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ON GRAPHS ALL OF WHOSE $\{C_3, T_3\}$-FREE ARC COLORATIONS ARE KERNEL-PERFECT

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Abstract

A digraph D is called a kernel-perfect digraph or KP-digraph when every induced subdigraph of D has a kernel.

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m distinct colours. A path P is monochromatic in D if all of its arcs are coloured alike in D. The closure of D, denoted by $\zeta(D)$, is the m-coloured digraph defined as follows:

$V(\zeta(D)) = V(D)$, and

$A(\zeta(D)) = \bigcup_i \{(u, v)$ with colour i: there exists a monochromatic path of colour i from the vertex u to the vertex v contained in $D\}$.

We will denoted by T_3 and C_3, the transitive tournament of order 3 and the 3-directed-cycle respectively; both of whose arcs are coloured with three different colours.

Let G be a simple graph. By an m-orientation-coloration of G we mean an m-coloured digraph which is an asymmetric orientation of G.

By the class E we mean the set of all the simple graphs G that for any m-orientation-coloration D without C_3 or T_3, we have that $\zeta(D)$ is a KP-digraph.

In this paper we prove that if G is a hamiltonian graph of class E, then its complement has at most one nontrivial component, and this component is K_3 or a star.

Keywords: kernel, kernel-perfect digraph, m-coloured digraph.

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1. Introduction

Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. An arc $(u,v) \in A(D)$ is called *asymmetrical* if $(v,u) \notin A(D)$. An arc $(u,v) \in A(D)$ is called *symmetrical* if $(v,u) \in A(D)$. The *asymmetrical part* of D, denoted by $\operatorname{asym}(D)$, is the spanning subdigraph of D whose arcs are the asymmetrical arcs of D. The *symmetrical part* of D, denoted by $\operatorname{sym}(D)$, is the spanning subdigraph of D whose arcs are the symmetrical arcs of D. A digraph D is called *asymmetrical* if $D = \operatorname{asym}(D)$.

If S is a nonempty subset of $V(D)$, then the subdigraph $D[S]$ *induced* by S is the digraph having vertex set S and whose arcs are all those arcs of D joining vertices of S.

A set $I \subseteq V(D)$ is *independent* in D if $A\left(D[I]\right) = \emptyset$. A set $I \subseteq V(D)$ is said to be *absorbent* in D if for each vertex $x \in V(D) \setminus I$, there exists a vertex $y \in I$ such that$(x,y) \in A(D)$. A set $I \subseteq V(D)$ will be called a *kernel* of D if I is an independent and absorbent set in D.

The set of all the independent (absorbent) sets in D is denoted by $\operatorname{ind}(D)$ $(\operatorname{abs}(D))$.

The set of all the kernels of D is denoted by $\ker(D)$, i.e., $\ker(D) = \operatorname{ind}(D) \cap \operatorname{abs}(D)$.

A digraph D is called a *critical-kernel-imperfect* digraph or CKI-digraph when D has no kernel but every proper induced subdigraph of D has a kernel; i.e., $\ker(D) = \emptyset$ and for every nonempty set of vertices $I \subseteq V(D)$, $I \neq V(D)$ implies $\ker(D[I]) \neq \emptyset$.

A digraph D is called *complete* if for every two different vertices $u, v \in V(D)$, $(u,v) \in A(D)$ or $(v,u) \in A(D)$. A *tournament* is a complete asymmetrical digraph.

If γ is a directed cycle and $x, y \in V(\gamma)$, then we denote by (x, γ, y) the directed path from x to y contained in γ.

We call the digraph D an *m-coloured digraph* if the arcs of D are coloured with m distinct colours.

By an *orientation* of a graph G we mean a digraph D such that $V(D) = V(G)$ and in which for every edge $[u,v] \in E(G)$ we have that at least one of the arcs $(u,,v)$ or (v,u) is in $A(D)$. An *asymmetrical orientation* of G is an orientation of G which is also an asymmetrical digraph.

If γ is a cycle of G, then a *chord* of γ is an edge $[u,v]$ between two nonconsecutive vertices of γ. The *length* of the chord $[u,v]$ in the cycle γ is the length of the shortest undirected path from u to v contained in γ. A graph G is *triangulated* if every cycle of G has at least one chord.

A *star* with m peaks is a simple graph with $m+1$ vertices in which there is a vertex, called the center of the star, adjacent to all the other vertices of the star (called peaks) and with no adjacencies among the peaks.

For a simple graph G, we define its *complement* G^c, as the following simple graph.
$V(G^c) = V(G)$ and $[u,v] \in E(G^c)$ if and only if $[u,v] \notin E(G)$.

2. Notation

Let G be a simple graph. We recall that by an *m-orientation-coloration* of G we mean an m-coloured digraph which is an asymmetrical orientation of G.

By the class E we mean the simple graphs that under any orientation coloration D without C_3 or T_3, we have that $\zeta(D)$ is a KP-digraph.

Our main task is to give a characterization of the class E. It has been proved before that complete graphs and graphs which miss an edge are of class E.

Theorem 2.1 [2]**.** *Complete graphs are of the class E.*

Theorem 2.2 [1]**.** *Complete graphs that miss an edge are of class E.*

We can consider only connected graphs. A directed graph is kernel-perfect if and only if every one of its connected components are kernel-perfect, so a graph G is of class E if and only if every one of its connected components are of class E.

3. A Special Class of Triangulated Hamiltonian Graphs

Theorem 3.1. *Let G be a graph with more than 3 vertices and γ a hamiltonian cycle of G. If G is triangulated, then there is a vertex, say x, such that γ has a chord between two neighbours of x in γ.*

Proof. As G is triangulated and γ is a cycle of order higher than three, γ must have a chord. Let $[u,v]$ be the shortest chord of γ. The length of the chord $[u,v]$ of the cycle γ must be two: otherwise there would be a cycle of order higher than three without chords, contradicting that G is triangulated. As $[u,v]$ is of length two, it is a chord between the two neighbours in γ of some vertex x. ■

Theorem 3.2. *Let G be a triangulated hamiltonian graph. If G^c does not have induced subgraphs isomorphic to $G_1, G_2, G_3, G_4, G_5, G_6, G_7$, then G^c has at most one nontrivial component, and it is K_3 or a star.*

Proof. We proceed by induction on the number of vertices of G.

For graphs up to three vertices the result is clear.

Let G be a triangulated hamiltonian graph with more than three vertices.

Assume the result is valid for all triangulated hamiltonian graphs with the number of vertices less than $|V(G)|$.

Let γ be a hamiltonian cycle of G. By Theorem 3.1, there is a vertex $x \in V(G)$ such that its two neighbours in γ are adjacent in G.

Let $\gamma = (x, v_1, v_2, \ldots, v_n, x)$ and $\delta = (v_1, v_2, \ldots, v_n, v_1)$. Note that δ is a hamiltonian cycle of $G - \{x\}$, so $G - \{x\}$ is a triangulated hamiltonian graph whose complement $(G - \{x\})^c$ does not have induced subgraphs isomorphic to $G_1, G_2, G_3, G_4, G_5, G_6, G_7$ and with less vertices than G. By the induction hypothesis, $(G - \{x\})^c$ has at most one nontrivial component, and this component is K_3 or a star.

We proceed by cases, considering $(G - \{x\})^c$. In each case we reach the conclusion of the theorem or a contradiction.

Case 1. $(G - \{x\})^c$ has only trivial components.
In this case, it is clear that G^c has at most one nontrivial component, and it is a star centered at x.

Case 2. $(G - \{x\})^c$ has one nontrivial component, and it is K_2 with $[u, v]$ as an edge.
We proceed by considering two subcases.

Case 2.1. For every vertex $y \in V(G) \setminus \{x, u, v\}$, $[x, y] \notin E(G^c)$.
Clearly $E(G^c) \subseteq \{[u, v], [v, x], [x, u]\}$. It follows that G^c has one nontrivial component, and it is K_3 or a star with one or two peaks.

Case 2.2. There is a vertex $y \in V(G) \setminus \{x, u, v\}$, such that $[x, y] \in E(G^c)$.
As $[x, y]$ and $[u, v]$ are two nonadjacent edges of G^c, and G^c does not have an induced G_1, we have that they should have a common adjacent edge in G^c. The vertex x must be adjacent in G^c to at least one of the two vertices u, v; as $G^c[\{y\}]$ is a connected component of $(G - \{x\})^c$.

If x is adjacent in G^c to exactly one of two vertices u, v we can assume, without loss of generality, that $[x, u] \in E(G^c)$, $[x, v] \notin E(G)$. Since G is hamiltonian, x must have at least two neighbours in G. As $[x, u] \in E(G^c)$ and $[x, y] \in E(G^c)$, there must be at least another vertex $z \in V(G)$ (besides v) such that $[x, z] \in E(G)$. Notice that $G^c[\{z\}]$ is a connected component of $(G-\{x\})^c$. But now $G^c[\{u, v, x, y, z, w\}] \cong G_2$, contradicting the hypothesis.

If x is adjacent to both u, v in G^c, then, as G is hamiltonian, x must have at least two neighbours in G. As $\{[x, u], [x, v], [x, y]\} \subseteq E(G^c)$, there must be at least another two vertices $z, w \in V(G)$ such that $[x, z] \in E(G)$ and $[x, w] \in E(G)$. Recall that $G^c[\{z\}]$ and $G^c[\{w\}]$ are different connected components of $(G - \{x\})^c$. But now $G^c[\{u, v, x, y, z\}] \cong G_4$, contradicting the hypothesis.

Case 3. $(G - \{x\})^c$ has one nontrivial component, and it is an star with m peaks, $m > 1$.
Let v be the center of the star, and $u_1, u_2, \ldots, u_m$ be its peaks.
We have again two cases, and proceed to consider each one.

Case 3.1. x is adjacent to v in G^c, i.e., $[x, v] \in E(G^c)$.
Here we have to consider three cases.

Case 3.1.1. x is adjacent in G^c to every peak of the star.
As G is hamiltonian, there are at least two vertices, say y and z, adjacent to x in G.

Clearly, y is not adjacent to v in G^c or to any peak to the star, as $G^c[\{y\}]$ is a connected component of $(G-\{x\})^c$. Similarly, z is not adjacent to v in G^c or to any peak of the star. Recaling that the star has at least two peaks. We have $G^c[\{x, y, z, v, u_1, u_2\}] \cong G_5$. This contradicts the original hypothesis.

Case 3.1.2. There is exactly one peak of the star, say u_j, which is not adjacent to x in G^c.

Since the star has at least two peaks, there is at least one peak, say u_i, which is adjacent to x in G^c. Moreover, there is a vertex, say $y\,(y \neq u_j)$, which is adjacent to x in G (because G is hamiltonian) and, there is another vertex, namely $z\,(z \neq y)$, adjacent to v in G.

If $[x, z] \in E(G^c)$, then $G^c[\{x, y, z, v, u_j\}] \cong G_2$, a contradiction.

If $[x, z] \notin E(G^c)$, then $G^c[\{x, y, z, v, u_j, u_i\}] \cong G_4$, another contradiction.

Case 3.1.3. There are at least two peaks of the star, namely u_j and u_i, which are not adjacent to x in G^c.

There are at least two vertices, say z_1 and z_2, adjacent to v in G, because G is hamiltonian.

Now we consider three possible cases.

Case 3.1.3.1. x is not adjacent in G^c to any vertex of $V(G) \setminus \{x, v, u_1, \ldots, u_m\}$.

If x is not adjacent in G^c to any peak of the star, then G^c has only one nontrivial component, and it is a star with $m+1$ peaks.

If x is adjacent to some peak of the star, say u_k, then $G^c[\{x, v, u_j, u_k, z_1, z_2\}] \cong G_4$, a contradiction with the hypothesis of the theorem.

Case 3.1.3.2. There is a vertex of $V(G) \setminus \{x, v, u_1, \ldots, u_m\}$ which is adjacent to x in G^c, and there is another vertex of $V(G) \setminus \{x, v, u_1, \ldots, u_m\}$ which is not adjacent to x in G^c.

Assume, without loss of generality, that $[x, z_1] \in E(G^c)$ and that $[x, z_2] \notin E(G^c)$. We have that $G^c[\{x, v, u_j, z_1, z_2\}] \cong G_2$, a contradiction.

Case 3.1.3.3. x is adjacent in G^c to every vertex of $V(G) \setminus \{x, v, u_1, \ldots, u_m\}$.

Now, $G^c[\{x, v, u_i, u_j, z_1, z_2\}] \cong G_6$, another contradiction.

Case 3.2. x is not adjacent to v in G^c, i.e., $[x, v] \notin E(G^c)$.
We consider two possible cases.

Case 3.2.1. There is a vertex $y \in V(G) \setminus \{x, v, u_1, u_m\}$ such that $[x, y] \in E(G^c)$.

We have that x must be adjacent in G^c to every peak of the star. Indeed: G^c does not have induced subgraph isomorphic to G_1, by the condition in (3.2) that $[x, v] \notin E(G^c)$, and $G^c[\{y\}]$ is connected component of $(G - \{x\})^c$. Further, there is another vertex, say z $(z \neq v)$, which is adjacent to x in G, because G is hamiltonian. As $G^c[\{z\}]$ also is a connected component of $(G - \{x\})^c$, we have that $G^c[\{x, y, v, u_1, z\}] \cong G_2$, another contradiction with the hypothesis.

Case 3.2.2. For every vertex $y \in V(G) \setminus \{x, v, u_1, \ldots, u_m\}$ we have $[x, y] \notin E(G^c)$.

We state that there is a vertex z, besides x, which is adjacent to v in G.

If x is not adjacent in G^c to any peak of the star, then G^c has exactly one nontrivial component, and it is a star with m peaks.

If there are two peaks of the star, u_i and u_j, such that $[x, u_i] \in E(G^c)$ and $[x, u_j] \notin E(G^c)$, then $G^c[\{x, v, z, u_i, u_j\}] \cong G_2$, a contradiction.

Finally, if x is adjacent in G^c to every peak of the star, then there must be one more vertex w ($w \neq v, w \neq z$) adjacent to at least x or v one of the vertices in G, because the hamiltonian cycle γ can not contain three arcs $[v, x], [x, z]$ and $[z, v]$. Now if w is adjacent to both we have $G^c[\{x, v, u_1, u_2, z, w\}] \cong G_3$, one contradiction. If w is adjacent to exactly one of the vertices x, v in G, say $wx \in E(G)$ and $wv \in E(G^c)$, then $G^c[\{w, v, u_1, x, z\}] \cong G_2$, a contradiction.

Case 4. $(G - \{x\})^c$ has one nontrivial component and it is K_3.
Let $G^c[\{u, v, w\}]$ be the unique nontrivial component of $(G - \{x\})^c$.

Case 4.1. There is a vertex $y \in V(G) \setminus \{x, u, v, w\}$ such that $[x, y] \in E(G^c)$.

We have that x must be adjacent in G^c to at least two of the three vertices u, v, w, because G^c does not have induced subgraph isomorphic to G_1.

If x is adjacent to exactly two of the three vertices in G^c, say u and v, then it must be another vertex z, besides w, adjacent to x in G. But now we have $G^c[\{v, w, x, y, z]\} \cong G_2$, a contradiction.

If x is adjacent to the three vertices u, v, w in G^c, then there are two vertices, say z_1 and z_2, adjacent to x in G. We have that $G^c[\{v, w, x, y, z_1, z_2\}] \cong G_4$, another contradiction. We discard this case (4.1).

Case 4.2. For every vertex $y \in V(G) \setminus \{x, u, v, w\}$ we have $[x, y] \notin EG(^c)$.

If x is not adjacent to any of the three vertices u, v, w in G^c, then G^c has only one nontrivial component, and it is K_3.

If x is adjacent to exactly one of the three vertices in G^c, say to u, then there must be two vertices z_1, z_2 adjacent to u in G, As $[x, z_1] \notin E(G^c)$ and $[x, z_2] \notin E(G^c)$, we have that $G^c[\{u, v, w, x, z_1, z_2\}] \cong G_4$, a contradiction.

If x is adjacent to exactly two of the three vertices in G^c, say to u and v, then there must be two vertices z_1, z_2 adjacent to u in G, As $[x, z_1] \notin E(G^c)$ and $[x, z_2] \notin E(G^c)$, we have that $G^c[\{u, v, w, x, z_1, z_2\}] \cong G_5$, a contradiction.

Finally, if x is adjacent to the three vertices in G^c, then there must be other four vertices z_1, z_2, z_3, z_4 in G. Indeed: G is hamiltonian and the vertices in the hamiltonian cycle can not be repeated vertices. Recall that each one of these four vertices z_1, z_2, z_3, z_4 is a connected component of $(G - \{x\})^c$. As x is not adjacent in G^c to any of them, $G^c[\{x, u, v, w, z_1, z_2, z_3, z_4\}] \cong G_7$, another contradiction.

The proof of Theorem 3.2 is complete. ■

Remark 3.3. In Theorem 3.2, the hypothesis that G^c does not have induced subgraph isomorphic to G_1 can be dispensed, because it is implied by the assumption that G is triangulated.

4. Necessary Conditions for a Graph to be of Class E

Theorem 4.1. *If the graph G has an induced subdigraph G' such that $G' \notin E$, then $G \notin E$.*

Proof. Let G' be an induced sugraph of G, and suppose that $G' \notin E$. We proceed by showing an m-orientation-coloration D of the graph G without C_3 or T_3, whose closure $\zeta(D)$ is not a KP-digraph.

Since $G' \notin E$, there is an m-orientation-coloration D' of G' that it has not C_3 or T_3 and such that $\zeta(D')$ is not a KP-digraph.

We choose a colour that appears in D', say black. We define D to be an m-orientation-coloration of G such that:

(i) $D[V(G')] = D'$.

(ii) If $x \in V(D) \setminus V(D')$, $y \in V(G')$ and they are adjacent in G, let $(x, y) \in A(D)$ be black.

(iii) All other arcs be black and in any direction.

First we note that D is an m-orientation-coloration of G without C_3 or T_3, because D' does not have C_3 or T_3, and we have added only black arcs. Now notice that $\zeta(D)[V(G')] = \zeta(D')$, as no arc of D has its initial vertex in $V(D')$ and its final vertex in $V(D) \setminus V(D')$ and no monochromatic paths are created among the vertices of D'. It follows that $\zeta(D)$ is not a KP-digraph because it has an induced subdigraph which is not a KP-digraph. We conclude that $G \notin E$. ■

Theorem 4.2. *If G is a cycle of order higher than 3, then $G \notin E$.*

Proof. We proceed by showing a 3-orientation-coloration D without C_3 or T_3 of G such that $\zeta(D)$ is not a KP-digraph.

Let the arcs of D induce a directed cycle γ. Take three consecutive vertices in order x, y, z of γ. Let $(x, y) \in A(D)$ be blue $(y, z) \in A(D)$ be red and (z, γ, x) be a monochromatic green path.

It is readily seen that D does not have C_3 or T_3, $\ker(\zeta(D)[\{x, y, z\}]) = \emptyset$, and $\zeta(D)$ is not a KP-digraph. ■

Theorem 4.3. *If $G \in E$, then G is triangulated.*

Proof. By Theorem 4.1 and 4.2, G does not have induced cycles of order higher than 3. ∎

Theorem 4.4. *Let G be a graph of class E. If its complement G^c has two nonadjacent edges $h, k \in E(G^c)$, then there is an edge $\ell \in E(G^c)$ such that ℓ is adjacent in G^c to both h and k.*

Proof. We proceed by contradiction. Assume h and k have no common adjacent edge in G^c.

As h and k are not adjacent, they have four different vertices. The subgraph of G induced by these four vertices is isomorphic to C_4. G is not triangulated, contradicting Theorem 4.3. ∎

Theorem 4.5. *If G is a graph of class E, then its complement G^c has at most one nontrivial component.*

Proof. We proceed by contradiction. Assume G^c has two nontrivial components.

Taking one edge of each one of the two nontrivial components, G^c has two nonadjacent edges which do not have any common adjacent edge in G^c (otherwise they would be in the same component in G^c), contradicting Theorem 4.4. ∎

Theorem 4.6. *If G is a graph of class E, then its complement G^c does not have induced subgraphs isomorphic to $G_1, G_2, G_3, G_4, G_5, G_6, G_7$.*

Proof. We proceed by contradiction. If G^c has induced subgraph G_i, $i \in \{1, 2, \ldots, 6, 7\}$, then G has induced subgraph G_i^c. But D_i is a 3-orientation-coloration of G_i^c without C_3 or T_3, and $\zeta(D_i)[\{x, y, z\}]$ does not have a kernel. $\zeta(D_i)$ is not kernel-perfect, and G_i^c is not of class E, and by Theorem 4.1, G is not of class E, contradicting the hypothesis (see Figure i, $i \in \{1, 2, \ldots, 6, 7\}$). ∎

Theorem 4.7. *If G is a hamiltonian graph of class E, then its complement has at most one nontrivial component, and this component is K_3 or a star.*

Proof. By Theorem 4.6, G^c does not have induced subgraphs isomorphic to $G_1, G_2, G_3, G_4, G_5, G_6, G_7$. By Theorem 4.3, G is triangulated, so by Theorem 3.2 the complement of G has at most one nontrivial component, and it is K_3 or a star. ∎

Remark 4.8. If, in Theorem 4.7, we ask only that G be of class E, and allow G to be not hamiltonian, the result does not hold, as shows the following example (see Figure 8).

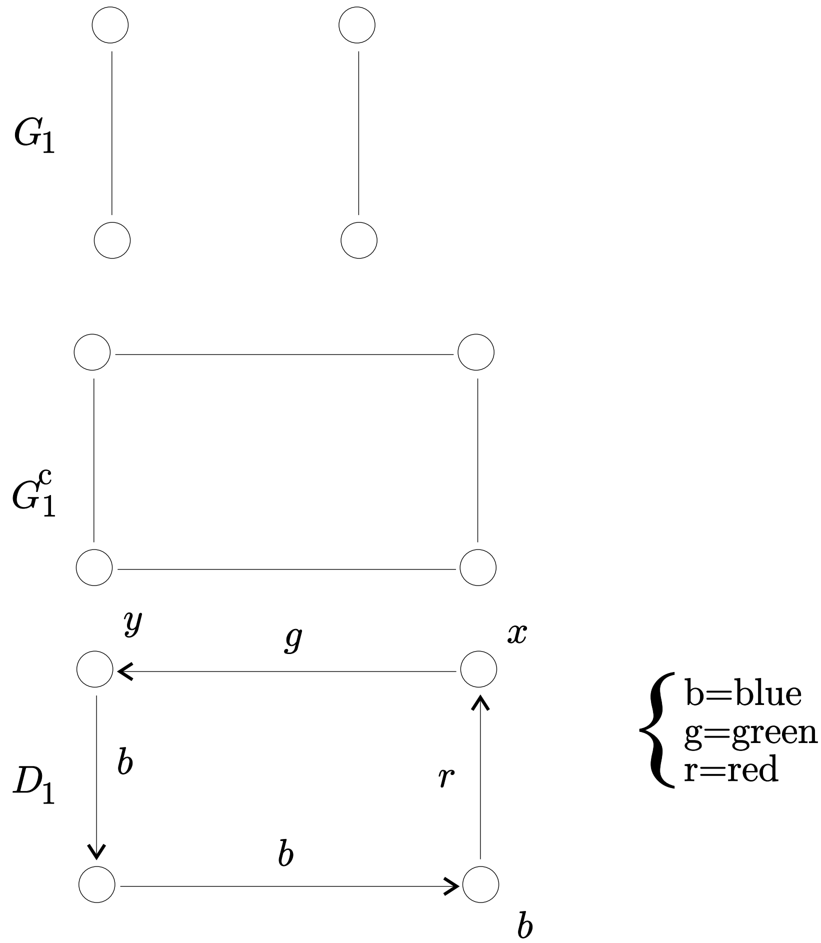


Figure 1

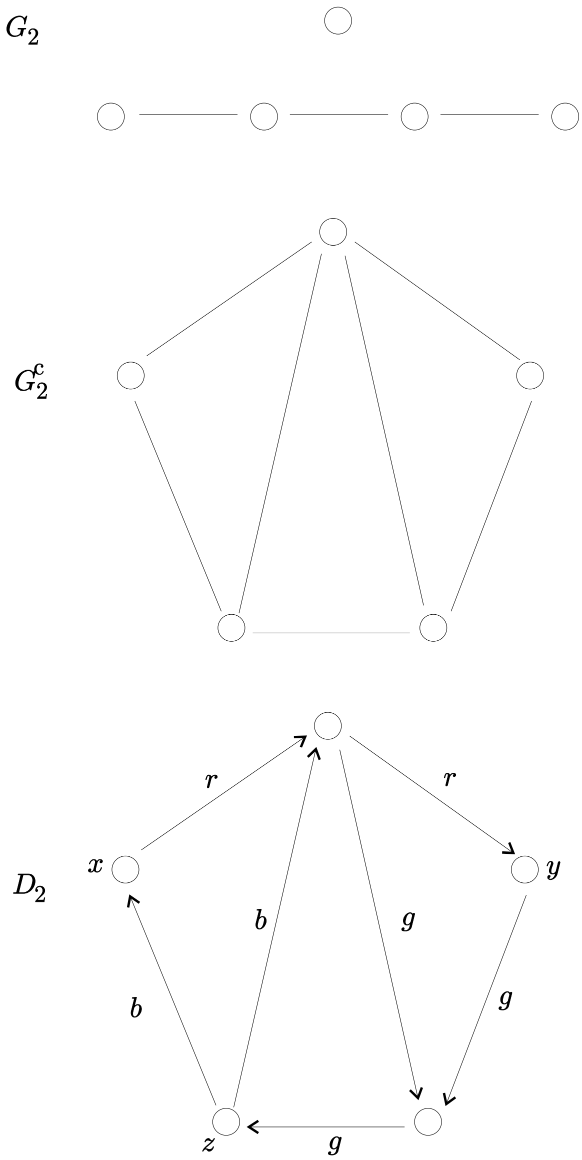
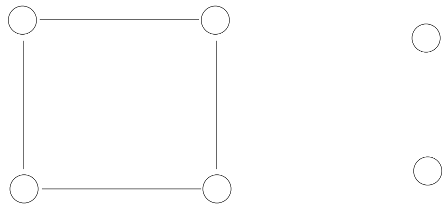
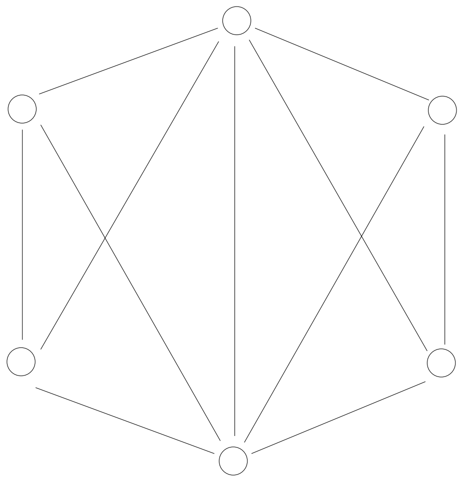


Figure 2

G_3



G_3^c



D_3

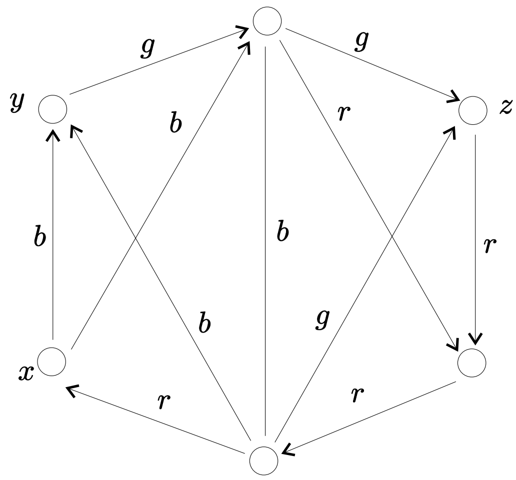


Figure 3

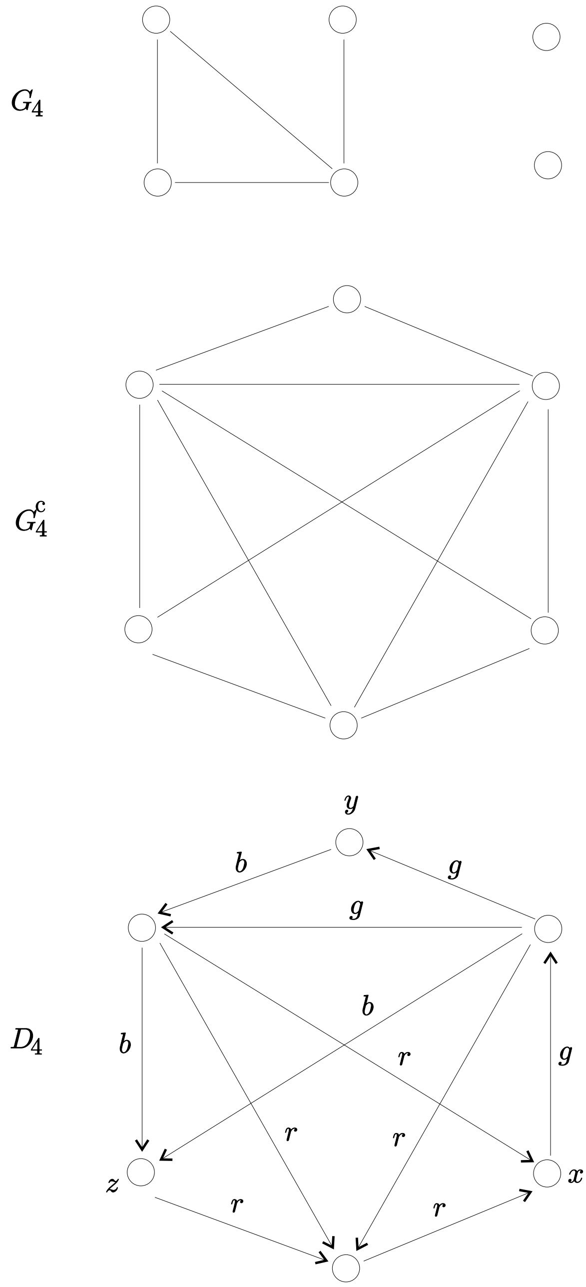


Figure 4

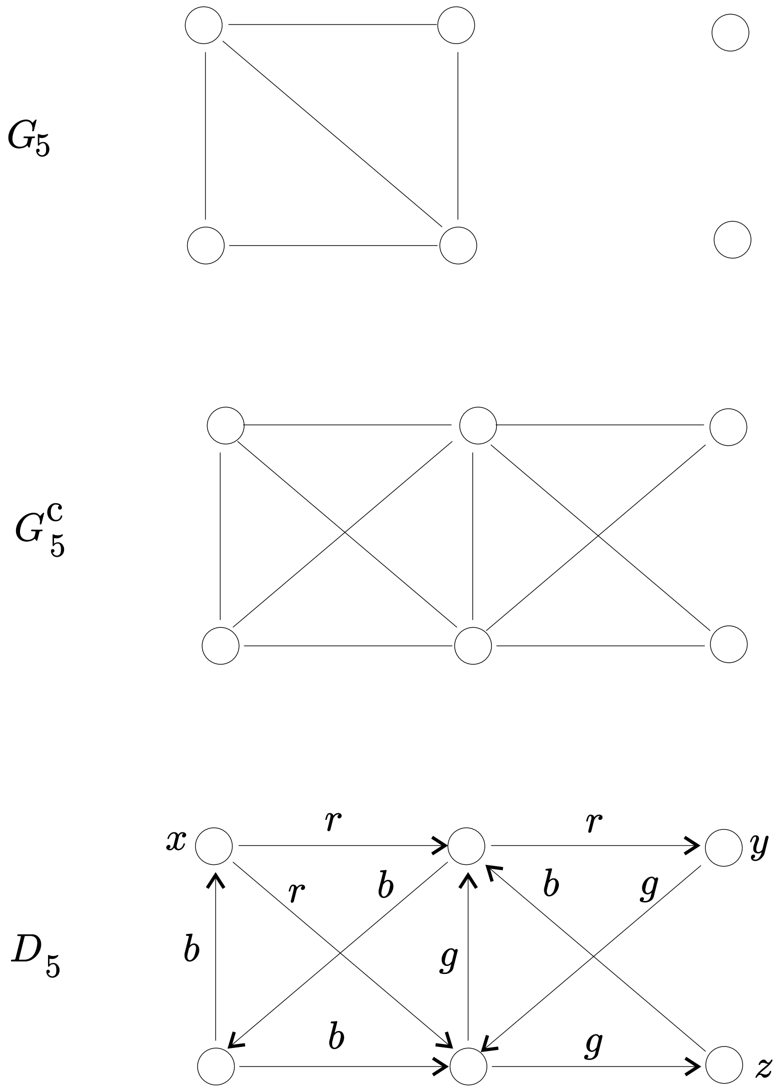


Figure 5

G_6

G_6^c

D_6

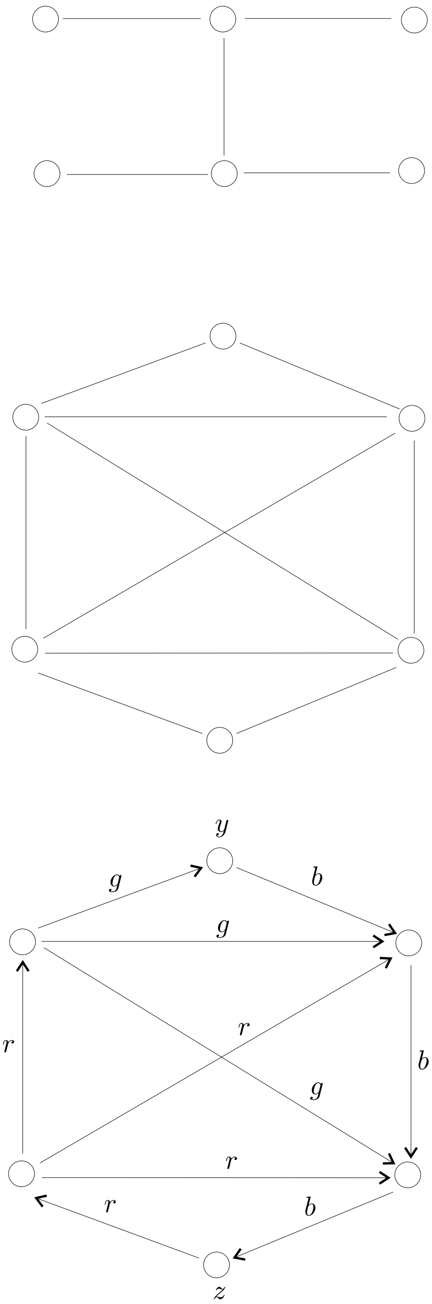


Figure 6

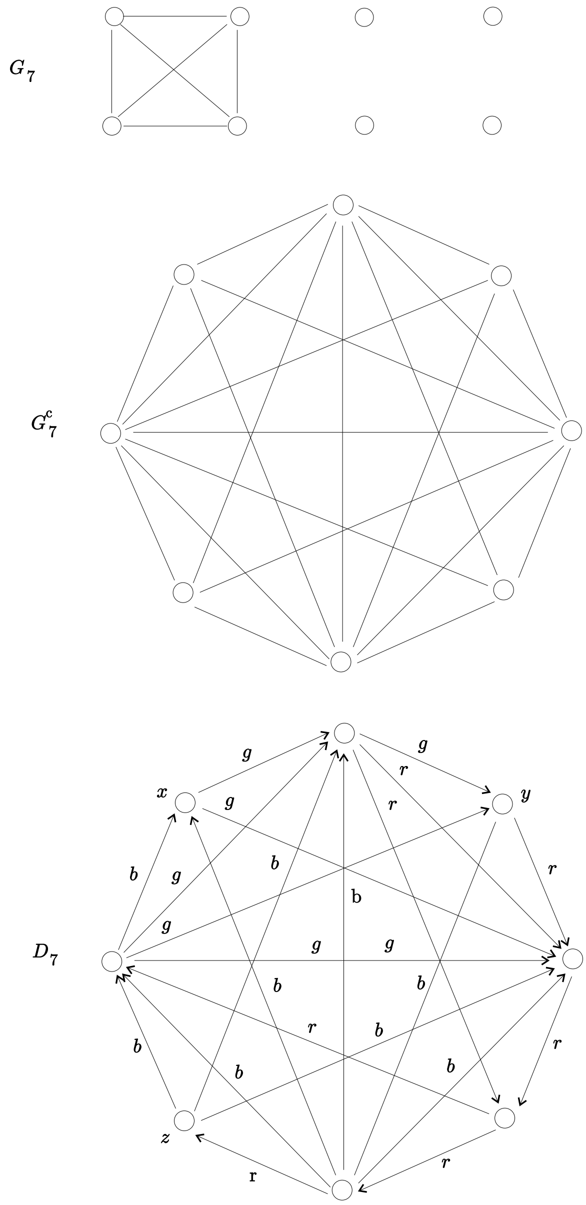


Figure 7

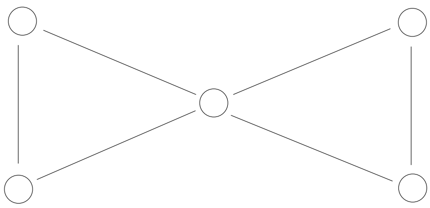


Figure 8

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References

[1] H. Galeana-Sánchez and J.J. García, *Kernels in the closure of coloured digraphs*, submitted.

[2] Shen Minggang, *On monochromatic paths in m-coloured tournaments*, J. Combin. Theory (B) **45** (1988) 108–111.

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