

## FULL DOMINATION IN GRAPHS

ROBERT C. BRIGHAM

*Department of Mathematics*  
*University of Central Florida, Orlando, FL 32816*

GARY CHARTRAND

*Department of Mathematics and Statistics*  
*Western Michigan University, Kalamazoo, MI 49008*

RONALD D. DUTTON

*Program of Computer Science*  
*University of Central Florida, Orlando, FL 32816*

AND

PING ZHANG<sup>1</sup>

*Department of Mathematics and Statistics*  
*Western Michigan University, Kalamazoo, MI 49008*

### Abstract

For each vertex  $v$  in a graph  $G$ , let there be associated a subgraph  $H_v$  of  $G$ . The vertex  $v$  is said to dominate  $H_v$  as well as dominate each vertex and edge of  $H_v$ . A set  $S$  of vertices of  $G$  is called a full dominating set if every vertex of  $G$  is dominated by some vertex of  $S$ , as is every edge of  $G$ . The minimum cardinality of a full dominating set of  $G$  is its full domination number  $\gamma_{FH}(G)$ . A full dominating set of  $G$  of cardinality  $\gamma_{FH}(G)$  is called a  $\gamma_{FH}$ -set of  $G$ . We study three types of full domination in graphs: full star domination, where  $H_v$  is the maximum star centered at  $v$ , full closed domination, where  $H_v$  is the subgraph induced by the closed neighborhood of  $v$ , and full open domination, where  $H_v$  is the subgraph induced by the open neighborhood of  $v$ .

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## 1. Introduction

A vertex  $v$  in a graph  $G$  is said to *dominate* itself and each of its neighbors. A set  $S \subseteq V(G)$  is called a *dominating set* for  $G$  if every vertex of  $G$  is dominated by some vertex of  $S$ . The minimum cardinality of a dominating set is the *domination number*  $\gamma(G)$  of  $G$ . A dominating set of cardinality  $\gamma(G)$  is a  $\gamma$ -set for  $G$ . There has been increased interest in recent years in the study of domination in graphs. Indeed, the books [2, 3] by Haynes, Hedetniemi, and Slater are devoted exclusively to this subject. In domination, a vertex dominates a set of vertices (according to some rule); while in covering, a vertex covers the edges incident with it. We combine these concepts to describe another variation of domination.

For a graph  $G$ , let  $H$  be a function that maps each vertex  $v$  of  $G$  into a subgraph  $H_v$  of  $G$ . In this context, the vertex  $v$  is said to *dominate*  $H_v$  as well as dominate each vertex and edge of  $H_v$ . A set  $S$  of vertices of  $G$  is called a *full dominating set* if every vertex and every edge of  $G$  is dominated by some vertex of  $S$ . For each full dominating set  $S$  of  $G$  and  $v \in V(G) - S$ , the set  $S \cup \{v\}$  is also a full dominating set. If  $G$  has no isolated vertices, then we need only be concerned with each edge of  $G$  being dominated by some vertex of  $S$ . The minimum cardinality of a full dominating set of  $G$  is its *full domination number* (with respect to the function  $H$ ) and is denoted by  $\gamma_{FH}(G)$ . A full dominating set of  $G$  of cardinality  $\gamma_{FH}(G)$  is called a  $\gamma_{FH}$ -set of  $G$ . Certainly,  $\gamma_{FH}(G)$  is defined for a graph  $G$  if and only if  $V(G)$  is a full dominating set for  $G$ .

In this paper we study three examples of full domination, namely *full star domination*, where  $H_v$  is the maximum star  $S_v$  centered at  $v$ , *full closed domination*, where  $H_v = \langle N[v] \rangle$ , the subgraph induced by the closed neighborhood of  $v$ , and *full open domination*, where  $H_v = \langle N(v) \rangle$ , the subgraph induced by the open neighborhood of  $v$ .

## 2. Full Star Domination in Graphs

We denote the *full star domination number* of a graph  $G$  by  $\gamma_{FS}(G)$ . Certainly,  $\gamma_{FS}(G)$  is defined for every graph  $G$ . Indeed, if  $G$  is a graph without isolated vertices, then  $\gamma_{FS}(G) = \alpha_o(G)$ , the vertex covering number of  $G$  (the minimum number of vertices that cover all edges of  $G$ ). If  $G$  has  $I(G)$  isolated vertices, then  $\gamma_{FS}(G) = \alpha_o(G) + I(G)$ . Therefore, the full star domination number is not a new parameter; it only provides a new setting for

an old one. A well-known theorem of Gallai [1] states that if  $G$  is a graph of order  $n$  without isolated vertices, then  $\alpha_o(G) + \beta_o(G) = n$ , where  $\beta_o(G)$  is the vertex independence number of  $G$ . This gives us the following.

**Observation 2.1.** *For every graph  $G$  of order  $n$  without isolated vertices,*

$$\gamma_{FS}(G) = n - \beta_o(G).$$

Since every full star dominating set of a graph is also a dominating set, it follows that  $\gamma(G) \leq \gamma_{FS}(G)$ . By Observation 2.1,

$$1 \leq \gamma(G) \leq \gamma_{FS}(G) \leq n - 1$$

for every graph  $G$  of order  $n$  with at most  $n - 2$  isolated vertices. We now consider the realizability of three integers  $a, b, n$  as the domination number, full star domination number, and order, respectively, of some graph without isolated vertices. Thus any such triple  $a, b, n$  described above must satisfy  $1 \leq a \leq b \leq n - 1$ . By Observation 2.1, however,  $\gamma_{FS}(G) = n - 1$  if and only if  $G = K_n$ , which implies that  $\gamma(G) = 1$ . Hence we may assume that  $1 \leq a \leq b \leq n - 2$ . On the other hand, the independent domination number  $i(G)$  satisfies

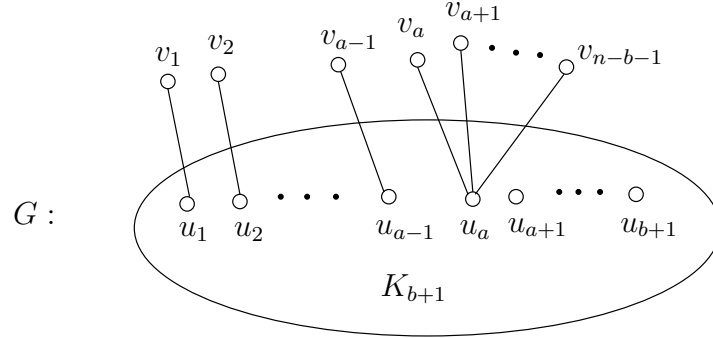
$$\gamma(G) \leq i(G) \leq \beta_o(G) = n - \gamma_{FS}(G).$$

This implies that  $\gamma(G) + \gamma_{FS}(G) \leq n$ , thereby obtaining Ore's [6] well-known inequality  $\gamma(G) \leq n/2$  for graphs  $G$  of order  $n$  without isolated vertices. We now present the desired realization result.

**Proposition 2.2.** *For every triple  $a, b, n$  of integers with  $n \geq 3$ ,  $1 \leq a \leq b \leq n - 2$ , and  $a + b \leq n$ , there exists a graph  $G$  of order  $n$  without isolated vertices such that  $\gamma(G) = a$  and  $\gamma_{FS}(G) = b$ .*

**Proof.** We consider two cases.

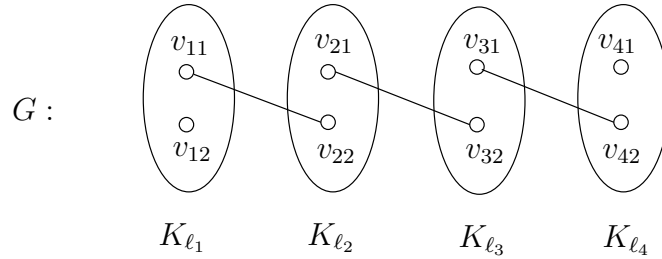
*Case 1.*  $a + b \leq n - 1$ . Let  $K_{b+1}$  be the complete graph with vertex set  $\{u_1, u_2, \dots, u_{b+1}\}$  and let  $G$  be the graph obtained from  $K_{b+1}$  by adding  $n - b - 1$  new vertices  $v_1, v_2, \dots, v_{n-b-1}$ , the  $a - 1$  edges  $u_i v_i$  ( $1 \leq i \leq a - 1$ ), and the  $n - b - a$  edges  $u_a v_i$  ( $a \leq i \leq n - b - 1$ ). The graph  $G$  is shown in Figure 1. Since  $\{u_1, u_2, \dots, u_a\}$  is a  $\gamma$ -set and  $\{u_1, u_2, \dots, u_b\}$  is a  $\gamma_{FS}$ -set for  $G$ , it follows that  $\gamma(G) = a$  and  $\gamma_{FS}(G) = b$ .

Figure 1. The graph  $G$  in Case 1

*Case 2.*  $a + b = n$ . Let  $K_{\ell_1}, K_{\ell_2}, \dots, K_{\ell_a}$  be complete graphs, where  $\ell_i \geq 2$  for all  $i$  and  $\sum_{i=1}^a \ell_i = n$ . Also, let  $v_{i1}$  and  $v_{i2}$  be distinct vertices in  $K_{\ell_i}$  ( $1 \leq i \leq a$ ). Then the graph  $G$  is obtained from the graph  $\bigcup_{i=1}^a K_{\ell_i}$  by adding the  $a - 1$  edges  $v_{i1} v_{i+1,2}$  for  $1 \leq i \leq a - 1$ . For  $a = 4$ , the graph  $G$  is shown in Figure 2. Since  $\{v_{11}, v_{21}, \dots, v_{a1}\}$  is a  $\gamma$ -set,  $\gamma(G) = a$ . On the other hand,

$$\bigcup_{i=1}^a V(K_{\ell_i}) - \{v_{12}, v_{22}, \dots, v_{a2}\}$$

is a  $\gamma_{FS}$ -set and so  $\gamma_{FS}(G) = n - a = b$ . ■

Figure 2. The graph  $G$  in Case 2 when  $a = 4$ 

### 3. Full Closed Domination in Graphs

Recall that a set  $S$  of vertices in a graph  $G$  is a *full closed dominating set* if every vertex and edge of  $G$  belongs to  $\langle N[v] \rangle$  for some  $v \in S$ . The minimum

cardinality of a full closed dominating set is the *full closed domination number*  $\gamma_{FC}(G)$ . A full closed dominating set of cardinality  $\gamma_{FC}(G)$  is referred to as a  $\gamma_{FC}$ -set. This parameter was first introduced by Sampathkumar and Neeralagi in [5], where it was called the neighborhood number of a graph, and further studied by Jayaram, Kwong, and Straight in [4]. The following two propositions appeared in [5].

**Proposition 3.1.** *For every graph  $G$ ,  $\gamma(G) \leq \gamma_{FC}(G) \leq \gamma_{FS}(G)$ .*

**Proposition 3.2.** *If  $G$  is a triangle-free graph, then  $\gamma_{FC}(G) = \gamma_{FS}(G)$ .*

The converse of Proposition 3.2 is not true in general unless  $\gamma_{FC}(G) = \gamma_{FS}(G) = 1$ , in which case  $G$  is a star. To see this, we recall that the *corona* of a graph  $G$  with  $V(G) = \{v_1, v_2, \dots, v_n\}$  is that graph of order  $2n$  obtained from  $G$  by adding  $n$  new vertices  $u_1, u_2, \dots, u_n$  and the  $n$  new edges  $u_i v_i$  ( $1 \leq i \leq n$ ). For  $n \geq 3$ , let  $G_n$  denote the corona of  $K_n$ . Define  $G_2$  as the graph obtained from  $G_3$  by deleting an end-vertex. Then  $\gamma_{FS}(G_n) = \gamma_{FC}(G_n) = n$  for  $n \geq 2$  but certainly  $G_n$  is not triangle-free.

If  $\gamma(G) = 1$ , then  $\gamma_{FC}(G) = 1$  while  $1 \leq \gamma_{FS}(G) \leq n - 1$ . For each integer  $k$  with  $1 \leq k \leq n - 1$ , the graph  $H$  obtained by deleting the edges of a complete subgraph of order  $n - k$  from  $K_n$  has  $\gamma(H) = \gamma_{FC}(H) = 1$  and  $\gamma_{FS}(H) = k$ . For  $\gamma(G) \geq 2$ , the following realization result appeared in [4].

**Theorem 3.3.** *For every triple  $a, b, c$  of integers with  $2 \leq a \leq b \leq c$ , there exists a graph  $G$  with  $\gamma(G) = a$ ,  $\gamma_{FC}(G) = b$ , and  $\gamma_{FS}(G) = c$ .*

It is often of interest to know how the value of a graphical parameter is affected when a small change is made in a graph. In this connection, we now consider this question in the case of  $\gamma_{FC}(G)$  when an edge is deleted from  $G$ . We show, in fact, that such an operation produces a graph whose full closed domination number differs from that of the original graph by at most 1.

**Proposition 3.4.** *For each edge  $e$  of a graph  $G$ ,*

$$|\gamma_{FC}(G) - \gamma_{FC}(G - e)| \leq 1.$$

**Proof.** Let  $e = uv$  be an edge of  $G$  and let  $S$  be a  $\gamma_{FC}$ -set of  $G - e$ . Then  $S \cup \{u\}$  is a full closed dominating set of  $G$ . So  $\gamma_{FC}(G) \leq |S \cup \{u\}| \leq \gamma_{FC}(G - e) + 1$ . Next we show that  $\gamma_{FC}(G - e) \leq \gamma_{FC}(G) + 1$ . We consider two cases.

*Case 1.* There exists a  $\gamma_{FC}$ -set  $S'$  of  $G$  such that  $u, v \notin S'$ . Then  $S'$  is a full closed dominating set of  $G - e$  as well. Therefore,  $\gamma_{FC}(G - e) \leq |S'| = \gamma_{FC}(G) < \gamma_{FC}(G) + 1$ .

*Case 2.* For every  $\gamma_{FC}$ -set  $S$  of  $G$ , at least one of  $u$  and  $v$  belongs to  $S$ . Since  $S \cup \{u, v\}$  is a full closed dominating set of  $G - e$ , it follows that

$$\gamma_{FC}(G - e) \leq |S \cup \{u, v\}| \leq |S| + 1 = \gamma_{FC}(G) + 1. \quad \blacksquare$$

The bounds presented in Proposition 3.4 are sharp. To see this, we consider the graph  $G$  of Figure 3, where  $\gamma_{FC}(G) = 3$  and the vertices of a  $\gamma_{FC}$ -set in  $G$  are indicated by solid circles. Observe that

$$\begin{aligned} \gamma_{FC}(G - e_0) &= \gamma_{FC}(G) = 3, \\ \gamma_{FC}(G - e_1) &= \gamma_{FC}(G) - 1 = 2, \\ \gamma_{FC}(G - e_2) &= \gamma_{FC}(G) + 1 = 4. \end{aligned}$$

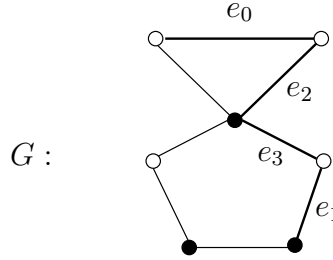


Figure 3. How the full closed domination number is affected by the removal of an edge

In view of Proposition 3.4, the edge set of a graph  $G$  can be partitioned into the following subsets:

$$\begin{aligned} E_0(G) &= \{e \in E(G) : \gamma_{FC}(G - e) = \gamma_{FC}(G)\}, \\ E_-(G) &= \{e \in E(G) : \gamma_{FC}(G - e) = \gamma_{FC}(G) - 1\}, \\ E_+(G) &= \{e \in E(G) : \gamma_{FC}(G - e) = \gamma_{FC}(G) + 1\}. \end{aligned}$$

The graph  $G$  of Figure 3 shows that it is possible for all three of these subsets to be nonempty for a single graph  $G$ . We now present some facts concerning elements in  $E_-(G)$  and  $E_+(G)$ .

**Proposition 3.5.** *Let  $G$  be a graph containing an edge  $e = uv$ . Then  $e \in E_-(G)$  if and only if for every  $\gamma_{FC}$ -set  $S'$  of  $G - e$ ,*

- (a) *neither  $u, v$ , nor any common neighbor of  $u$  and  $v$  belongs to  $S'$ , and*
- (b) *for each  $w \in N_G[u] \cap N_G[v]$ , the set  $S' \cup \{w\}$  is a  $\gamma_{FC}$ -set of  $G$ .*

**Proof.** Suppose that  $e = uv \in E_-(G)$ . Let  $S'$  be a  $\gamma_{FC}$ -set of  $G - e$ . We first verify (a). Assume, to the contrary, that either  $u$ ,  $v$ , or some common neighbor of  $u$  and  $v$  belongs to  $S'$ . Thus  $S'$  is also a full closed dominating set of  $G$ . So  $\gamma_{FC}(G) \leq |S'|$ , a contradiction. To verify (b), let  $w \in N_G[u] \cap N_G[v]$ . Then  $w \notin S'$  by (a). Let  $S = S' \cup \{w\}$ . Thus  $|S| = |S'| + 1 = \gamma_{FC}(G)$ . Since  $S$  is a full closed dominating set of  $G$ , it follows that  $S$  is a  $\gamma_{FC}$ -set for  $G$  and so (b) holds.

For the converse, let  $S'$  be a  $\gamma_{FC}$ -set of  $G - e$ , satisfying (a) and (b). It then follows from (a) that  $S' \cap (N_G[u] \cap N_G[v]) = \emptyset$ . Let  $w \in N_G[u] \cap N_G[v]$ . By (b) the set  $S = S' \cup \{w\}$  is a  $\gamma_{FC}$ -set of  $G$  and so  $|S| = |S'| + 1 = \gamma_{FC}(G)$ . Thus  $|S| = \gamma_{FC}(G) - 1$ , implying that  $e \in E_-(G)$ . ■

**Proposition 3.6.** *Let  $G$  be a graph containing an edge  $e = uv$ . Then  $e \in E_+(G)$  if and only if for every  $\gamma_{FC}$ -set  $S$  of  $G$ ,*

- (a)  $|S \cap \{u, v\}| = 1$ , and
- (b)  $S \cup \{u, v\}$  is a  $\gamma_{FC}$ -set of  $G - e$ .

**Proof.** Let  $e = uv \in E_+(G)$  and let  $S$  be a  $\gamma_{FC}$ -set of  $G$ . First we verify (a). Assume, to the contrary, that  $|S \cap \{u, v\}| \neq 1$ . If  $S \cap \{u, v\} = \emptyset$ , then, since  $e$  is dominated by some vertex in  $S$ , there is a vertex  $w \in S$  adjacent to both  $u$  and  $v$ . However, then,  $S$  is a full closed dominating set for  $G - e$ , contradicting the fact that  $e \in E_+(G)$ . On the other hand, if  $\{u, v\} \subseteq S$ , then, once again,  $S$  is full closed dominating set for  $G - e$ , a contradiction. Next we verify (b). Certainly,  $S \cup \{u, v\}$  is a full closed dominating set of  $G - e$ . By (a), however,

$$|S \cup \{u, v\}| = |S| + 1 = \gamma_{FC}(G) + 1 = \gamma_{FC}(G - e)$$

and so  $S \cup \{u, v\}$  is a  $\gamma_{FC}$ -set for  $G - e$ .

For the converse, let  $S$  be a  $\gamma_{FC}$ -set of  $G$  that satisfies (a) and (b). By (a),  $S$  contains exactly one of  $u$  and  $v$ . Let  $S' = S \cup \{u, v\}$  and so  $|S'| = |S| + 1$ . By (b),  $S'$  is a  $\gamma_{FC}$ -set of  $G - e$ . Thus  $\gamma_{FC}(G - e) = |S'| = |S| + 1 = \gamma_{FC}(G) + 1$ , implying that  $e \in E_+(G)$ . ■

If we were to delete two edges from  $G$ , one belonging to  $E_-(G)$  and the other belonging to  $E_+(G)$ , then the full closed domination number of the resulting graph is the same as  $\gamma_{FC}(G)$ .

**Proposition 3.7.** *Let  $G$  be a graph. If  $e_1 \in E_-(G)$  and  $e_2 \in E_+(G)$ , then*

$$\gamma_{FC}(G - e_1 - e_2) = \gamma_{FC}(G)$$

**Proof.** Removing  $e_1$  first and then  $e_2$  shows that  $\gamma_{FC}(G - e_1 - e_2) \leq \gamma_{FC}(G)$ ; while removing  $e_2$  first and then  $e_1$  produces the inequality  $\gamma_{FC}(G - e_2 - e_1) \geq \gamma_{FC}(G)$ . ■

Accordingly, if the edges  $e_1$  and  $e_2$  of graph  $G$  of Figure 3 are deleted, then  $\gamma_{FC}(G - e_1 - e_2) = 3$  since  $\gamma_{FC}(G) = 3$ . Observe that the edges  $e_1$  and  $e_2$  of this graph are not adjacent. This is no coincidence as we next show.

**Proposition 3.8.** *Let  $G$  be a graph. If  $e_1 \in E_-(G)$  and  $e_2 \in E_+(G)$ , then  $e_1$  and  $e_2$  are not adjacent in  $G$ .*

**Proof.** Assume, to the contrary, that there exists a graph  $G$  containing adjacent edges  $e_1$  and  $e_2$  with  $e_1 \in E_-(G)$  and  $e_2 \in E_+(G)$ . Let  $e_1 = uv$  and  $e_2 = vw$ . Let  $S'$  be a  $\gamma_{FC}$ -set for  $G - uv$ . By Proposition 3.5,  $v \notin S'$ . Let  $S'' = S' \cup \{v\}$  and consider the graph  $G - vw$ . The edge  $uv$  is dominated by  $v \in S''$ . Since  $vw$  is dominated by some vertex of  $S'$ , it follows that either  $vw$  is dominated by  $w \in S'$  or dominated by some  $x \in S'$ , where  $x$  is adjacent to both  $v$  and  $w$ . In either case,  $w$  is dominated in  $G - vw$  by some vertex of  $S''$ . Hence,  $S''$  is a full closed dominating set for  $G - vw$ . However,

$$|S''| = \gamma_{FC}(G - uv) + 1 = \gamma_{FC}(G) < \gamma_{FC}(G - vw),$$

which is impossible. ■

By Proposition 3.8 then, for the graph  $G$  of Figure 3, it follows that  $e_3 \in E_0(G)$ . Indeed, if  $G$  is a connected graph in which  $E_+(G) \neq \emptyset$  and  $E_-(G) \neq \emptyset$ , then  $E_0(G) \neq \emptyset$ . There are numerous graphs  $G$  in which every edge of  $G$  belongs to  $E_0(G)$ , such as even cycles and  $K_n$  ( $n \geq 3$ ). There is, however, no graph  $G$  in which every edge belongs to  $E_+(G)$ .

**Proposition 3.9.** *No graph  $G$  exists every edge of which belongs to  $E_+(G)$ .*



**Proof.** Assume, to the contrary, that there exists a graph  $G$  such that  $E(G) = E_+(G)$ . Let  $S$  be a  $\gamma_{FC}$ -set of  $G$ . Then, by Proposition 3.6, for every edge  $uv$  in  $G$ , one of  $u$  and  $v$  belongs to  $S$  and the other to  $V(G) - S$ . This implies that  $G$  is a bipartite graph with partite sets  $S$  and  $V(G) - S$ . Thus  $G$  is triangle-free. By Observation 2.1,  $\gamma_{FC}(G) = \gamma_{FS}(G) = \alpha_o(G)$ . However,

$$\gamma_{FC}(G) + 1 = \gamma_{FC}(G - e) = \alpha_o(G - e) \leq \alpha_o(G) = \gamma_{FC}(G)$$

for every edge  $e$  in  $G$ , which contradicts the fact that  $e \in E_+(G)$ . ■

There are graphs  $G$ , though, every edge of which belongs to  $E_-(G)$ . For example, odd cycles of order at least 5 have this property.

For a set  $S$  of vertices of a graph  $G$  and a vertex  $v$  of  $G$ , the *distance between  $v$  and  $S$*  is defined as

$$d(v, S) = \min\{d(v, u) : u \in S\}.$$

The *diameter* of  $S$  is defined as

$$\text{diam}S = \max\{d(u, v) : u, v \in S\}.$$

Thus  $\text{diam}V(G) = \text{diam}G$ .

For a nonempty set  $S$  of vertices in a connected graph  $G$ , a *Steiner  $S$ -tree* is a tree of minimum size in  $G$  containing  $S$ . Certainly, every end-vertex of a Steiner  $S$ -tree belongs to  $S$ . An edge  $e = uv$  in a Steiner  $S$ -tree  $T$  is called  *$S$ -free* if both  $u \notin S$  and  $v \notin S$ .

**Lemma 3.10.** *For every  $\gamma_{FC}$ -set  $S$  of a connected graph  $G$ , there exists a Steiner  $S$ -tree containing no  $S$ -free edges.*

**Proof.** Assume, to the contrary, that there is a connected graph  $G$  and a  $\gamma_{FC}$ -set  $S$  of  $G$  such that every Steiner  $S$ -tree in  $G$  contains  $S$ -free edges. Among all Steiner  $S$ -trees, let  $T$  be a Steiner  $S$ -tree containing a minimum number of  $S$ -free edges. Then  $T$  contains an  $S$ -free edge  $e = uv$  and a vertex  $x \in S$  such that  $x, u, v$  is a path in  $T$ . Since  $S$  is a  $\gamma_{FC}$ -set of  $G$ , it follows that  $e$  is dominated by some vertex in  $S$ . If  $e$  is dominated by  $x$ , then necessarily  $x$  is adjacent to both  $u$  and  $v$ . Hence  $(T - uv) + xv$  is a Steiner  $S$ -tree in  $G$  containing fewer  $S$ -free edges than  $T$ , which is impossible. Thus  $e$  is dominated by some vertex  $w \in S$ , where  $w \neq x$ . Let  $T_u$  and  $T_v$  be the two components of  $T - uv$ , where  $T_u$  contains  $u$  and  $T_v$  contains  $v$ . Necessarily,  $w$  belongs to exactly one of  $T_u$  and  $T_v$ , say  $T_u$ . Then  $(T - uv) + wv$  is a Steiner  $S$ -tree in  $G$  containing fewer  $S$ -free edges than  $T$ , again an impossibility. ■

**Lemma 3.11.** *If  $S$  is a  $\gamma_{FC}$ -set of a connected graph  $G$ , then the order of any Steiner  $S$ -tree is at most  $2\gamma_{FC}(G) - 1$ .*

**Proof.** Let  $T$  be a Steiner  $S$ -tree containing no  $S$ -free edges. Then  $V(T) = S \cup W$ , where  $S \cap W = \emptyset$ . Let  $|W| = a$ . Thus the order of  $T$  is  $a + \gamma_{FC}(G)$  and the size of  $T$  is  $a + \gamma_{FC}(G) - 1$ . Assume that  $a \geq \gamma_{FC}(G)$ . Since  $T$  is a Steiner  $S$ -tree, every end-vertex in  $T$  belongs to  $S$ . Thus every vertex in  $W$  has degree at least 2 in  $T$ . Also, since  $T$  has no  $S$ -free edge, every vertex in  $W$  is adjacent only to vertices of  $S$  in  $T$ .

Therefore, the size of  $T$  is at least

$$\sum_{w \in W} \deg_T w \geq 2a \geq a + \gamma_{FC},$$

producing a contradiction. ■

**Corollary 3.12.** *If  $S$  is a  $\gamma_{FC}$ -set in a connected graph  $G$ , then*

$$\text{diam}S \leq 2\gamma_{FC}(G) - 2.$$

**Proof.** Let  $T$  be a Steiner  $S$ -tree of  $G$  and suppose that the order of  $T$  is  $k$ . By Lemma 3.11,  $k \leq 2\gamma_{FC}(G) - 1$ . Among all trees of order  $k$ , the path  $P_k$  has the greatest diameter, namely  $k - 1$ , and  $k - 1 \leq 2\gamma_{FC}(G) - 2$ . ■

**Theorem 3.13.** *If  $G$  is a graph of diameter  $d$ , then*

$$\gamma_{FC}(G) \geq d/2.$$

**Proof.** Let  $x$  and  $y$  be vertices of  $G$  such that  $d(x, y) = d$  and let  $S$  be a  $\gamma_{FC}$ -set in  $G$ . Then  $x$  is dominated by some vertex  $u \in S$  and  $y$  is dominated by some  $v \in S$ . Either  $u = x$  or  $ux \in E(G)$ . Similarly, either  $v = y$  or  $vy \in E(G)$ . Thus, using Corollary 3.12, we have

$$d = \text{diam}G = d(x, y) \leq d(u, v) + 2 \leq \text{diam}S + 2 \leq 2\gamma_{FC}(G),$$

producing the desired result. ■

To show that the bound presented in Theorem 3.13 is sharp, let  $G = P_{2k+1}$  be the path of order  $2k + 1$ . Then  $\text{diam}G = 2k$  and  $\gamma_{FC}(G) = k$ , as desired.

A set  $S$  of vertices in a graph  $G$  is an *open dominating set* (or *total dominating set*) if every vertex of  $G$  is adjacent to at least one vertex of  $S$ . An open dominating set of minimum cardinality is a *minimum open dominating*

set and its cardinality is the *open domination number*  $\gamma_t(G)$ , also called the *total domination number*. The open domination number is also referred to as the *total domination number*. No graph containing isolated vertices has an open dominating set.

In order to obtain a relationship between the open domination number and the full closed domination number, we present the following lemma.

**Lemma 3.14.** *For every  $\gamma_{FC}$ -set  $S$  in a connected graph and each vertex  $v \in S$ ,*

$$d(v, S - \{v\}) \leq 2.$$

**Proof.** Assume, to the contrary, that there is a  $\gamma_{FC}$ -set  $S$  in a connected graph  $G$  and a vertex  $v \in S$  such that  $d(v, S - \{v\}) = k \geq 3$ . Let  $w \in S$  such that  $d(v, w) = d(v, S - \{v\})$  and let  $P : v = u_0, u_1, \dots, u_k = w$  be a  $v - w$  geodesic in  $G$ . Thus, neither  $u_1$  nor  $u_2$  is in  $S$ ; for otherwise,  $d(v, S - \{v\}) \leq 2 < k$ . Hence the edge  $e = u_1 u_2$  is dominated by some vertex  $y \in S$  (that is necessarily adjacent to both  $u_1$  and  $u_2$ ). Hence  $d(v, y) \leq 2 < k$ , a contradiction. ■

We have already seen (in Corollary 3.12) that if  $S$  is a  $\gamma_{FC}$ -set in a connected graph  $G$ , then  $\text{diam} S \leq 2\gamma_{FC}(G) - 2$ . We now show that  $\gamma_t(G)$  has a similar upper bound.

**Theorem 3.15.** *For every connected graph  $G$ ,*

$$\gamma_t(G) \leq 2\gamma_{FC}(G) - 1.$$

**Proof.** Let  $S$  be a  $\gamma_{FC}$ -set. Since  $S$  is also a dominating set for  $G$ , every vertex in  $V(G) - S$  is dominated by and therefore adjacent to some vertex in  $S$ . Consequently,  $S$  openly dominates all vertices in  $V(G) - S$ . By Lemma 3.14, for every vertex  $u \in S$ , there is a vertex  $v (\neq u)$  in  $S$  such that  $d(u, v) \leq 2$ . If every vertex in  $S$  is adjacent to some vertex in  $S$ , then  $S$  is also an open dominating set and so  $\gamma_t(G) \leq \gamma_{FC}(G)$ . On the other hand, suppose that there is a vertex  $x \in S$  that is adjacent to no vertex in  $S$ . Then there is a vertex  $y \in S$  such that  $d(x, y) = 2$ . Let  $w$  be a vertex of  $G$  adjacent to  $x$  and  $y$ . Hence  $w \notin S$ . For each vertex  $u \in S - \{x, y\}$ , let  $u'$  be a vertex of  $G$  that is adjacent to  $u$ . So  $u'$  may or may not be in  $S$ . Let  $S' = \{u' \mid u \in S - \{x, y\}\}$ . Then  $S \cup S' \cup \{w\}$  is an open dominating set,  $|S \cup S' \cup \{w\}| \leq 2\gamma_{FC}(G) - 1$ , and so  $\gamma_t(G) \leq 2\gamma_{FC}(G) - 1$ . ■

To see that the upper bound in Theorem 3.15 is sharp, we show that for each integer  $k \geq 5$ , there exists a connected graph  $G_k$  such that  $\gamma_{FC}(G_k) = k$

and  $\gamma_t(G) = 2k - 1$ . Let  $G_k$  be the graph obtained from a cycle  $C_k : v_0, v_1, v_2, \dots, v_{k-1}, v_0$  by (i) adding a vertex  $a_0$  and the edges  $a_0v_0$  and  $v_0v_i$  for  $2 \leq i \leq k - 2$  and (ii) adding the vertices  $a_i$  and  $b_i$  ( $1 \leq i \leq k - 1$ ) and the edges  $a_ib_i$  and  $b_iv_i$  for  $1 \leq i \leq k - 1$ . The graph  $G_5$  is shown in Figure 4.

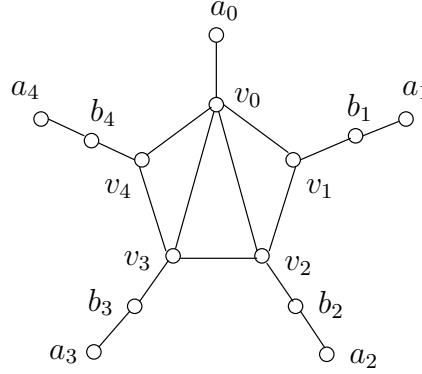


Figure 4. The graph  $G_5$

#### 4. Full Open Domination in Graphs

A vertex  $v$  in a graph  $G$  *openly dominates* the subgraph  $\langle N(v) \rangle$  induced by the (open) neighborhood  $N(v)$  of  $v$ , but  $v$  does not openly dominate itself or any edge incident with it. A set  $S$  of vertices in  $G$  is a *full open dominating set* if every vertex and every edge of  $G$  belongs to  $\langle N(v) \rangle$  for some  $v \in S$ . The minimum cardinality of a full open dominating set is the *full open domination number*  $\gamma_{FO}(G)$ . A full open dominating set of cardinality  $\gamma_{FO}(G)$  is referred to as a  $\gamma_{FO}$ -set. Note that a graph  $G$  has a full open dominating set if and only if  $G$  contains no isolated vertices and every edge of  $G$  lies on a triangle in  $G$ . Consequently, we have the following.

**Observation 4.1.** *Let  $S$  be a full open dominating set in a graph  $G$ . Every vertex of  $S$  (and consequently every edge joining two vertices of  $S$ ) belongs to a triangle every vertex of which belongs to  $S$ .*

To illustrate these concepts, consider the graphs  $G_1 = P_5 + K_1$  and  $G_2 = K_{2,2,2}$  shown in Figure 5. In  $G_1$ , since each edge  $v_i v_{i+1}$  ( $1 \leq i \leq 4$ ) is openly dominated only by  $u$ , the edge  $uv_1$  is openly dominated only by  $v_2$ , and the edge  $uv_5$  is openly dominated only by  $v_4$ , it follows that  $u, v_2, v_4$

belong to every  $\gamma_{FO}$ -set of  $G_1$ . However, the set  $\{u, v_2, v_4\}$  is not a full open dominating set of  $G_1$  as the edges  $uv_2$  and  $uv_4$  are not openly dominated by any vertex in  $\{u, v_2, v_4\}$ . Since  $S_1 = \{u, v_2, v_3, v_4\}$  is a full open dominating set,  $S_1$  is a  $\gamma_{FO}$ -set of  $G_1$  and so  $\gamma_{FO}(G_1) = 4$ . In  $G_2$ , the set  $S_2 = \{x_1, x_2, x_3\}$  is a full open dominating set. Moreover, there is no 2-element full open dominating set. Thus,  $S_2$  is a  $\gamma_{FO}$ -set of  $G_2$  and  $\gamma_{FO}(G_2) = 3$ .

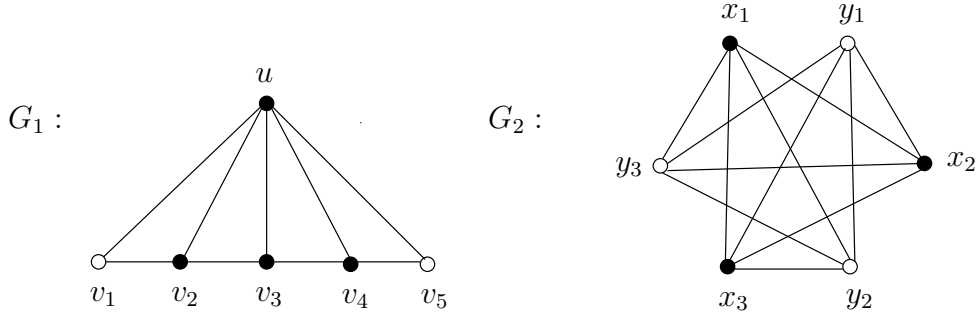


Figure 5. Graphs  $G_1 = P_5 + K_1$  and  $G_2 = K_{2,2,2}$

By Observation 4.1, every full open dominating set of a graph  $G$  must contain at least three vertices and so  $\gamma_{FO}(G) \geq 3$ . Certainly, every full open dominating set of a graph  $G$  is also a full closed dominating set and so  $\gamma_{FO}(G) \geq \gamma_{FC}(G)$ . This observation yields the following lower bound for  $\gamma_{FO}(G)$ .

**Corollary 4.2.** *For a graph  $G$  without isolated vertices and in which every edge belongs to a triangle,*

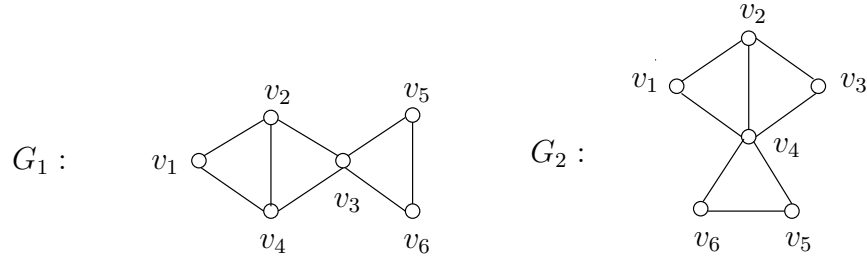
$$\gamma_{FO}(G) \geq \max\{3, \gamma_{FC}(G)\}.$$

Certainly, if  $G$  is a nontrivial connected graph of order  $n$ , then  $1 \leq \gamma(G) \leq \gamma_{FC}(G) \leq \gamma_{FS}(G) \leq n - 1$ . Hence no nontrivial connected graph  $G$  of order  $n$  has  $\gamma_{FC}(G) = n$  or  $\gamma_{FS}(G) = n$ . However, this is not true for  $\gamma_{FO}(G)$ , as we show next. For a graph  $G$  consisting of  $k$  ( $k \geq 1$ ) disjoint copies of a graph  $H$ , we write  $G = kH$ . In particular,  $G = H$  for  $k = 1$ .

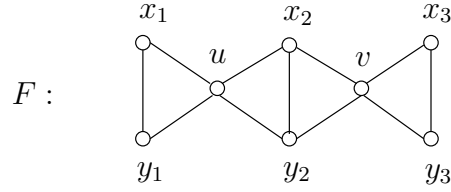
**Theorem 4.3.** *For  $n \geq 3$ , there exists a connected graph  $G$  of order  $n$  such that  $\gamma_{FO}(G) = n$  if and only if  $n \notin \{4, 6\}$ .*

**Proof.** We first show that for  $n = 4$  or  $n = 6$ , there is no connected graph  $G$  of order  $n$  with  $\gamma_{FO}(G) = n$ . If  $n = 4$ , then  $K_4 - e$  and  $K_4$  are the

only graphs of order 4 in which every edge belongs to a triangle. However,  $\gamma_{FO}(K_4 - e) = \gamma_{FO}(K_4) = 3$ . Next we show that there is no connected graph of order 6 with full open domination number 6. Assume, to the contrary, that there is a connected graph  $G$  of order 6 such that  $\gamma_{FO}(G) = 6$ . Let  $V(G) = \{v_1, v_2, \dots, v_6\}$ . Since every edge of  $G$  belongs to a triangle in  $G$ , there exist at least two triangles in  $G$  that have a common edge. Thus  $G$  contains  $K_4 - e$  as a subgraph. Assume, without loss of generality, that  $v_2v_4 \in E(G)$ ,  $v_i v_{i+1} \in E(G)$  for  $i = 1, 2, 3$ , and  $v_1v_4 \in E(G)$ . Since  $\{v_1, v_2, v_3, v_4\}$  openly dominates the induced subgraph  $\langle\{v_1, v_2, v_3, v_4\}\rangle$ , which is  $K_4 - e$  or  $K_4$ , the vertex  $v_5$  must openly dominate  $v_6$  or some edge incident with  $v_6$ . If  $v_5$  openly dominates  $v_6$ , then  $v_5v_6 \in E(G)$ . However, then, the edge  $v_5v_6$  must be openly dominated by some vertex  $v_i$  for  $i \in \{1, 2, 3, 4\}$ . If  $v_5$  openly dominates some edge  $e$  that is incident with  $v_6$ , then  $e = v_i v_6$  for some  $i \in \{1, 2, 3, 4\}$  and  $v_5$  is adjacent to both  $v_i$  and  $v_6$ . In either case,  $G$  contains a triangle  $v_i, v_5, v_6, v_i$  for some  $i \in \{1, 2, 3, 4\}$ . Thus, we may assume that  $G$  contains at least one of the two graphs  $G_1$  or  $G_2$  of Figure 6 as a subgraph. Let  $S = V(G) - \{v_1\}$ . Since  $\langle S \rangle$  contains  $2K_2 + K_1$  as a subgraph, every edge of  $\langle S \rangle$  belongs to a triangle. Thus  $S$  is a full open dominating set of  $\langle S \rangle$ . Moreover, the vertex  $v_1$  and all edges incident with  $v_1$  are openly dominated by  $S$ . This implies that  $S$  is also a full open dominating set of  $G$ . Therefore,  $\gamma_{FO}(G) \leq |S| = 5$ , which is a contradiction.

Figure 6. Subgraphs  $G_1$  and  $G_2$ 

For the converse, assume that  $n \geq 3$  but  $n \neq 4, 6$ . We construct a graph  $G$  of order  $n$  with  $\gamma_{FO}(G) = n$ . If  $n = 2k + 1$  for some integer  $k \geq 1$ , let  $G = kK_2 + K_1$  for some positive integer  $k$ . Then the order of  $G$  is  $2k + 1$ . Since  $V(G)$  is the only full open dominating set,  $\gamma_{FO}(G) = 2k + 1$ . Now let  $n = 2k$  some integer  $k \geq 4$ . For  $k = 4$ , let  $F$  be the graph of Figure 7. Note that for every vertex  $w$  in  $G$ , there is an edge that is only openly dominated by  $w$ . Hence  $V(G)$  is the only full open dominating set and so  $\gamma_{FO}(G) = 8$ .

Figure 7. A graph  $F$  of order 8 with  $\gamma_{FO}(F) = 8$ 

For  $k \geq 5$ , let  $G$  be obtained from the graph  $F$  of Figure 7 and the graph  $(k-4)K_2$  by joining every vertex of  $(k-4)K_2$  to the vertex  $v$  in  $F$ . Then the order of  $G$  is  $2k$  and  $\gamma_{FO}(G) = 2k$ , as desired. ■

Since there is no graph  $G$  of order 4 and  $\gamma_{FO}(G) = 4$  while the disconnected graph  $G = 2K_3$  has order 6 and  $\gamma_{FO}(G) = 6$ , we have the following corollary.

**Corollary 4.4.** *For  $n \geq 3$ , there exists a graph  $G$  of order  $n$  such that  $\gamma_{FO}(G) = n$  if and only if  $n \neq 4$ .*

We have seen that if  $G$  is a graph in which every edge belongs to a triangle and  $\gamma_{FC}(G) = a$  and  $\gamma_{FO}(G) = b$ , then  $1 \leq a \leq b$  and  $b \geq 3$ . Next we show that the converse of this fact is also true.

**Theorem 4.5.** *For each pair  $a, b$  of integers with  $1 \leq a \leq b$  and  $b \geq 3$ , there exists a connected graph  $G$  in which every edge belongs to a triangle with  $\gamma_{FC}(G) = a$  and  $\gamma_{FO}(G) = b$ .*

**Proof.** We consider three cases, according to whether  $a = 1$ ,  $a = 2$ , or  $a \geq 3$ .

*Case 1.*  $a = 1$ . Suppose, first, that  $b$  is odd. Then  $b = 2k + 1$  for some integer  $k \geq 1$ . Let  $G = kK_2 + K_1$ , where  $\deg_G u = 2k$ . Since  $\{u\}$  is a full closed dominating set,  $\gamma_{FC}(G) = 1$ . Moreover,  $V(G)$  is the only full open dominating set, so  $\gamma_{FO}(G) = |V(G)| = 2k + 1 = b$ . Next suppose that  $b$  is even. Then  $b = 2k$  for some integer  $k \geq 2$ . Here we let  $G = (P_5 \cup (k-2)K_2) + K_1$ , where  $P_5 : v_1, v_2, v_3, v_4, v_5$  and  $\deg_G u = 2k + 1$ . Then  $G = P_5 + K_1$  (shown in Figure 5) for  $k = 2$ . Again,  $\{u\}$  is a full closed dominating set and so  $\gamma_{FC}(G) = 1$ . On the other hand, the set  $\{u, v_2, v_3, v_4\} \cup V((k-2)K_2)$  is a  $\gamma_{FO}$ -set, implying that  $\gamma_{FO}(G) = 2k$ .

*Case 2.*  $a = 2$ . For  $b = 3$ , let  $G = K_{2,2,2}$  (shown in Figure 5). Then  $\gamma_{FC}(G) = 2$  and  $\gamma_{FO}(G) = 3$ . For  $b = 2k + 1$ , where  $k \geq 2$ , let  $G$  be

obtained from  $K_{2,2,2}$  and  $F = (k-1)K_2 + K_1$ , where  $\deg_F u = 2k-2$ , by identifying some vertex in  $K_{2,2,2}$  and the vertex  $u$  in  $F$ . Then  $\gamma_{FC}(G) = 2$  and  $\gamma_{FO}(G) = 3 + 2(k-1) = 2k+1 = b$ .

Now suppose that  $b$  is even. Then  $b = 2k$  for some integer  $k \geq 2$ . Let  $G_1 = P_5 + \bar{K}_2$ , where  $P_5 : v_1, v_2, v_3, v_4, v_5$  and the remaining two vertices of  $G_1$  are  $u$  and  $v$ ; and for  $k \geq 3$ , let  $G_2 = (k-2)K_2 + K_1$ , where  $\deg_{G_2} x = 2k-4$ . For  $k = 2$ , let  $G = G_1$  and for  $k \geq 3$ , let  $G$  be the graph obtained from  $G_1$  and  $G_2$  by identifying  $u$  and  $x$ . Then  $\gamma_{FC}(G) = 2$ , while  $\gamma_{FO}(G) = 4 + 2(k-2) = 2k = b$ .

*Case 3.*  $a \geq 3$ . We consider three subcases here, depending on whether  $b = a$ ,  $b = a+1$ , or  $b \geq a+2$ .

*Subcase 3.1.*  $b = a$ . Let  $F_0$  be a copy of the complete graph  $K_a$  with  $V(F_0) = \{u_1, u_2, \dots, u_a\}$ . For each  $i$  with  $1 \leq i \leq a$ , let  $F_i : x_i, y_i$  be a copy of  $K_2$ . Then let  $G$  be the graph obtained from the graphs  $F_i$  ( $0 \leq i \leq a$ ) by adding the  $4a$  new edges  $u_{i-1}x_i$ ,  $u_i x_i$ ,  $u_i y_i$ , and  $u_{i+1}y_i$  for all  $1 \leq i \leq a$ , where each subscript is one of the integers  $1, 2, \dots, a$  modulo  $a$ . The graph  $G$  is shown in Figure 8 for  $a = 3, 4$ . We show that  $\gamma_{FO}(G) = \gamma_{FC}(G) = a$ .

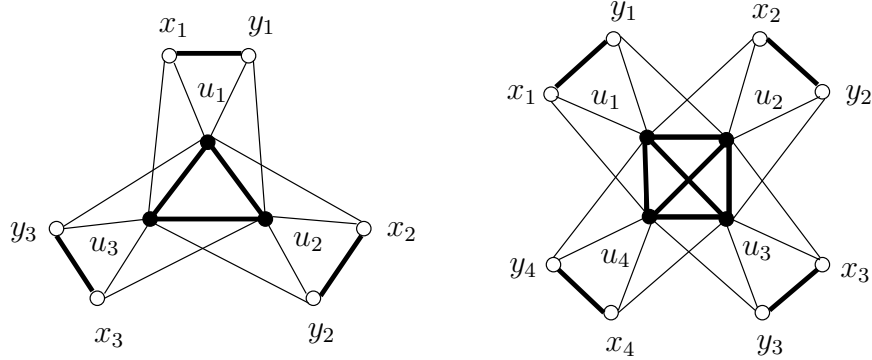


Figure 8. Graphs  $G$  with  $\gamma_{FO}(G) = \gamma_{FC}(G) = a$  for  $a = 3, 4$

Let  $S = \{u_1, u_2, \dots, u_a\}$ . Since  $S$  is a full closed and full open dominating set of  $G$ , it follows that  $\gamma_{FC}(G) \leq a$  and  $\gamma_{FO}(G) \leq a$ . On the other hand, each edge  $x_i y_i$  ( $1 \leq i \leq k$ ) in  $G$  is dominated only by  $x_i$ ,  $y_i$ , or  $u_i$ . Hence every full closed dominating set of  $G$  must contain at least one vertex from each set  $\{u_i, x_i, y_i\}$  for all  $1 \leq i \leq a$ . Thus  $\gamma_{FC}(G) \geq a$ . Since  $\gamma_{FO}(G) \geq \gamma_{FC}(G)$ , it follows that  $\gamma_{FO}(G) \geq a$ . Therefore,  $\gamma_{FO}(G) = \gamma_{FC}(G) = a$ .



*Subcase 3.2.*  $b = a + 1$ . Let  $G$  be obtained from the graph  $G$  constructed in Subcase 3.1 by first subdividing the edge  $x_1y_1$  into  $x_1z$  and  $zy_1$  and then adding the edge  $zu_1$ . The graph  $G$  is shown in Figure 9 for  $a = 3$ . Since  $S = \{u_1, u_2, \dots, u_a\}$  is a  $\gamma_{FC}$ -set of  $G$ , it follows that  $\gamma_{FC}(G) = a$ . On the other hand,  $S \cup \{x_1\}$  is a  $\gamma_{FO}$ -set and so  $\gamma_{FO}(G) = a + 1$ .

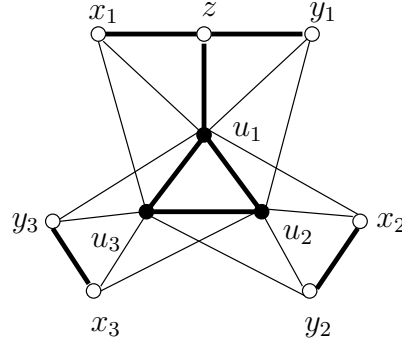


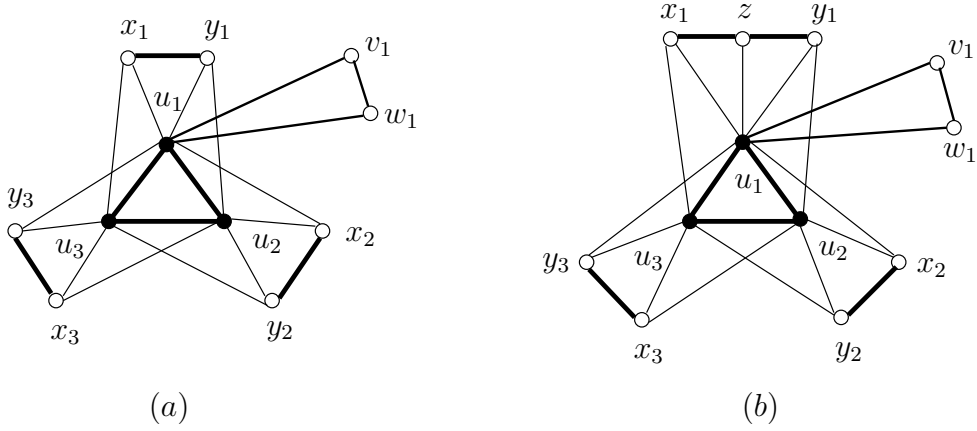
Figure 9. A graph  $G$  with  $\gamma_{FO}(G) = 4$  and  $\gamma_{FC}(G) = 3$

*Subcase 3.3.*  $b \geq a + 2$ . Suppose first that  $b = a + 2k$ , where  $k \geq 1$ . For each integer  $i$  with  $1 \leq i \leq k$ , let  $H_i : v_i, w_i$  be a copy of  $K_2$ . Now let  $H$  be the graph obtained from the graph  $G$  in Subcase 3.1 and the graphs  $H_i$  ( $1 \leq i \leq k$ ) by adding the  $2k$  edges  $u_1v_i, u_1w_i$  for all  $i$  with  $1 \leq i \leq k$ . For  $a = 3$  and  $k = 1$  (so  $b = 5$ ), the graph  $H$  is shown in Figure 10(a).

Next, suppose that  $b = a + 2k + 1$ , where  $k \geq 1$ . Let  $H$  be the graph obtained from the graph  $G$  in Subcase 3.2 and the graphs  $H_i$  ( $1 \leq i \leq k$ ) by adding the  $2k$  edges  $u_1v_i, u_1w_i$  for all  $1 \leq i \leq k$ . For  $a = 3$  and  $k = 1$  (so  $b = 6$ ), the graph  $H$  is shown in Figure 10(b).

Since  $S = \{u_1, u_2, \dots, u_a\}$  is a  $\gamma_{FC}$ -set in  $H$  for all  $b \geq a + 2$ , it follows that  $\gamma_{FC}(H) = a$ . Next we show that  $\gamma_{FO}(H) = b$ . Let  $V = \{v_1, v_2, \dots, v_k\}$  and  $W = \{w_1, w_2, \dots, w_k\}$ . For  $b = a + 2k$ , the set  $S_1 = S \cup V \cup W$  is a  $\gamma_{FO}$ -set of  $H$  and so  $\gamma_{FO}(H) = |S_1| = a + 2k = b$ . For  $b = a + 2k + 1$ , the set  $S_2 = S \cup \{x_1\} \cup V \cup W$  is  $\gamma_{FO}$ -set of  $H$  and so  $\gamma_{FO}(H) = |S_2| = a + 1 + 2k = b$ . ■

Certainly, every full open dominating set of a graph  $G$  is also an open dominating set. Thus if  $G$  is a graph without isolated vertices in which every edge is in a triangle, then  $\gamma_{FO}(G) \geq \gamma_t(G)$ . Next we show that there is no graph  $G$  with  $\gamma_{FO}(G) = \gamma_t(G)$ .

Figure 10. Graphs in Subcase 3.3 for  $a = 3$  and  $b = 5, 6$ 

**Proposition 4.6.** *If  $G$  is a graph without isolated vertices in which every edge is in a triangle, then  $\gamma_{FO}(G) > \gamma_t(G)$ .*

**Proof.** Assume, to the contrary, that there exists a graph  $G$  with  $\gamma_t(G) = \gamma_{FO}(G)$ . Let  $S$  be a  $\gamma_{FO}$ -set of  $G$ . Since every full open dominating set is also an open dominating set and  $\gamma_t(G) = \gamma_{FO}(G)$ , it follows that  $S$  is also a  $\gamma_t$ -set in  $G$ . Let  $u \in S$ . We consider two cases.

*Case 1.* *There exists a vertex  $x$  that is openly dominated by vertex  $u \in S$  but not by any vertex in  $S - \{u\}$ .* This implies that  $x$  is adjacent to  $u$ , but  $x$  is not adjacent to any vertex in  $S - \{u\}$ . On the other hand, since  $S$  is a  $\gamma_{FO}$ -set of  $G$ , the edge  $ux$  is openly dominated by some vertex in  $v \in S - \{u\}$ . Hence  $ux$  belongs to  $\langle N(v) \rangle$ , implying that  $x$  is adjacent to  $v \in S - \{u\}$ , a contradiction.

*Case 2.* *Each vertex in  $G$  that is openly dominated by  $u$  is also openly dominated by some vertex in  $S - \{u\}$ .* Since  $S$  is a  $\gamma_t$ -set in  $G$ , there is a vertex  $v$  adjacent to  $u$  such that  $v$  is not adjacent to any other vertex in  $S - \{u\}$ . However, then, the edge  $uv$  is not openly dominated by any vertex in  $S$ , a contradiction. ■

Next we show that every pair  $a, b$  of integers with  $2 \leq a < b$  is realizable as the open domination number and the full open domination number, respectively, of some graph.

**Theorem 4.7.** For every pair  $a, b$  of integers with  $2 \leq a < b$ , there exists a graph  $G$  with  $\gamma_t(G) = a$  and  $\gamma_{FO}(G) = b$ .

**Proof.** We consider two cases.

*Case 1.*  $b = a + 1$  or  $b = a + 2$ . Let  $G_a$  be the graph obtained from  $K_{a+1}$  with  $V(K_{a+1}) = \{v_1, v_2, \dots, v_{a+1}\}$  by adding, for each edge  $e_{ij} = v_i v_j$ , where  $1 \leq i < j \leq a+1$ , a new vertex  $w_{ij}$  and joining it to  $v_i$  and  $v_j$ . The graphs  $G_2$  and  $G_3$  are shown in Figure 11. Let  $H_a = G_{a+1} - w_{12}$ . The graph  $H_2$  is shown in Figure 11. Since  $\{v_1, v_2, \dots, v_a\}$  is a  $\gamma_t$ -set of  $G_a$  and  $\{v_1, v_2, \dots, v_{a+1}\}$  is a  $\gamma_{FO}$ -set of  $G_a$ , it follows that  $\gamma_t(G_a) = a$  and  $\gamma_{FO}(G_a) = a + 1$ . Moreover, since  $\{v_3, v_4, \dots, v_{a+2}\}$  is a  $\gamma_t$ -set of  $H_a$  and  $\{v_1, v_2, \dots, v_{a+2}\}$  is a  $\gamma_{FO}$ -set of  $H_a$ , we have  $\gamma_t(H_a) = a$  and  $\gamma_{FO}(H_a) = a + 2$ .

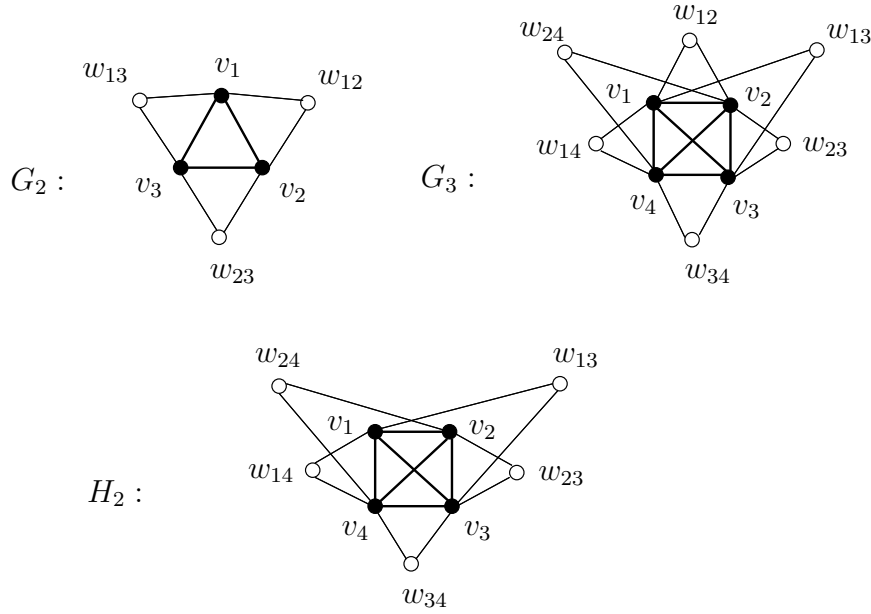


Figure 11. The graphs  $G_2$ ,  $G_3$ , and  $H_2$

*Case 2.*  $b \geq a + 3$ . Suppose, first, that  $b = a + 2k + 1$  ( $k \geq 1$ ). Let  $G$  be the graph obtained from the graph  $G_a$  (from Case 1) and  $kK_2$  by joining  $v_1$  to each of the vertices of  $kK_2$ . If  $b = a + 2k$  ( $k \geq 2$ ), then let  $G$  be the graph obtained from the graph  $H_a$  (of Case 1) and  $(k - 1)K_2$  by joining  $v_3$  to each vertex of  $(k - 1)K_2$ . It is routine to verify that  $\gamma_t(G) = a$  and  $\gamma_{FO}(G) = b$ . ■

We conclude this paper with a problem.

**Problem 4.8.** *Determine all triples  $a, b, c$  of integers with  $\max\{a, b\} \leq c$ ,  $a \geq 2$ ,  $c \geq 3$ , and  $a < c$  for which there exists a graph  $G$  with  $\gamma_t(G) = a$ ,  $\gamma_{FC}(G) = b$ , and  $\gamma_{FO}(G) = c$ .*

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