceil \frac{\eta}{l} \rceil$ ).*

Let a set  $J \subset V(C_m)$  be such that  $|J| = s$ . It is easy to observe that if  $J$  is  $k$ -stable in  $C_m$ , but not  $l$ -dominating in  $C_m$ , then  $|N_{C_m}^l(J)| \geq m - s(l + 1)$ . In that case in view of Theorem ??, we can formulate the following corollary.

**Corollary 3.6.** *Let  $C_m$  be such that it does not have a  $(k, l)$ -kernel and  $D$  be a spanning superdigraph of  $C_m$ . If  $J \subset V(D)$  is a  $(k, l)$ -kernel of  $D$ , with  $|J| = s$ , then  $D$  has at least  $\lceil \frac{m-s}{l} \rceil - s$  chords.*

If  $s$  is an integer such that  $1 \leq s \leq n$ , it is clear that the expression  $\lceil \frac{m-s}{l} \rceil - s$  has the smallest value for  $s = n$ . This implies the next corollary.

**Corollary 3.7.** *Let  $D$  be a spanning superdigraph of  $C_m$ . If  $D$  has a  $(k, l)$ -kernel, then  $C_m$  also has a  $(k, l)$ -kernel or  $D$  possesses at least  $\lceil \frac{m-n}{l} \rceil - n$  chords.*

**Lemma 3.8.** *If  $J$  is a  $(k, l)$ -kernel of spanning superdigraph  $D$  of  $C_m$ , then for every two consecutive vertices  $x, y$  in  $J$  we have  $d_{C_m}(x, y) \leq 2l + 1$ .*

**Proof.** Suppose to the contrary that there exist two consecutive vertices in  $J$ , say  $x_i, x_j$  such that  $d_{C_m}(x_i, x_j) > 2l + 1$ . As a consequence  $d_{C_m}(x_{i+1}, x_j) > 2l$ . Let us remark that the existence of short chords in  $D$  leads to inequality  $d_D(x_{i+1}, x_j) \geq \frac{1}{2}d_{C_m}(x_{i+1}, x_j)$ . Combining the above facts we deduce that  $d_D(x_{i+1}, x_j) > l$ . But this contradicts the assumption that  $J$  is  $l$ -dominating in  $D$ , hence the lemma is proved. ■

In what follows  $D$  will be a spanning superdigraph of  $C_m$  containing only short chords, where  $m = nk + r$  with  $0 \leq r < k$ .

Recall that if  $n = 0$  and  $r > l + 1$ , then  $C_{m=r}$  has no  $(k, l)$ -kernel. It is easy to observe that if additionally  $r \leq 2l + 1$ , then every spanning superdigraph  $D$  of  $C_m$  having a  $(k, l)$ -kernel has at least  $r - l - 1$  short chords. For  $n \geq 1$  we state the next assertion.

**Lemma 3.9.** *If  $C_m$  contains no  $(k, l)$ -kernel, then every spanning superdigraph  $D$  of  $C_m$  having a  $(k, l)$ -kernel for  $k \geq 2$ ,  $l \geq 1$  and  $n \geq 1$  has at least  $m - n(l + 1)$  chords.*

**Proof.** Let  $J$  be a  $(k, l)$ -kernel of  $D$ . Since  $C_m$  has no  $(k, l)$ -kernel, hence  $r > n(l - k + 1)$  or  $k > l + 1$  in view of Theorem ???. This means that  $m = nk + r > n(l + 1)$ . Let  $|J| = s$ . As it was remarked, we deduce that at least  $m - s(l + 1)$  vertices are not  $l$ -dominated by  $J$  in  $C_m$ . Assume that  $s \geq 2$ , hence there exist two consecutive vertices in  $J$ , say  $x_i, x_j$ , with  $i < j$  and  $d_{C_m}(x_i, x_j) > l + 1$ . Then it follows easily from the above that  $N = \{x_{i+1}, x_{i+2}, \dots, x_{j-l-1}\} \subseteq N_{C_m}^l(J)$ . Let  $\eta$  denote the number of short chords of  $D$ , whose endpoints are vertices  $x_t$ , where  $i < t \leq j$ . We shall prove that  $\eta \geq |N| = j - i - l - 1$ . Assume this cannot occur i.e.,  $\eta < j - i - l - 1$ . Since  $d_{C_m}(x_{i+1}, x_j) = j - i - 1$ , hence  $d_D(x_{i+1}, x_j) \geq \eta + (j - i - 1 - 2\eta) = j - i - 1 - \eta > l$ . This means that  $x_{i+1}$  is not  $l$ -dominated by  $x_j$  in  $D$ . Hence  $x_{i+1}$  cannot be dominated by  $J$  in  $D$ , contradicting the assumption that  $J$  is a  $(k, l)$ -kernel of  $D$ . This contradiction proves that  $\eta \geq j - i - l - 1$ . Taking all vertices not  $l$ -dominating by  $J$  in  $C_m$  into consideration, we get that  $D$  has at least  $|N_{C_m}^l(J)|$  chords. In case when  $s = 1$  we take  $m + i$  instead of  $j$  and proceed as above.

If  $s$  is an integer such that  $1 \leq s \leq n$ , it is clear that the expression  $m - s(l + 1)$  achieves the smallest value for  $s = n$ . This completes the proof. ■

If  $k > l + 1$ , then any superdigraph  $D$  of  $C_m$  cannot have a  $(k, l)$ -kernel of cardinality more than one. Indeed, because of  $k > l + 1$  every  $k$ -stable subset

of  $C_m$  is not  $l$ -dominating in view of Theorem ???. Short chords of  $D$  can cause that arbitrary  $k$ -stable set of  $C_m$  will be  $l$ -dominating in  $D$  but not  $k$ -stable in  $D$ . Moreover, taking the condition  $k > l+1$  into consideration there exists a spanning superdigraph  $D$  of  $C_m$  having a  $(k, l)$ -kernel if  $m \leq 2l+1$ .

**Theorem 3.10.** *If  $C_m$  does not have a  $(k, l)$ -kernel with  $k \leq l+1$  and  $r \leq n(2l-k+1)$ , then there exists a spanning superdigraph  $D$  of  $C_m$  having a  $(k, l)$ -kernel.*

**Proof.** Since  $C_m$  does not have a  $(k, l)$ -kernel and  $k \leq l+1$ , then  $r > n(l-k+1)$  see Theorem ???. Moreover,  $m = nk + r > n(l+1)$ . Now, we shall show that there exists an integer  $p > 0$  such that  $m = n(k+p) + r_p$ , where  $0 \leq r_p < n$  and  $p \geq l-k+1$ . On the contrary, let  $p \leq l-k$ . Hence  $m = n(k+p) + r_p \leq n(k+l-k) + r_p = nl + r_p < n(l+1)$ . On the other hand we have  $m = nk + r > nk + n(l-k+1) = n(l+1)$ , a contradiction. Notice that if  $r_p = 0$ , then  $p > l-k+1$  (if  $r_p = 0$  and  $p = l-k+1$ , then  $m = n(k+p) + r_p = n(l+1)$ , contrary to  $m > n(l+1)$ ).

For  $r_p = 0$  (i.e.,  $m = n(k+p)$ ), the subset  $J = \{x_1, x_{1+(k+p)}, x_{1+2(k+p)}, \dots, x_{1+(n-1)(k+p)}\}$  is  $k$ -stable in  $C_m$ . In order to show it, it suffices to observe that  $d_{C_m}(x_{1+(n-1)(k+p)}, x_1) = m + 1 - [1 + (n-1)(k+p)] = k+p \geq k$ . Let  $N_j = \{x_{2+j(k+p)}, x_{3+j(k+p)}, x_{4+j(k+p)}, \dots, x_{k+p-l+j(k+p)}\}$ , where  $0 \leq j \leq n-1$ . It is clear that  $2+j(k+p) \leq k+p-l+j(k+p)$  owing to  $p > l-k+1$ . We can observe that for every  $x \in N_j$  we have  $d_{C_m}(x, J) = d_{C_m}(x, x_{1+(j+1)(k+p)}) \geq d_{C_m}(x_{k+p-l+j(k+p)}, x_{1+(j+1)(k+p)}) = l+1$ . This means that no vertex from  $N_j$  is  $l$ -dominated by stable set  $J$ . Moreover, it is not difficult to see that  $\bigcup_{j=0}^{n-1} N_j = N_{C_m}^l(J)$ . Let  $D$  be a spanning superdigraph of  $C_m$  with  $A(D) = A(C_m) \cup A_0$ , where  $A_0 = \{a_{i,j} : 1 \leq i \leq k+p-l-1 \wedge 0 \leq j \leq n-1\}$  and  $a_{i,j} = (x_{2i+j(k+p)}, x_{2i+2+j(k+p)})$ . We can show that the indices of all endpoints  $x_t$  of chords  $a_{i,j}$  meet the condition  $1+j(k+p) < t \leq 1+(j+1)(k+p)$  for each  $j$ . In order to show it, it suffices to observe that  $a_{1,j} = (x_{2+j(k+p)}, x_{4+j(k+p)})$  and  $a_{k+p-l-1,j} = (x_{2(k+p-l-1)+j(k+p)}, x_{2+2(k+p-l-1)+j(k+p)})$  have endpoints whose indices satisfy the condition mentioned. Hence for every  $x \in N_j$  we have  $d_D(x, J) \leq d_D(x_{2+j(k+p)}, J) = d_D(x_{2+j(k+p)}, x_{1+(j+1)(k+p)}) = d_D(x_{2+j(k+p)}, x_{2+2(k+p-l-1)+j(k+p)}) + d_D(x_{2+2(k+p-l-1)+j(k+p)}, x_{1+(j+1)(k+p)}) = \frac{2+2(k+p-l-1)+j(k+p)-[2+j(k+p)]}{2} + 1 + (j+1)(k+p) - [2+2(k+p-l-1)+j(k+p)] = (k+p-l-1) + (1-k-p+2l) = l$  (see Figure 3).

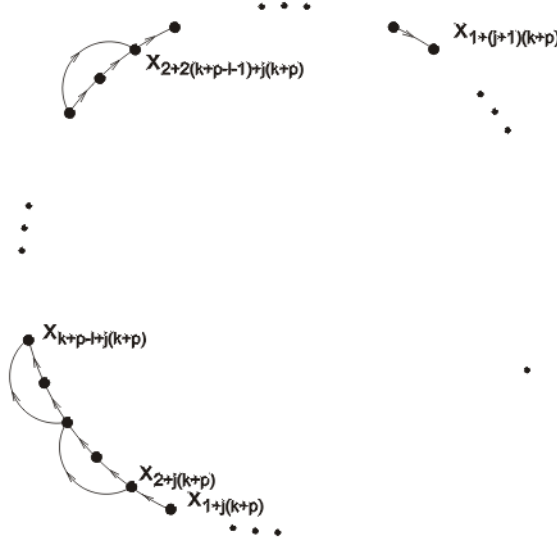


Figure 3

This means that all  $x \in N_{C_m}^l(J)$  are  $l$ -dominated by  $k$ -stable set  $J$ , so  $J$  is a  $(k, l)$ -kernel of  $D$ . Notice that  $|A_0| = (k + p - l - 1)n = n(k + p) - n(l + 1) = m - n(l + 1)$ . Hence  $D$  (in view of Lemma ??) is a spanning superdigraph of  $C_m$  with the minimum number of short chords. For  $r_p > 0$  the subset  $J = \{x_1, x_{1+(k+p)}, \dots, x_{1+(n-r_p)(k+p)}, x_{1+(n-r_p+1)(k+p)+1}, x_{1+(n-r_p+2)(k+p)+2}, \dots, x_{1+(n-1)(k+p)+r_p-1}\}$  is  $k$ -stable in  $C_m$ .

Put  $M_j = N_j \cup \{x_{k+p-l+1}\}$ , for  $n - r_p \leq j \leq n - 1$ . If  $p = l - k + 1$ , then  $N_j = \emptyset$  for  $0 \leq j \leq n - r_p - 1$  and  $M_j \neq \emptyset$  for  $n - r_p \leq j \leq n - 1$ . It is easy to observe that  $\bigcup_{j=0}^{n-r_p-1} N_j \cup \bigcup_{j=n-r_p}^{n-1} M_j = N_{C_m}^l(J)$ . Similarly, as for  $r_p = 0$  we can show that for every  $x \in N_{C_m}^l(J)$  we have  $d_{C_m}(x, J) \geq l + 1$ . This means that no vertex from  $N_{C_m}^l(J)$  is  $l$ -dominated by stable set  $J$ . Let  $D$  be a spanning superdigraph of  $C_m$  with  $A(D) = A(C_m) \cup A_1$ , where  $A_1 = \{a_{i,j} : (1 \leq i \leq k + p - l - 1 \wedge 0 \leq j \leq n - r_p - 1) \text{ or } (1 \leq i \leq k + p - l \wedge n - r_p \leq j \leq n - 1)\}$ .

It is not difficult to see that for  $0 \leq j \leq n - 1$  all endpoints  $x_t$  of chords  $a_{i,j}$  meet the condition  $1 + j(k + p) < t \leq 1 + (j + 1)(k + p)$ . It is easy to calculate (similarly as for  $r_p = 0$ ) that for every  $x \in N_{C_m}^l(J)$  we have  $d_D(x, J) \leq l$ . This means that all  $x \in N_{C_m}^l(J)$  are  $l$ -dominated

by stable set  $J$ , so  $J$  is a  $(k, l)$ -kernel of  $D$ . At the same time  $|A_1| = (k + p - l - 1)(n - r_p) + (k + p - l)r_p = n(k + p) + r_p - n(l + 1) = m - n(l + 1)$ . This means that  $D$  is a spanning superdigraph of  $C_m$  with a minimum number of short chords in view of Lemma ??.

#### 4. On $(k, l)$ -Kernels of Graphs

In this section, the notation  $C_m$  means an directed graph defined analogously as the circuit  $C_m$ . In this case  $d_{C_m}(x, y) = d_{C_m}(y, x)$ .

Recall that  $m = nk + r$ ,  $n \geq 0$  and  $0 \leq r < k$ . It is not difficult to observe that if  $n = 0$ , then  $C_m$  has a  $(k, l)$ -kernel iff  $r \leq 2l + 2$ . If  $n \geq 1$ , then  $C_m$  has no strong  $(k, l)$ -kernel.

**Theorem 4.1.** *Let  $n \geq 2$ .  $C_m$  has a strong  $(k, l)$ -kernel if and only if at least one of the following conditions is fulfilled:*

- (8)  $m - k - 2l = 0$ ,
- (9)  $m - k - 2l - 1 = 0$ ,
- (10)  $m - k - 2l \geq k$  and  $C_{m-k-2l}$  has a  $(k, l)$ -kernel,
- (11)  $m - k - 2l - 1 \geq k$  and  $C_{m-k-2l-1}$  has a  $(k, l)$ -kernel.

**Proof.** The sufficient condition of existence of a strong  $(k, l)$ -kernel we prove on the contrary using the method from Part I of the proof of Theorem ?? and considering two conditions:

- (a)  $0 < m - k - 2l < k$  or  $m - k - 2l > 0$  and  $C_{m-k-2l}$  has no  $(k, l)$ -kernel,
- (b)  $0 < m - k - 2l - 1 < k$  or  $m - k - 2l - 1 > 0$  and  $C_{m-k-2l-1}$  has no  $(k, l)$ -kernel.

Proceeding as in the second part of the proof of Theorem ??, we can prove the necessary condition of the theorem.

Theorem ?? is a generalization of the result announced in [?] and concerning a strong  $(k, k - 2)$ -kernel of  $C_m$ .

Noting that a symbol  $\lfloor p \rfloor$  denotes the greatest integer less than or equal to  $p$ , we prove the following.

**Theorem 4.2.** *The cycle  $C_m$ , where  $m = nk + r$  and  $n \geq 1$ , has a  $(k, l)$ -kernel if and only if  $k \leq 2l + 1$  and  $r \leq n(2l - k + 1)$ .*

**Proof.** I. Let  $k \leq 2l + 1$  and  $r \leq n(2l - k + 1)$ . At first, notice that if  $x_i, x_j \in J$  are consecutive in  $J$ , then for each integer  $t$  such that  $i < t < j$  we have  $\max_t d_{C_m}(x_t, J) = \left\lfloor \frac{j-i}{2} \right\rfloor$ . It is easy to observe that if  $r = 0$  (i.e.,  $m = nk$ ), then the set  $J = \{x_1, x_{1+k}, x_{1+2k}, \dots, x_{1+(n-1)k}\}$  is a  $(k, l)$ -kernel of  $C_m$ . Indeed, for every two vertices  $x_{1+(i-1)k}, x_{1+ik}$  consecutive in  $J$  we have  $d_{C_m}(x_{1+(i-1)k}, x_{1+ik}) = k$ , where  $i = 1, \dots, n-1$  and  $d_{C_m}(x_{1+(n-1)k}, x_1) = m + 1 - [1 + (n-1)k] = k$ , which means that  $J$  is  $k$ -stable. We have also for each  $x \in V(D) \setminus J$  that  $d_{C_m}(x, J) \leq \frac{k}{2} \leq \frac{2l+1}{2} = l + \frac{1}{2}$ . Since  $d_{C_m}(x, J)$  is an integer number, then finally  $d_{C_m}(x, J) \leq l$ .

Now let  $r > 0$ . We state that there exists an integer  $s$  such that  $0 \leq s \leq 2l - k + 1$  and  $m = n(k + s) + r_s$ , where  $0 \leq r_s < n$ . Suppose on the contrary that for every  $s$ ,  $0 \leq s \leq 2l - k + 1$  we have  $r_s > n$ . Let  $s = 2l - k$  and  $m = n(k + s) + r_s = 2nl + r_{l-k}$ . Since  $r_{l-k} > n$ , then  $m > n(2l + 1)$ . On the other hand, we have  $m = nk + r \leq nk + n(2l - k + 1) = n(2l + 1)$ , a contradiction.

It is not difficult to observe that for  $r_s = 0$  the subset  $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \dots, x_{1+(n-1)(k+s)}\}$  is a  $(k, l)$ -kernel of  $C_m$ . If  $r_s > 0$ , then  $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \dots, x_{1+(n-r_s)(k+s)}, x_{1+(n-r_s+1)(k+s)+1}, x_{1+(n-r_s+2)(k+s)+2}, \dots, x_{1+(n-r_s+2)(k+s)+r_s-2}, x_{1+(n-1)(k+s)+r_s-1}\}$  is a  $(k, l)$ -kernel of  $C_m$ . Indeed,  $d_{C_m}(x_{1+(n-1)(k+s)}, x_1) = m + 1 - [1 + (n-1)(k+s) + r_s - 1] = k + s + 1 > k$ . We have also for every  $x \in V(D) \setminus J$ ,  $d_{C_m}(x, J) \leq \frac{k+s}{2} < \frac{2l+2}{2} = l + 1$ , where the existence of such an integer  $s$  is assured.

II. Assume that  $C_m$  has a  $(k, l)$ -kernel  $J$ , but  $k > 2l + 1$  or  $r > n(2l - k + 1)$ . If  $|J| = 1$ , then  $n = 1$  and  $J = \{x_i\}$ , where  $1 \leq i \leq m$ . Moreover, if  $m = k + r$  is an even number, then  $d_{C_m}(x_{i+\frac{m}{2}}, J) = d_{C_m}(x_{i+\frac{m}{2}}, x_i) = \frac{m}{2} = \frac{k+r}{2}$ . From the assumption that  $k > 2l + 1$  or  $r > n(2l - k + 1)$  we have that  $\frac{k+r}{2} > l + \frac{1}{2} > l$ . Thus the vertex  $x_{i+\frac{m}{2}}$  is not  $l$ -dominated by  $J$ , which contradicts the assumption that  $J$  is a  $(k, l)$ -kernel of  $C_m$ . If  $m$  is odd, then  $d_{C_m}(x_{i+\frac{m-1}{2}}, J) = d_{C_m}(x_{i+\frac{m-1}{2}}, x_i) = \frac{m-1}{2} = \frac{k+r-1}{2} > l$  and the vertex  $x_{i+\frac{m-1}{2}}$  is not  $l$ -dominated by  $J$ , a contradiction with the assumption. It remains to consider the case when  $|J| \geq 2$ . Let  $x_i, x_j \in J$  be two consecutive vertices in  $J$ . If  $k > 2l + 1$  and  $j - i$  is even, then it follows from the structure of  $C_m$  that  $d_{C_m}(x_{\frac{j+i}{2}}, J) = d_{C_m}(x_{\frac{j+i}{2}}, x_j) = d_{C_m}(x_i, x_{\frac{j+i}{2}}) = \frac{j-i}{2} \geq \frac{k}{2} > \frac{2l+1}{2} > l$ . Further for odd  $j - i$ :  $d_{C_m}(x_{\frac{j+i+1}{2}}, J) = d_{C_m}(x_{\frac{j+i+1}{2}}, x_j) = d_{C_m}(x_{\frac{j+i}{2}}, J) = \frac{j-i-1}{2} \geq \frac{k-1}{2} > \frac{2l+1-1}{2} = l$ . Then it follows easily from the above that  $J$  is not  $l$ -dominating, a contradiction.

If  $r > n(2l-k+1)$ , then  $m = nk+r > n(2l+1)$ . Since  $|J| \leq n$  the existence of two consecutive vertices, say  $x_i, x_j$  such that  $d_{C_m}(x_i, x_j) > 2l+1$  is assured. Using a technique similar to that in the case when  $k > 2l+1$  we conclude the following: for even  $j-i$ ,  $d_{C_m}(x_{\frac{j+i}{2}}, J) = d_{C_m}(x_{\frac{j+i}{2}}, x_j) = d_{C_m}(x_i, x_{\frac{j+i}{2}}) = \frac{j-i}{2} > \frac{2l+1}{2} > l$  and for odd  $j-i$ ,  $d_{C_m}(x_{\frac{j+i+1}{2}}, J) = d_{C_m}(x_{\frac{j+i+1}{2}}, x_j) = d_{C_m}(x_{\frac{j+i+1}{2}}, J) = \frac{j-i-1}{2} > \frac{2l+1-1}{2} = l$ . This means that there exists some vertex, which is not  $l$ -dominated by  $J$ . This leads to a conclusion that  $J$  is not a  $(k, l)$ -kernel of  $C_m$  and completes the proof of the theorem. ■

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