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ON (k, l)-KERNELS OF SPECIAL SUPERDIGRAPHS OF P_m AND C_m

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Abstract

The concept of (k, l)-kernels of digraphs was introduced in [2]. Next, H. Galeana-Sanchez [**?**] proved a sufficient condition for a digraph to have a (k, l)-kernel. The result generalizes the well-known theorem of P. Duchet and it is formulated in terms of symmetric pairs of arcs. Our aim is to give necessary and sufficient conditions for digraphs without symmetric pairs of arcs to have a (k, l)-kernel. We restrict our attention to special superdigraphs of digraphs P_m and C_m.

Keywords: kernel, semikernel, (k, l)-kernel.

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1. Introduction

For general concepts we refer the reader to [**?**]. Let D denote a finite, directed graph without loops and multiple arcs (for short: a *digraph*), where $V(D)$ is the set of vertices of D and $A(D)$ is the set of arcs of D. We restrict our considerations to digraphs not having symmetric pairs of arcs. A *path* is a digraph P_m with $V(P_m) = \{x_1, x_2, \ldots, x_m\}$ and $A(P_m) = \{x_i x_{i+1} : i = 1, \ldots, m-1\}$ for $m \geq 2$. A circuit C_m is a digraph with $V(C_m) = V(P_m)$ and $A(C_m) = A(P_m) \cup \{x_m x_1\}$, for $m \geq 3$. For simplicity, $x_{m+i} = x_i$, with $1 \leq i \leq m$. The cardinality of $A(P_m)$ and $A(C_m)$ we call the *length of* P_m and C_m, respectively. We denote by $d_D(x, y)$

the length of the shortest path from x to y in D. This path is meant as a subdigraph of D isomorphic to P_m, where $x_1 = x$ and $x_m = y$. For any $X \subseteq V(D)$ and $x \in V(D) \setminus X$ we put $d_D(x,X) = \min_{y \in X} d_D(x,y)$, $d_D(X,x) = \min_{y \in X} d_D(y,x)$ and $N_D^l(X) = \{x \in V(D) \setminus X : d_D(x,X) > l\}$. For the sake of clarity, we introduce the following notations. A *spanning superdigraph of* D is a digraph H such that $V(H) = V(D)$ and $A(H) \supset A(D)$. If H is a spanning superdigraph of P_m (C_m), then an arc $a \in A(H) \setminus A(C_m)$ ($a \in A(H) \setminus A(C_m)$) will be called a *chord* of D and a chord $x_i x_{i+2}$ we will call a *short chord* of D. Two vertices $x_i, x_j \in X \subset V(P_m) = V(C_m)$ with $i < j$ are called *consecutive in* X if for every integer t with $i < t < j$, we have that $x_t \in V(P_m) \setminus X$. If $i > j$, then instead of j we take $j + m$ and we define that x_i, x_j are consecutive in X as the above. Let k, l be fixed positive integers, $k \geq 2$ and $l \geq 1$. A subset $J \subseteq V(D)$ is called a (k,l)-*kernel of* D if

(1) for each $x, y \in J$ and $x \neq y, d_D(x,y) \geq k$ and

(2) for each $x \in V(D) \setminus J$ there exists $y \in J$ that $d_D(x,y) \leq l$.

The concept of a (k,l)-kernel of a digraph was introduced in [**?**] and considered in [**?**] and [**?**]. It may be to noted that for $k = 2$ and $l = 1$ we obtain the definition of a kernel of D in the sense of Berge [**?**]. If J satisfies the condition (1), then we say that J is k-*stable in* D. Moreover, we assume that the subset including exactly one vertex is also k-stable in D. We say that the vertex x is l-*dominated by* J *in* D or J l-*dominates* x *in* D or J is l-*dominating in* D, when the condition (2) is fulfilled. A subset $J \subseteq V(D)$ is a *strong* (k,l)-kernel of D if J is a (k,l)-kernel of D and

(3) there exist $x, y \in J$, $x \neq y$ that $d_D(x,y) = k$ and

(4) there exists $x \in V(D) \setminus J$ that $d_D(x,J) = l$.

Notice that a (k,l)-kernel consisting of exactly one vertex cannot be a strong (k,l)-kernel. A subset $J \subset V(D)$ is a (k,l)-*semikernel* of D if J is k-stable in D and

(5) for each $x \in V(D) \setminus J$ for which $d_D(J,x) \leq l$, there must be $d_D(x,J) \leq l$.

It is clear that if J is a (k,l)-kernel of D, then J is a (k,l)-semikernel of D. For $k = 2$ and $l = 1$ we obtain the definition of semikernel [**?**].

All definitions are similar for undirected graphs, which are also considered.

2. The Existence of (k,l)-Kernels in P_m and its Spanning Superdigraph

For a fixed $k \geq 2$ we can write an arbitrary positive integer number $m \geq 2$ in the form $m = nk + r$, where $n \geq 0$ and $0 \leq r < k$. By the way, if J is a (k,l)-kernel of P_m, then $|J| \leq n+1$.

First, we give a necessary and sufficient condition for a digraph P_m to have a (k,l)-kernel. If $n = 0$, then P_m has a (k,l)-kernel if and only if $r \leq l+1$. For $n \geq 1$ we have the following result.

Theorem 2.1. *Let P_m be a digraph of order $m = nk + r$ and $n \geq 1$. Then P_m has a (k,l)-kernel if and only if $k \leq l+1$.*

Proof. Let $k \leq l+1$. It is not difficult to observe that $J = \{x_r, x_{r+k}, x_{r+2k}, \ldots, x_{r+(n-1)k}, x_{r+nk=m}\}$ is a (k,l)-kernel of P_m. Indeed, J is k-stable and for every $x \in V(P_m) \setminus J$ we have $d_{P_m}(x, J) \leq k-1 \leq l$.

Now suppose on the contrary that P_m has a (k,l)-kernel J, but $k > l+1$. Then for every two consecutive vertices $x_i, x_j \in J$, $d_{P_m}(x_i, x_j) \geq k > l+1$. Moreover, $d_{P_m}(x_{i+1}, J) = d_{P_m}(x_{i+1}, x_j) \geq k-1 > l$ and this means that x_{i+1} is not l-dominated by J. This contradicts the assumption that J is a (k,l)-kernel of P_m and completes the proof. ■

It is natural to ask whether adding a new arc (the opposite arcs are not a taken into consideration) to P_m guarantees the existence of a (k,l)-kernel in an obtained spanning superdigraph, for $k > l+1$. We shall calculate the smallest number of chords of a spanning superdigraph of P_m having a (k,l)-kernel for the case, when $k > l+1$. In order to do it, we start with a simple assertion noting that throughout all sections we assume $m = nk + r$, $n \geq 1$ and $0 \leq r < k$.

Lemma 2.2. *Let D be a spanning superdigraph of P_m such that $|A(D) \setminus A(P_m)| = 1$. Then, for any $X \subset V(D)$, $|N^l_{P_m}(X) \setminus N^l_D(X)| \leq l$.*

Proof. Let D be a spanning superdigraph of P_m having exactly one additional arc from $A(D) \setminus A(P_m)$. We extend the numbering of the vertices in the natural fashion assuming that the sequence $(x_1, x_2, \ldots, x_m)$ constitutes the path P_m. Suppose for an indirect proof that there exists a subset $X \subset V(D)$ such that $|N^l_{P_m}(X) \setminus N^l_D(X)| \geq l+1$. Certainly, $N^l_D(X) \subset N^l_{P_m}(X)$. For convenience, we put $\eta = |N^l_{P_m}(X) \setminus N^l_D(X)|$.

Further, let $x_s x_t$ denote a unique arc belonging to the set $A(D) \setminus A(P_m)$ with $|s-t| \geq 2$. Notice that $x_s \in V(D) \setminus X$. Otherwise, it would be $N^l_{P_m}(X) = N^l_D(X)$. Hence $\eta = 0$ but this is a contradiction to the assumption that $\eta \geq l+1$. Choose a vertex $x_{u_0} \in N^l_{P_m}(X) \setminus N^l_D(X)$ such that $d_{P_m}(x_{u_0}, X) = \max_{x_u \in N^l_{P_m}(X) \setminus N^l_D(X)} d_{P_m}(x_u, X)$. It follows from the choice of x_{u_0} that if $x_u \in N^l_{P_m}(X) \setminus N^l_D(X)$, then $u_0 \leq u \leq u_0 + \eta - 1$ and $d_{P_m}(x_{u_0}, x_s) \geq \eta - 1$. As it was noted $x_{u_0} \in N^l_{P_m}(X) \setminus N^l_D(X)$, so $d_{P_m}(x_{u_0}, X) > l$ and $d_D(x_{u_0}, X) \leq l$. This means that the shortest path from x_{u_0} to the set X includes the arc $x_s x_t$. Therefore, we can conclude that $d_D(x_{u_0}, X) = d_D(x_{u_0}, x_s) + d_D(x_s, x_t) + d_D(x_t, X) = d_{P_m}(x_{u_0}, x_s) + 1 + d_{P_m}(x_t, X) \geq (\eta - 1) + 1 + d_{P_m}(x_t, X) \geq \eta \geq l+1$. Finally we obtain that $d_D(x_{u_0}, X) \leq l+1$, a contradiction.

Note that the Lemma **??** shows that adding exactly one arc to P_m creates superdigraph D such that the number s of l-dominated vertices by a fixed subset $X \subset V(P_m)$ in D is more than the number p of l-dominated vertices by X in P_m. Moreover, $s - p \leq l$. This leads to the following corollary.

Corollary 2.3. *Let $X \subseteq V(P_m)$, such that $|N^l_{P_m}(X)| = \eta > 0$. Then every spanning superdigraph D of P_m, in which X is l-dominating, has to possess at least $\lceil \frac{\eta}{l} \rceil$ additional arcs (i.e., $|A(D) \setminus A(P_m)| \geq \lceil \frac{\eta}{l} \rceil$), where $\lceil p \rceil$ denotes the smallest integer greater than or equal to p.*

It may be noted that if $X \subset V(P_m)$ and $|X| = 1$, then X can l-dominate at most l vertices of P_m. Moreover, if $|X| = s$, then X can l-dominate at most $s \cdot l$ vertices of P_m. Now we discuss the case when $k > l+1$ with respect to the existence of a (k,l)-kernel in spanning superdigraph D of P_m. More precisely, we estimate a number of additional arcs which are needed for a superdigraph D having a (k,l)-kernel with $k > l+1$.

Theorem 2.4. *Let D be a spanning superdigraph of P_m. If $k > l+1$ and D has a (k,l)-kernel, then $|A(D) \setminus A(P_m)| \geq \lceil \frac{m-n-r}{l} \rceil - n$ for $r \leq l+1$ and $|A(D) \setminus A(P_m)| \geq \left\lceil \frac{m-n-1}{l} \right\rceil - n - 1$ for $r > l+1$.*

Proof. Let J be a (k,l)-kernel of D. Since J is k-stable in D, then it is k-stable in P_m, too. Moreover, from the assumption that $k > l+1$ we have that J is not a (k,l)-kernel of P_m (see Theorem **??**). Thus J is not l-dominating in P_m. Then $N^l_{P_m}(J) \neq \emptyset$. We can present the set of vertices as a sum of disjoint subsets, namely $V(P_m) = J \cup \{y \in V(P_m) \setminus J : d_{P_m}(y, J) \leq l\} \cup$

$N^l_{P_m}(J)$. Hence if we take the cardinalities of these sets into consideration, we have the following equality: $m = |J| + |\{y \in V(P_m) \setminus J : d_{P_m}(y, J) \leq l\}| + |N^l_{P_m}(J)|$. Moreover, $|\{y \in V(P_m) \setminus J : d_{P_m}(y, J) \leq l\}| \leq l|J|$. Then $m \leq |J| + l|J| + |N^l_{P_m}(J)|$. As it was mentioned earlier, $|J| \leq n+1$. This means that $m \leq (n+1)(l+1) + |N^l_{P_m}(J)|$ i.e., $|N^l_{P_m}(J)| \geq m - (n+1)(l+1)$. As a consequence $|A(D) \setminus A(P_m)| \geq \left\lceil \frac{m-(n+1)(l+1)}{l} \right\rceil = \left\lceil \frac{m-n-1}{l} \right\rceil - n - 1$ in view of Corollary **??**. If $r \leq l+1$ we can give a better estimate. We shall show that in this case $|N^l_{P_m}(J)| \geq m - n(l+1) - r$. Assume that $|N^l_{P_m}(J)| < m - n(l+1) - r$. Combining the upper bound of m (given above) and the last inequality we deduce that $m < (n+1)(l+1) + m - n(l+1) - r = l + 1 - n(l+1) = (1-n)(l+1)$. If $n = 0$, then $m = r < l+1$ i.e., J is a (k,l)-kernel of P_m, which contradicts the assumption. If $n \geq 1$, then $m < 0$, the next contradiction. Thus we conclude that $\left|N^l_{P_m}(J)\right| \geq m - n(l+1) - r$. This means that $|A(D) \setminus A(P_m)| \geq \left\lceil \frac{m-n(l+1)-r}{l} \right\rceil = \lceil \frac{m-n-r}{l} \rceil - n$ in view of Corollary **??** and completes the proof. ■

3. Special Kinds of (k,l)-Kernels in C_m and its Superdigraphs

At the beginning, we prove the relationship between the existence of (k,l)-kernel and (k,l)-semikernel in C_m. We extend the numbering of the vertices in the natural fashion around the circuit C_m i.e., the sequence $(x_1, x_2, \ldots, x_m)$ constitutes the digraph C_m.

Theorem 3.1. *Let $m \geq 3$. Then C_m has a (k,l)-semikernel if and only if it has a (k,l)-kernel.*

Proof. Let J be a (k,l)-semikernel of C_m. To prove that J is a (k,l)-kernel of C_m it is enough to show that J is l-dominating in C_m. Let $x_i, x_j \in J$ be any consecutive vertices in J. If $i > j$, then instead of j we take $j+m$. Since $d_{C_m}(x_i, x_{i+1}) = 1 \leq l$, then we have that $d_{C_m}(x_{i+1}, x_j) \leq l$. Hence J is l-dominating in C_m. As it was remarked in Introduction, each (k,l)-kernel of a digraph is a (k,l)-semikernel of the digraph which completes the proof. ■

Recall that $m = nk + r$, $n \geq 0$ and $0 \leq r < k$. It is not difficult to see that if J is a (k,l)-kernel of C_m, then $|J| \leq n$, for $n \geq 1$ or $|J| = 1$, for $n = 0$.

Moreover, if $n = 0$, then C_m has a (k,l)- kernel J iff $r \leq l+2$. If $n \geq 1$, then we have the following theorem.

Theorem 3.2. *Let C_m be given with $m = nk + r$, $n \geq 1$. Then C_m has a (k,l)-kernel if and only if $k \leq l+1$ and $r \leq n(l-k+1)$.*

Proof. I. Let $k \leq l+1$ and $r \leq n(l-k+1)$. It is easy to observe that if $r = 0$ (i.e., $m = nk$), then the subset $J = \{x_1, x_{1+k}, x_{1+2k}, \ldots, x_{1+(n-1)k}\}$ is a (k,l)-kernel of C_m.

Assuming that $r > 0$ we shall prove that there exists an integer s such that $0 \leq s \leq l-k+1$ and $m = n(k+s) + r_s$, where $0 \leq r_s < n$. Assume that this is not true, or in other words for every s with $0 \leq s \leq l-k+1$ we have $r_s > n$. Taking $s = l-k$ we have $m = n(k+s) + r_s = nl + r_{l-k}$. Since $r_{l-k} > n$, so $m > n(l+1)$. But at the same time we have $m = nk + r \leq nk + n(l-k+1) = n(l+1)$, a contradiction.

Now, we shall show that the existence of a (k,l)-kernel in C_m is assured. For $r_s = 0$ the set $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \ldots, x_{1+(n-1)(k+s)}\}$ is a (k,l)-kernel of C_m. For $r_s > 0$, we put $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \ldots, x_{1+(n-r_s)(k+s)}, x_{1+(n-r_s+1)(k+s)+1}, x_{1+(n-r_s+2)(k+s)+2}, \ldots, x_{1+(n-r_s+(r_s-2))(k+s)+r_s-2}, x_{1+(n-1)(k+s)+r_s-1}\}$. In order to show that J is k-stable in C_m it suffices to observe that $d_{C_m}(x_{1+(n-1)(k+s)}, x_1) = m + 1 - [1 + (n-1)(k+s) + r_s - 1] = k + s + 1 > k$. We have also for every $x \in V(C_m) \setminus J$ that $d_{C_m}(x, J) \leq k + s < l + 1$, what proves that J is l-dominating in C_m. Consequently, J is a (k,l)-kernel of C_m and the first part of the theorem is proved.

II. Assume that J is a (k,l)-kernel of C_m, but $k > l+1$ or $r > n(l-k+1)$. If $|J| = 1$, then it can be verified that $n = 1$ and $J = \{x_i\}$, where $1 \leq i \leq m$. As a consequence $d_{C_m}(x_{i+1}, J) = d_{C_m}(x_{i+1}, x_i) = m-1 = k+r-1$. Further, from the assumption that $k > l+1$ or $r > n(l-k+1)$ it follows that $k + r - 1 > l$. This means that the vertex x_{i+1} is not l-dominated by J and contradicts our assumption that J is a (k,l)-kernel of C_m. Now we consider the case when $|J| \geq 2$. Let $x_i, x_j \in J$ be two consecutive vertices in J. If $k > l+1$, then $d_{C_m}(x_{i+1}, x_j) = d_{C_m}(x_i, x_j) - 1 \geq k - 1 > l$. This means that x_{i+1} is not l-dominated by J, a contradiction to the assumption that J is a (k,l)-kernel of C_m. If $r > n(l-k+1)$, then $m = nk + r > n(l+1)$. From this and in fact that $|J| \leq n$, the existence of two consecutive vertices in J, say x_i, x_j such that $d_{C_m}(x_i, x_j) > l+1$ is assured. Hence $d_{C_m}(x_{i+1}, x_j) = d_{C_m}(x_i, x_j) - 1 > l$. This means that x_{i+1} is not l-dominated by J i.e., J is not a (k,l)-kernel of C_m. This contradiction completes the proof of the theorem. ■

Certainly, if $n \leq 1$, then each k-stable set of C_m contains exactly one vertex. Therefore, we conclude that C_m does not have a strong (k,l)-kernel, since the condition (3) is not satisfied. Now, we give a necessary and sufficient condition for C_m to have a strong (k,l)-kernel.

Theorem 3.3. *The digraph C_m possesses a strong (k,l)-kernel if and only if:*

(6) $m-k-l-1=0$ *or*

(7) $m-k-l-1 \geq k$ *and $C_{m-k-l-1}$ has a (k,l)-kernel.*

Proof. I. Let J be a strong (k,l)-kernel of C_m. This implies that it must be $k \leq l+1$, by Theorem **??**. By the way, it is easy to observe that $m-k-l-1 \geq 0$. Suppose on the contrary that $m-k-l-1<0$. This is equivalent to $m<k+l+1 \leq 2k$, since $k \leq l+1$. In conclusion there must be $|J|=1$, which is impossible by the assumption that J is a strong (k,l)-kernel of C_m. Finally, we state $m-k-l-1 \geq 0$. Next, assume on the contrary that both conditions (6) and (7) do not hold simultaneously. In other words (by the condition $m-k-l-1 \geq 0$) there must hold: (a) $0<m-k-l-1<k$ or (b) $m-k-l-1>0$ and $C_{m-k-l-1}$ has no (k,l)-kernel. Suppose that the condition (a) holds. Since J is a strong (k,l)-kernel of C_m, then there exist $x_q, x_p \in J$ and $x_s \in V(C_m) \setminus J$ such that $d_{C_m}(x_q,x_p)=k$ and $d_{C_m}(x_s,J)=l$. Without loss of generality, let $q<p$ (if $q>p$, then take $p+m$ instead of p). If $q<s<p$, then $s=q+1$ and $d_{C_m}(x_q,x_p)=l+1$, hence $k=l+1$. In conclusion, the condition (a) is equivalent to the expression $0<m-2k<k$. This means that $m=2k+r$, where $r>0$. On the other hand, since C_m has a (k,l)-kernel, then $r \leq n(l-k+1)$ in view of Theorem **??**. Therefore, putting $k=l+1$ we have $r \leq 0$, contrary to the conclusion that $r>0$. If $s<q$ or $s>p$, there exists a vertex $x_t \in J$, such that $t \neq q$ and $t \neq p$. Figure 1 illustrates the positions of the vertex x_t with respect to the vertex x_s.

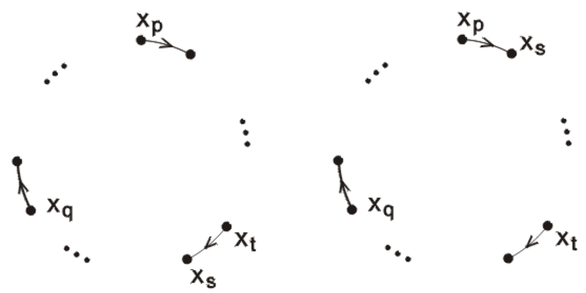

Figure 1

Otherwise (i.e., $J = \{x_q, x_p\}$), we would have $m = k + l + 1$ or equivalently $m - k - l - 1 = 0$, which is impossible by (a). Thus, $t \neq q$ and $t \neq p$. This means that $d_{C_m}(x_p, x_q) = d_{C_m}(x_p, x_t) + d_{C_m}(x_t, x_s) + d_{C_m}(x_s, x_q)$ or $d_{C_m}(x_p, x_q) = d_{C_m}(x_p, x_s) + d_{C_m}(x_s, x_t) + d_{C_m}(x_t, x_q)$ (see Figure 1). As it was noted $x_q, x_p, x_t \in J$, where J is k-stable and $d_{C_m}(x_s, J) = l$, then $d_{C_m}(x_p, x_q) \geq k + l + 1$. Using the last inequality we can write that $m = d_{C_m}(x_p, x_q) + d_{C_m}(x_q, x_p) \geq k + (k+l+1) = 2k+l+1$. Thus $m-k-l-1 \geq k$, which is a contradiction to (a). Assume that the condition (b) holds. Since $m > k+l+1$, then $|J| \geq 3$. Otherwise, (i.e., $|J| \leq 2$) the subset J could not be a strong (k,l)-kernel of C_m. Finally $|J| \geq 3$. Therefore, we may assume without loss of generality that $x_{m-k-l-1}, x_{m-k}, x_m \in J$ (see Figure 2).

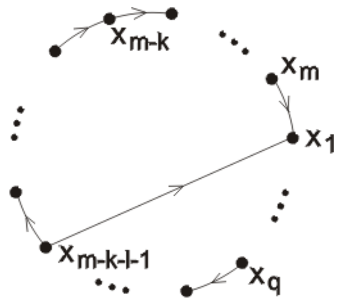

Figure 2

Create a spanning superdigraph D of C_m adding a new arc $x_{m-k-l-1}x_1$ to C_m. Thus a subdigraph H of D induced by the set $\{x_1, x_2, \ldots, x_{m-k-l-1}\}$ is isomorphic to $C_{m-k-l-1}$. Then H has no (k,l)-kernel either. We define $J_0 = J \setminus \{x_{m-k}, x_m\}$. Since J is a (k,l)-kernel of C_m, then for $1 \leq s \leq m - k - l - 1$ we have $d_{C_m}(x_s, J) \leq l$. Moreover, $d_{C_m}(x_s, \{x_{m-k}, x_m\}) > l$. This means that $d_H(x_s, J_0) = d_{C_m}(x_s, J) \leq l$. Hence J_0 is l-dominating in H. Now we show that J_0 is k-stable in H. Choose a vertex $x_q \in J$ that x_m, x_q are consecutive in J (of course $q < m$). In order to show that J_0 is k- stable in H, it is enough to observe that $d_H(x_{m-k-l-1}, x_q) \geq k$. Indeed, $d_H(x_{m-k-l-1}, x_q) = q = d_{C_m}(x_m, x_q) \geq k$, since $x_m, x_q \in J$. Thus J_0 is a (k,l)-kernel of H, what is a required contradiction and proves the first part of the theorem.

II. Let $m - k - l - 1 = 0$. Thus $J = \{x_1, x_{1+k}\}$ is a strong (k,l)-kernel of C_m. Indeed, $d_{C_m}(x_1, x_{1+k}) = k$ and $d_{C_m}(x_{1+k}, x_1) = m + 1 - (1 + k) = m - k = l + 1$, hence $d_{C_m}(x_{2+k}, x_1) = l$. Now let $m - k - l - 1 \geq k$ and

$C_{m-k-l-1}$ has a (k,l)-kernel. Let a subdigraph H be defined in the same way as in Part I of the proof. Then H has a (k,l)-kernel, too. We denote it by J_0 and assume without loss of generality that $x_1 \in J_0$. We show that $J = J_0 \cup \{x_{m-k-l}, x_{m-l}\}$ is a strong (k,l)-kernel of C_m. Observe that because of the structure of C_m we have $d_{C_m}(x_{m-k-l}, x_{m-l}) = k$ and $d_{C_m}(x_{m-l}, x_1) = l+1$. Thus $d_{C_m}(x_{m-l+1}, x_1) = l$. This means that if J is a (k,l)-kernel of C_m, then it also is a strong (k,l)-kernel of C_m. If $|J_0| = 1$ (i.e., $J_0 = \{x_1\}$), then $d_{C_m}(x_1, x_{m-k-l}) = m-k-l-1 \geq k$ and $J = \{x_1, x_{m-k-l}, x_{m-l}\}$ is a (k,l)-kernel of C_m. If $|J_0| > 1$, then there exists $x_q \in J_0$, such that x_q, x_1 are consecutive in J_0. Since $d_H(x_q, x_1) = m-k-l-q \geq k$, then $d_{C_m}(x_q, x_{m-k-l}) = m-k-l-q \geq k$. Thus J is k-stable and l-dominating in C_m i.e., J is a (k,l)-kernel of C_m and this completes the proof of the theorem. ■

Proceeding by the same argument as for P_m in the proof of Lemma **??** and Corollary **??** we state two assertions with respect to C_m.

Theorem 3.4. *Let D be a spanning superdigraph of C_m including only one chord and $X \subset V(D)$. Then $|N^l_{C_m}(X) \setminus N^l_D(X)| \leq l$.*

Corollary 3.5. *Let $X \subset V(C_m)$, where $|N^l_{C_m}(X)| = \eta > 0$. Then every spanning superdigraph D of C_m, in which X is l-dominating, has at least $\lceil \frac{\eta}{l} \rceil$ additional arcs (i.e., $|A(D) \setminus A(C_m)| \geq \lceil \frac{\eta}{l} \rceil$).*

Let a set $J \subset V(C_m)$ be such that $|J| = s$. It is easy to observe that if J is k-stable in C_m, but not l-dominating in C_m, then $|N^l_{C_m}(J)| \geq m - s(l+1)$. In that case in view of Theorem **??**, we can formulate the following corollary.

Corollary 3.6. *Let C_m be such that it does not have a (k,l)-kernel and D be a spanning superdigraph of C_m. If $J \subset V(D)$ is a (k,l)-kernel of D, with $|J| = s$, then D has at least $\lceil \frac{m-s}{l} \rceil - s$ chords.*

If s is an integer such that $1 \leq s \leq n$, it is clear that the expression $\lceil \frac{m-s}{l} \rceil - s$ has the smallest value for $s = n$. This implies the next corollary.

Corollary 3.7. *Let D be a spanning superdigraph of C_m. If D has a (k,l)-kernel, then C_m also has a (k,l)-kernel or D possesses at least $\lceil \frac{m-n}{l} \rceil - n$ chords.*

Lemma 3.8. *If J is a (k,l)-kernel of spanning superdigraph D of C_m, then for every two consecutive vertices x, y in J we have $d_{C_m}(x,y) \leq 2l+1$.*

Proof. Suppose to the contrary that there exist two consecutive vertices in J, say x_i, x_j such that $d_{C_m}(x_i, x_j) > 2l + 1$. As a consequence $d_{C_m}(x_{i+1}, x_j) > 2l$. Let us remark that the existence of short chords in D leads to inequality $d_D(x_{i+1}, x_j) \geq \frac{1}{2} d_{C_m}(x_{i+1}, x_j)$. Combining the above facts we deduce that $d_D(x_{i+1}, x_j) > l$. But this contradicts the assumption that J is l-dominating in D, hence the lemma is proved. ■

In what follows D will be a spanning superdigraph of C_m containing only short chords, where $m = nk + r$ with $0 \leq r < k$.

Recall that if $n = 0$ and $r > l + 1$, then $C_{m=r}$ has no (k, l)-kernel. It is easy to observe that if additionally $r \leq 2l + 1$, then every spanning superdigraph D of C_m having a (k, l)-kernel has at least $r - l - 1$ short chords. For $n \geq 1$ we state the next assertion.

Lemma 3.9. *If C_m contains no (k, l)-kernel, then every spanning superdigraph D of C_m having a (k, l)-kernel for $k \geq 2$, $l \geq 1$ and $n \geq 1$ has at least $m - n(l + 1)$ chords.*

Proof. Let J be a (k, l)-kernel of D. Since C_m has no (k, l)-kernel, hence $r > n(l - k + 1)$ or $k > l + 1$ in view of Theorem **??**. This means that $m = nk + r > n(l + 1)$. Let $|J| = s$. As it was remarked, we deduce that at least $m - s(l + 1)$ vertices are not l-dominated by J in C_m. Assume that $s \geq 2$, hence there exist two consecutive vertices in J, say x_i, x_j, with $i < j$ and $d_{C_m}(x_i, x_j) > l + 1$. Then it follows easily from the above that $N = \{x_{i+1}, x_{i+2}, \ldots, x_{j-l-1}\} \subseteq N^l_{C_m}(J)$. Let η denote the number of short chords of D, whose endpoints are vertices x_t, where $i < t \leq j$. We shall prove that $\eta \geq |N| = j - i - l - 1$. Assume this cannot occur i.e., $\eta < j - i - l - 1$. Since $d_{C_m}(x_{i+1}, x_j) = j - i - 1$, hence $d_D(x_{i+1}, x_j) \geq \eta + (j - i - 1 - 2\eta) = j - i - 1 - \eta > l$. This means that x_{i+1} is not l-dominated by x_j in D. Hence x_{i+1} cannot be dominated by J in D, contradicting the assumption that J is a (k, l)-kernel of D. This contradiction proves that $\eta \geq j - i - l - 1$. Taking all vertices not l-dominating by J in C_m into consideration, we get that D has at least $|N^l_{C_m}(J)|$ chords. In case when $s = 1$ we take $m + i$ instead of j and proceed as above.

If s is an integer such that $1 \leq s \leq n$, it is clear that the expression $m - s(l + 1)$ achieves the smallest value for $s = n$. This completes the proof. ■

If $k > l + 1$, then any superdigraph D of C_m cannot have a (k, l)-kernel of cardinality more than one. Indeed, because of $k > l+1$ every k-stable subset of C_m is not l-dominating in view of Theorem **??**. Short chords of D can cause that arbitrary k-stable set of C_m will be l-dominating in D but not k-stable in D. Moreover, taking the condition $k > l+1$ into consideration there exists a spanning superdigraph D of C_m having a (k,l)-kernel if $m \leq 2l+1$.

Theorem 3.10. *If C_m does not have a (k,l)-kernel with $k \leq l+1$ and $r \leq n(2l-k+1)$, then there exists a spanning superdigraph D of C_m having a (k,l)-kernel.*

Proof. Since C_m does not have a (k,l)-kernel and $k \leq l+1$, then $r > n(l-k+1)$ see Theorem **??**. Moreover, $m = nk + r > n(l+1)$. Now, we shall show that there exists an integer $p > 0$ such that $m = n(k+p) + r_p$, where $0 \leq r_p < n$ and $p \geq l-k+1$. On the contrary, let $p \leq l-k$. Hence $m = n(k+p) + r_p \leq n(k+l-k) + r_p = nl + r_p < n(l+1)$. On the other hand we have $m = nk + r > nk + n(l-k+1) = n(l+1)$, a contradiction. Notice that if $r_p = 0$, then $p > l-k+1$ (if $r_p = 0$ and $p = l-k+1$, then $m = n(k+p) + r_p = n(l+1)$, contrary to $m > n(l+1)$).

For $r_p = 0$ (i.e., $m = n(k+p)$), the subset $J = \{x_1, x_{1+(k+p)}, x_{1+2(k+p)}, \ldots, x_{1+(n-1)(k+p)}\}$ is k-stable in C_m. In order to show it, it suffices to observe that $d_{C_m}(x_{1+(n-1)(k+p)}, x_1) = m + 1 - [1 + (n-1)(k+p)] = k + p \geq k$. Let $N_j = \{x_{2+j(k+p)}, x_{3+j(k+p)}, x_{4+j(k+p)}, \ldots, x_{k+p-l+j(k+p)}\}$, where $0 \leq j \leq n-1$. It is clear that $2 + j(k+p) \leq k + p - l + j(k+p)$ owing to $p > l-k+1$. We can observe that for every $x \in N_j$ we have $d_{C_m}(x, J) = d_{C_m}(x, x_{1+(j+1)(k+p)}) \geq d_{C_m}(x_{k+p-l+j(k+p)}, x_{1+(j+1)(k+p)}) = l+1$. This means that no vertex from N_j is l-dominated by stable set J. Moreover, it is not difficult to see that $\bigcup_{j=0}^{n-1} N_j = N^l_{C_m}(J)$. Let D be a spanning superdigraph of C_m with $A(D) = A(C_m) \cup A_0$, where $A_0 = \{a_{i,j} : 1 \leq i \leq k+p-l-1 \wedge 0 \leq j \leq n-1\}$ and $a_{i,j} = (x_{2i+j(k+p)}, x_{2i+2+j(k+p)})$. We can show that the indices of all endpoints x_t of chords $a_{i,j}$ meet the condition $1 + j(k+p) < t \leq 1 + (j+1)(k+p)$ for each j. In order to show it, it suffices to observe that $a_{1,j} = (x_{2+j(k+p)}, x_{4+j(k+p)})$ and $a_{k+p-l-1,j} = (x_{2(k+p-l-1)+j(k+p)}, x_{2+2(k+p-l-1)+j(k+p)})$ have endpoints whose indices satisfy the condition mentioned. Hence for every $x \in N_j$ we have $d_D(x, J) \leq d_D(x_{2+j(k+p)}, J) = d_D(x_{2+j(k+p)}, x_{1+(j+1)(k+p)}) = d_D(x_{2+j(k+p)}, x_{2+2(k+p-l-1)+j(k+p)}) + d_D(x_{2+2(k+p-l-1)+j(k+p)}, x_{1+(j+1)(k+p)}) = \frac{2+2(k+p-l-1)+j(k+p)-[2+j(k+p)]}{2} + 1 + (j+1)(k+p) - [2+2(k+p-l-1)+j(k+p) = (k+p-l-1) + (1-k-p+2l) = l$ (see Figure 3).

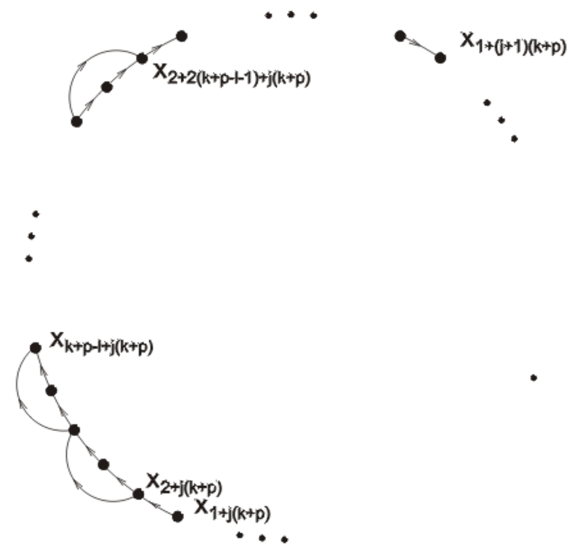

Figure 3

This means that all $x \in N^l_{C_m}(J)$ are l-dominated by k-stable set J, so J is a (k,l)-kernel of D. Notice that $|A_0| = (k+p-l-1)n = n(k+p) - n(l+1) = m - n(l+1)$. Hence D (in view of Lemma **??**) is a spanning superdigraph of C_m with the minimum number of short chords. For $r_p > 0$ the subset $J = \{x_1, x_{1+(k+p)}, \ldots, x_{1+(n-r_p)(k+p)}, x_{1+(n-r_p+1)(k+p)+1}, x_{1+(n-r_p+2)(k+p)+2}, \ldots, x_{1+(n-1)(k+p)+r_p-1}\}$ is k-stable in C_m.

Put $M_j = N_j \cup \{x_{k+p-l+1}\}$, for $n - r_p \leq j \leq n-1$. If $p = l-k+1$, then $N_j = \emptyset$ for $0 \leq j \leq n - r_p - 1$ and $M_j \neq \emptyset$ for $n - r_p \leq j \leq n-1$. It is easy to observe that $\bigcup_{j=0}^{n-r_p-1} N_j \cup \bigcup_{j=n-r_p}^{n-1} M_j = N^l_{C_m}(J)$. Similarly, as for $r_p = 0$ we can show that for every $x \in N^l_{C_m}(J)$ we have $d_{C_m}(x,J) \geq l+1$. This means that no vertex from $N^l_{C_m}(J)$ is l-dominated by stable set J. Let D be a spanning superdigraph of C_m with $A(D) = A(C_m) \cup A_1$, where $A_1 = \{a_{i,j} : (1 \leq i \leq k+p-l-1 \wedge 0 \leq j \leq n-r_p-1)$ or $(1 \leq i \leq k+p-l \wedge n-r_p \leq j \leq n-1)\}$.

It is not difficult to see that for $0 \leq j \leq n-1$ all endpoints x_t of chords $a_{i,j}$ meet the condition $1 + j(k+p) < t \leq 1 + (j+1)(k+p)$. It is easy to calculate (similarly as for $r_p = 0$) that for every $x \in N^l_{C_m}(J)$ we have $d_D(x,J) \leq l$. This means that all $x \in N^l_{C_m}(J)$ are l-dominated by stable set J, so J is a (k,l)-kernel of D. At the same time $|A_1| = (k+p-l-1)(n-r_p)+(k+p-l)r_p = n(k+p)+r_p-n(l+1) = m-n(l+1)$. This means that D is a spanning superdigraph of C_m with a minimum number of short chords in view of Lemma **??**. ■

4. On (k,l)-Kernels of Graphs

In this section, the notation C_m means an directed graph defined analogously as the circuit C_m. In this case $d_{C_m}(x,y) = d_{C_m}(y,x)$.

Recall that $m = nk + r$, $n \geq 0$ and $0 \leq r < k$. It is not difficult to observe that if $n = 0$, then C_m has a (k,l)-kernel iff $r \leq 2l + 2$. If $n \leq 1$, then C_m has no strong (k,l)-kernel.

Theorem 4.1. *Let $n \geq 2$. C_m has a strong (k,l)-kernel if and only if at least one of the following conditions is fulfilled:*

(8) $m - k - 2l = 0$,

(9) $m - k - 2l - 1 = 0$,

(10) $m - k - 2l \geq k$ *and* C_{m-k-2l} *has a* (k,l)*-kernel,*

(11) $m - k - 2l - 1 \geq k$ *and* $C_{m-k-2l-1}$ *has a* (k,l)*-kernel.*

Proof. The sufficient condition of existence of a strong (k,l)-kernel we prove on the contrary using the method from Part I of the proof of Theorem **??** and considering two conditions:

(a) $0 < m - k - 2l < k$ or $m - k - 2l > 0$ and C_{m-k-2l} has no (k,l)-kernel,

(b) $0 < m - k - 2l - 1 < k$ or $m - k - 2l - 1 > 0$ and $C_{m-k-2l-1}$ has no (k,l)-kernel.

Proceeding as in the second part of the proof of Theorem **??**, we can prove the necessary condition of the theorem. ■

Theorem **??** is a generalization of the result announced in [**?**] and concerning a strong $(k,k-2)$-kernel of C_m.

Noting that a symbol $\lfloor p \rfloor$ denotes the greatest integer less than or equal to p, we prove the following.

Theorem 4.2. *The cycle C_m, where $m = nk + r$ and $n \geq 1$, has a (k,l)-kernel if and only if $k \leq 2l + 1$ and $r \leq n(2l - k + 1)$.*

Proof. I. Let $k \leq 2l+1$ and $r \leq n(2l-k+1)$. At first, notice that if $x_i, x_j \in J$ are consecutive in J, then for each integer t such that $i < t < j$ we have $\max_t d_{C_m}(x_t, J) = \left\lfloor \frac{j-i}{2} \right\rfloor$. It is easy to observe that if $r = 0$ (i.e., $m = nk$), then the set $J = \{x_1,\ x_{1+k},\ x_{1+2k}, \ldots,\ x_{1+(n-1)k}\}$ is a (k,l)-kernel of C_m. Indeed, for every two vertices $x_{1+(i-1)k}, x_{1+ik}$ consecutive in J we have $d_{C_m}(x_{1+(i-1)k}, x_{1+ik}) = k$, where $i = 1, \ldots, n-1$ and $d_{C_m}(x_{1+(n-1)k}, x_1) = m+1-[1+(n-1)k] = k$, which means that J is k-stable. We have also for each $x \in V(D) \setminus J$ that $d_{C_m}(x, J) \leq \frac{k}{2} \leq \frac{2l+1}{2} = l + \frac{1}{2}$. Since $d_{C_m}(x, J)$ is an integer number, then finally $d_{C_m}(x, J) \leq l$.

Now let $r > 0$. We state that there exists an integer s such that $0 \leq s \leq 2l-k+1$ and $m = n(k+s) + r_s$, where $0 \leq r_s < n$. Suppose on the contrary that for every s, $0 \leq s \leq 2l-k+1$ we have $r_s > n$. Let $s = 2l-k$ and $m = n(k+s) + r_s = 2nl + r_{l-k}$. Since $r_{l-k} > n$, then $m > n(2l+1)$. On the other hand, we have $m = nk + r \leq nk + n(2l-k+1) = n(2l+1)$, a contradiction.

It is not difficult to observe that for $r_s = 0$ the subset $J = \{x_1, x_{1+(k+s)},\ x_{1+2(k+s)}, \ldots, x_{1+(n-1)(k+s)}\}$ is a (k,l)-kernel of C_m. If $r_s > 0$, then $J = \{x_1, x_{1+(k+s)}, x_{1+2(k+s)}, \ldots, x_{1+(n-r_s)(k+s)}, x_{1+(n-r_s+1)(k+s)+1}, x_{1+(n-r_s+2)(k+s)+2}, \ldots, x_{1+(n-r_s+2)(k+s)+r_s-2}, x_{1+(n-1)(k+s)+r_s-1}\}$ is a (k,l)-kernel of C_m. Indeed, $d_{C_m}(x_{1+(n-1)(k+s)}, x_1) = m+1-[1+(n-1)(k+s) + r_s - 1] = k+s+1 > k$. We have also for every $x \in V(D) \setminus J$, $d_{C_m}(x, J) \leq \frac{k+s}{2} < \frac{2l+2}{2} = l+1$, where the existence of such an integer s is assured.

II. Assume that C_m has a (k,l)-kernel J, but $k > 2l+1$ or $r > n(2l-k+1)$. If $|J| = 1$, then $n = 1$ and $J = \{x_i\}$, where $1 \leq i \leq m$. Moreover, if $m = k+r$ is an even number, then $d_{C_m}(x_{i+\frac{m}{2}}, J) = d_{C_m}(x_{i+\frac{m}{2}}, x_i) = \frac{m}{2} = \frac{k+r}{2}$. From the assumption that $k > 2l+1$ or $r > n(2l-k+1)$ we have that $\frac{k+r}{2} > l + \frac{1}{2} > l$. Thus the vertex $x_{i+\frac{m}{2}}$ is not l-dominated by J, which contradicts the assumption that J is a (k,l)-kernel of C_m. If m is odd, then $d_{C_m}(x_{i+\frac{m-1}{2}}, J) = d_{C_m}(x_{i+\frac{m-1}{2}}, x_i) = \frac{m-1}{2} = \frac{k+r-1}{2} > l$ and the vertex $x_{i+\frac{m-1}{2}}$ is not l-dominated by J, a contradiction with the assumption. It remains to consider the case when $|J| \geq 2$. Let $x_i, x_j \in J$ be two consecutive vertices in J. If $k > 2l+1$ and $j-i$ is even, then it follows from the structure of C_m that $d_{C_m}(x_{\frac{j+i}{2}}, J) = d_{C_m}(x_{\frac{j+i}{2}}, x_j) = d_{C_m}(x_i, x_{\frac{j+i}{2}}) = \frac{j-i}{2} \geq \frac{k}{2} > \frac{2l+1}{2} > l$. Further for odd $j-i: d_{C_m}(x_{\frac{j+i+1}{2}}, J) = d_{C_m}(x_{\frac{j+i+1}{2}}, x_j) = d_{C_m}(x_{\frac{j+i}{2}}, J) = \frac{j-i-1}{2} \geq \frac{k-1}{2} > \frac{2l+1-1}{2} = l$. Then it follows easily from the above that J is not l-dominating, a contradiction.

If $r > n(2l-k+1)$, then $m = nk+r > n(2l+1)$. Since $|J| \leq n$ the existence of two consecutive vertices, say x_i, x_j such that $d_{C_m}(x_i, x_j) > 2l+1$ is assured. Using a technique similar to that in the case when $k > 2l+1$ we conclude the following: for even $j-i$, $d_{C_m}(x_{\frac{j+i}{2}}, J) = d_{C_m}(x_{\frac{j+i}{2}}, x_j) = d_{C_m}(x_i, x_{\frac{j+i}{2}}) = \frac{j-i}{2} > \frac{2l+1}{2} > l$ and for odd $j-i$, $d_{C_m}(x_{\frac{j+i+1}{2}}, J) = d_{C_m}(x_{\frac{j+i+1}{2}}, x_j) = d_{C_m}(x_{\frac{j+i}{2}}, J) = \frac{j-i-1}{2} > \frac{2l+1-1}{2} = l$. This means that there exists some vertex, which is not l-dominated by J. This leads to a conclusion that J is not a (k,l)-kernel of C_m and completes the proof of the theorem. ■

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