Discussiones Mathematicae
Graph Theory 21 (2001) 13–30

ON 2-PERIODIC GRAPHS OF A CERTAIN GRAPH OPERATOR

Ivan Havel †

Mathematical Institute, Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1, Czech Republic

and

Bohdan Zelinka

Technical University
Voroněžská 13, 461 17 Liberec, Czech Republic

e-mail: Bohdan.Zelinka@vslib.cz

Abstract

We deal with the graph operator $\overline{\mathrm{Pow}_2}$ defined to be the complement of the square of a graph: $\overline{\mathrm{Pow}_2}(G) = \overline{\mathrm{Pow}_2(G)}$. Motivated by one of many open problems formulated in [6] we look for graphs that are 2-periodic with respect to this operator. We describe a class $\mathcal{G}$ of bipartite graphs possessing the above mentioned property and prove that for any $m, n \geq 6$, the complete bipartite graph $K_{m,n}$ can be decomposed in two edge-disjoint factors from $\mathcal{G}$. We further show that all the incidence graphs of Desarguesian finite projective geometries belong to $\mathcal{G}$ and find infinitely many graphs also belonging to $\mathcal{G}$ among generalized hypercubes.

Keywords: graph operator, power and complement of a graph, Desarguesian finite projective geometry, decomposition of a complete bipartite graph, generalized hypercube.

2000 Mathematics Subject Classification: 05C38, 05C75.

1. Introduction and Notation

The aim of this paper is to investigate the graph operator $\overline{\mathrm{Pow}_2}$ defined and studied (among many other graph operators) in [6].

We start by definitions. We give some fundamental definitions of the graph theory and some special definition from [6] concerning graph operators.

Our graphs are finite, undirected, having neither loops nor multiple edges. If G is a graph, then $V(G)$ ($E(G)$) denotes the vertex (edge) set of G. We denote by $d_G(u)$ the degree of the vertex u in G, by $d_G(u,v)$ the distance of the vertices u and v in G and by $\mathrm{diam}(G)$ the diameter of G. If G_1, and G_2 are graphs, w shall write $G_1 = G_2$ if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$; we shall write $G_1 \cong G_2$ if G_1 and G_2 are isomorphic. By i, j, k, l, m and n we denote integers.

We take from a general theory of graph operators the following concepts (cf. [6]). Let ϕ be an operator and G a graph such that $\phi^n(G)$ is defined for every $n \geq 1$. We say that G is *convergent* under ϕ if $\{\phi^n(G) : n \geq 1 \text{ is finite}\}$. We say that G is *periodic* if there is an integer n such that $G \cong \phi^n(G)$. (Observe that here only an isomorphy, not equality of G and $\phi^n(G)$ is required.) The smallest n with this property is called the *period* of G in ϕ and G is called *n-periodic* in ϕ. A 1-periodic graph G is called *ϕ-fixed* or a *fixed point* of ϕ. A *circuit* is any sequence of the form $(G, \phi(G), \ldots, \phi^{n-1}(G), \phi^n(G))$, where $G \cong \phi^n(G)$. Notice that a subsequence of a circuit may also be a circuit.

For $k \geq 2$, the k-th *power* $\mathrm{Pow}_k(G)$ of a graph G is defined as follows: $V(\mathrm{Pow},(G)) = V(G)$, $E(\mathrm{Pow}_k(G)) = \{\{u,v\} : u \in V(G), v \in V(G),\ 0 < d_G(u,v) \leq k\}$. The second power of G is also called the *square* of G. The *complement* $\overline{G}$ of a graph G is defined by $V(\overline{G}) = V(G), E(\overline{G}) = \{\{u,v\} : u \in V(G), v \in V(G), d_G(u,v) > 1\}$. The graph operator $\overline{\mathrm{Pow}_k}$ is defined by $\overline{\mathrm{Pow}_k}(G) = \overline{\mathrm{Pow}_k(G)}$.

Observe that for any $k \geq 2$ and G, $V(\mathrm{Pow}_k(G)) = V(G)$, hence G is convergent under $\overline{\mathrm{Pow}_k}$. The trivial circuits are (nK_1, K_r, nK_1) (where K_m denotes the complete graph on m vertices) and the main question is what are further circuits.

From the literature certain $\overline{\mathrm{Pow}_2}$-fixed graphs are known. First, for general $k \geq 2$ the graphs $\{G_k^{(l)} : l \geq 0\}$ defined below are $\overline{\mathrm{Pow}_k}$-fixed. $G_k^{(0)}$ is the cycle C_{2k+3}, and for $l \geq 1$, $G_k^{(l)}$ is constructed as follows: let u and v be two vertices of $G_k^{(b)} = C_{2k+3}$ at distance 2; add l new vertices to $C_k^{(0)}$ and join each to u and v. (The graphs $G_k^{(l)}$ are constructed in [1], $G_k^{(0)}$ also in [6].)

For $k = 2$ have two more graphs that are $\overline{\mathrm{Pow}_2}$-fixed. Both of them are given in [1], the second one also in [6]. Apparently we are still far from the exhaustive solution of the problem of $\overline{\mathrm{Pow}_2}$-fixed graphs.

The open problem No. 36 [6] (p. 194) asks to determine $\overline{\mathrm{Pow}_2}$-fixed graphs and to say something about periods (under this operator) greater than 1.

Below we will be dealing with the second part of this problem. We are going to describe by simple means (using only the concept of a diameter of a graph) a class $\mathcal{G}$ of bipartite graphs such that

$$G \in \mathcal{G} \Rightarrow \overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G)) = G.$$

Hence the graphs from $\mathcal{G}$ are 2-periodic with respect to $\overline{\mathrm{Pow}_2}$. Looking for examples of graphs $\mathcal{G}$ we show that

- for any $m, n \geq 6$, the complete bipartite graph $K_{m,n}$ can be decomposed into two edge-disjoint factors from $\mathcal{G}$,
- $\mathcal{G}$ contains all incidence graphs of Desarguesian projective geometries (cf. e.g. [3]),
- in the class of the s.c. generalized hypercubes (cf. [2]) there are infinitely many graphs from $\mathcal{G}$.

2. Bipartite Graphs that are 2-Periodic with Respect to $\overline{\mathrm{Pow}_2}$

We start our search for graphs that are 2-periodic with respect to $\overline{\mathrm{Pow}_2}$ with the following statement:

Lemma 1. *If a graph G is not connected, then $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$ consists of isolated vertices.*

The proof is straightforward and we omit it.

Because of Lemma 1 we can limit ourselves, in what follows, only to connected graphs. (We will, however, formulate the condition of connectedness explicitly any time we need it.)

The concept of a *bipartite graph* will be used in its usual sense; we will always assume that a bipartite graph has at least 2 vertices. Observe that for a connected bipartite graph G the vertex set $V(G)$ partitions into the independent subsets in a unique way. We will say that vertices $u, v \in V(G)$ are of the *same* (*different*) *parity* in G if $d_G(u, v)$ is even (odd). One can define the s.c. *complementary bipartite graph* of G (we denote it by $\mathrm{Compl}B(G)$) as follows:

$$V(\mathrm{Compl}B(G)) = V(G),$$

$$E(\mathrm{Compl}B(G)) = \{\{u, v\}; u, v \in V(\mathrm{Compl}B(G)) \text{ and } \{u, v\} \notin E(G), \text{ and } u, v \text{ are of different parity}\}.$$

Observe that $\mathrm{Compl}B(\mathrm{Compl}B(G)) = G$.

Lemma 2. *Let G be connected and bipartite, let $H = \mathrm{Compl}B(G)$. Then*

(i) $E(H) \subseteq E(\overline{\mathrm{Pow}_2}(G))$,

(ii) $E(H) \cap E(\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))) = \emptyset$,

(iii) *if* $\mathrm{diam}(G) = 3$, *then* $\overline{\mathrm{Pow}_2}(G) = H$,

(iv) *if* $\mathrm{diam}(G) > 3$, *then* $E(G) \setminus E(\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}}(G))) \neq \emptyset$,

(v) *if* $\mathrm{diam}(G) = 3$ *and* $\mathrm{diam}(H) > 3$, *then there is an edge in* $E(\overline{\mathrm{Pow}_2}(\mathrm{Pow}_2(G)))$ *joining two vertices of the same parity in* G.

Proof. Let G be connected and bipartite, let V_1, V_2 be the bipartition classes of $V(G)$, let $H = \mathrm{Compl}B(G)$.

(i) If $\{u, v\} \in E(H)$ then $d_G(u, v) \geq 3$ and therefore $\{u, v\} \in E(\overline{\mathrm{Pow}_2}(G))$.

(ii) Following (i) we have $E(H) \subseteq E(\overline{\mathrm{Pow}_2}(G))$ and obviously also $E(\overline{\mathrm{Pow}_2}(G)) \subseteq E(\mathrm{Pow}_2(\overline{\mathrm{Pow}_2}(G))$, hence

$$E(H) \subseteq E(\mathrm{Pow}_2(\overline{\mathrm{Pow}_2}(G))$$

and this is what we need.

(iii) Observe that $\mathrm{diam}(G) = 3$ yields

$$u, v \in V_j \text{ and } u \neq v \Rightarrow d_G(u, v) = 2, \quad j = 1, 2,$$
$$u \in V_1 \text{ and } u \in V_2 \Rightarrow d_G(u, v) = 1 \text{ or } d_G(u, v) = 3.$$

Hence $Pow_2(G)$ is the union of G and of the complete graphs on the vertex sets V_1 and V_2. This implies

$$\overline{\mathrm{Pow}_2}(G) = H.$$

(iv) Let $\mathrm{diam}(G) > 3$. Then there are vertices u, v of the same parity in G such that $d_G(u, v) > 2$. We have $\{u, v\} \notin \mathrm{Pow}_2(G)$ and thus $\{u, v\} \in \overline{\mathrm{Pow}_2}(G)$. The graph $\mathrm{Pow}_2(G)$ is the union of G and of the graphs L_1, L_2 such that $V(L_1) = V_1$, $V(L_2) = V_2$ and at least one of L_1, L_2 is a non-complete graph. The complement $\overline{\mathrm{Pow}_2}(G)$ is then the union of H and of the complements of L_1 and L_2 with respect to complete graphs on V_1 and V_2. We have $\{u, v\} \notin \mathrm{Pow}_2(G)$ and thus $\{u, v\} \in \overline{\mathrm{Pow}_2}(G)$. Evidently then $\{u, v\} \in \mathrm{Pow}_2(\overline{\mathrm{Pow}_2}(G))$ and $\{u, v\} \notin \overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$. Finally, $\{u, v\} \in E(G) \setminus E(\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$ and this yields the assertion.

(v) Let $\mathrm{diam}(G) = 3$ and $\mathrm{diam}(H) > 3$. As in (iii), we have $\overline{\mathrm{Pow}_2}(G) = H$. Analogously as in (iii) there exists vertices u, v of the same parity such that $d_H(u, v) > 3$. We have $\{u, v\} \in E(\mathrm{Pow}_2(H))$ and thus $\{u, v\} \in E(\overline{\mathrm{Pow}_2}(H)) = E(\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$ and the assertion is true.

Remark 1. For a bipartite graph G, $\mathrm{diam}(G) < 3$ if and only if G is a complete bipartite graph.

Theorem 1. *Let G be a connected bipartite graph with at least 3 vertices. Then the following two assertions are equivalent:*

(i) $\mathrm{diam}(G) = \mathrm{diam}(\mathrm{Compl}B(G)) = 3$,

(ii) $\overline{Pow_2}(\overline{\mathrm{Pow}_2}(G)) = G$.

Proof. Let G be connected and bipartite, let $|V(G)| \geq 3$, put

$$H = \mathrm{Compl}B(G).$$

We first prove (i) $\Rightarrow$ (ii), so assume $\mathrm{diam}(G) = \mathrm{diam}(H) = 3$. Using (iii) of Lemma 2 we get $\overline{\mathrm{Pow}_2}(G) = H$ (from $\mathrm{diam}(G) = 3$) and also $\overline{\mathrm{Pow}_2}(H) = G$ (from $\mathrm{diam}(H) = 3$). Hence

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G)) = \overline{\mathrm{Pow}_2}(H) = G.$$

To prove $\neg$(i) $\Rightarrow$ $\neg$(ii) assume that (i) does not hold. We may assume (because of Remark 1) that either $\mathrm{diam}(G) > 3$ or $\mathrm{diam}(H) > 3$.

(a) Let first $\mathrm{diam}(G) > 3$. We use (iv) of Lemma 2 and get

$$E(G) \setminus E(\overline{\mathrm{Pow}_2}(\mathrm{Pow}_2)(G))) \neq \emptyset.$$

Hence the proof is finished in the case a).

(b) Assuming $\mathrm{diam}(G) = 3$ and $\mathrm{diam}(H) > 3$ we use (v) of Lemma 2 and get

$$G \neq \overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$$

as well. ■

An infinite number of examples of bipartite graphs satisfying (i) of Theorem 1 are yielded by the incidence graphs of Desarguesian finite projective geometries (see e.g. [3]).

An incidence graph of a given finite projective geometry $\mathcal{G}$ is the graph whose vertex set is the union of the point set P and the line set L of $\mathcal{G}$ and

in which a point and a line are adjacent if and only if they are incident in $\mathcal{G}$. Let G be such a graph. Obviously, G is a bipartite graph with vertex classes P and L. The axioms of the projective geometry guarantee that to any two distinct points there exists exactly one line incident to both of them and to any two distinct lines there exists exactly one point incident to both of them. This implies that $\mathrm{diam}(G) = 3$. On the other hand, in every Desarguesian projective geometry $\mathcal{G}$ each line is incident with at least three points and each point is incident with at least three lines. Let $H = ComplB(G)$. In H, a point and a line are adjacent if and only if they are not incident in $\mathcal{G}$. To any two distinct points p_1, p_2 there exists at least one line l incident with none of them: it suffices to take the line l' joining p_1, p_2, to choose a point p' on l' distinct from both p_1 and p_2 and to choose as l another line incident with p'. Analogously, to any two distinct lines there exists at least one point incident with none of them. Hence $\mathrm{diam}(H) = 3$, (i) of Theorem 1 is fulfilled and we have the following

Corollary 1. *Let G be the incidence graph of a Desarguesian finite projective geometry. Then*

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G)) = G.$$

Observe that if G is a connected bipartite graph with vertex classes V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, then G and $ComplB(G)$ are edge-disjoint factors of the complete bipartite graph $K_{m,n}$ whose union is $K_{m,n}$. It is natural to ask, in connection with Theorem 1, for which integers m, n there is a decomposition of $K_{m,n}$ into two edge-disjoint factors of diameter 3.

The answer is given by the following

Theorem 2. *A complete bipartite graph $K_{m,n}$ can be decomposed into two edge-disjoint factors of diameter* 3 *if and only if* $\min(m, n) \geq 6$.

First we prove the following lemmas:

Lemma 3. *Let $m, n \geq 1$. If the complete bipartite graph $K_{m,n}$ can be decomposed into two edge-disjoint factors of diameter* 3 *then so can be decomposed both $K_{m+1,n}$ and $K_{m,n+1}$.*

Proof. Let $m, n \geq 1$, assume that $K_{m,n}$ can be decomposed into two edge-disjoint factors of diameter 3. Obviously, it suffices to show that such a decomposition exists also for $K_{m+1,n}$. Assume that $K_{m+1,n}$ has vertex classes V_1 and V_2, where $|V_1| = m + 1$ and $|V_2| = n$. Choose two different

vertices $x, y \in V_1$ and consider the complete bipartite graph $K_{m,n}$ with vertex classes $V_1 \setminus \{y\}$ and V_2. Use the assumption of the statement and consider G_1, G_2, two edge-disjoint factors of $K_{m,n}$ fulfilling $\mathrm{diam}(G_1) = \mathrm{diam}(G_2) = 3$. Let G_1' (G_2') be the graph obtained from G_1 (G_2) by adding the vertex y and joining it by new edges with exactly those vertices of V_2 which are adjacent in G_1 (G_2, respectively) to x. Then, for any two vertices $u, v \in V(G_1), d_{G_1'}(u, v) = d_{G_1'}(u, u)$. If $u \in V(G_1)$, $u \neq x$, then $d_{G_1'}(u, y) = d_{G_1'}(u, x)$. Obviously, $d_{G_1'}(x, y) = 2$. Since $\mathrm{diam}(G_1) = 3$, $\mathrm{diam}(G_1') = 3$ as well. One shows analogously also $\mathrm{diam}(G_2') = 3$. ■

Lemma 4. *Let $n \geq 1$. The complete bipartite graph $K_{5,n}$ cannot be decomposed into two edge-disjoint factors of diameter* 3.

Proof. The statement obviously holds for $n = 1, 2$. Assume that for certain $n \geq 3$ there is a decomposition of $K_{5,n}$ into two edge-disjoint factors G_1, G_2 such that

$$\mathrm{diam}(G_1) = \mathrm{diam}(G_2) = 3.$$

We will show that this assumption leads to a contradiction. Let V_1 and V_2 be the vertex classes of $K_{5,n}$, assume

$$V_1 = \{u_1, u_2, u_3, u_4, u_5\}, \qquad |V_2| = n.$$

First we show that for every $v \in V_2$, $2 \leq d_{G_1}(u) \leq 3$ and $2 \leq d_{G_2}(u) \leq 3$. In fact, $d_{G_1}(v) = 0$ means that G_1 is not connected, $d_{G_1}(u) = 1$ implies that the only neighbor of v in G_1 is adjacent in G_1 to all the vertices of V_2 (since $\mathrm{diam}(G_1) = 3$) hence it is an isolated vertex of G_2. Finally, $d_{G_1}(u) \in \{4, 5\}$ is equivalent to $d_{G_2}(v) \in \{0, 1\}$ and we exclude it in a similar way. Put

$$D_2 = \{v \in V_2;\ d_{G_1}(v) = 2\}, \quad D_3 = \{v \in V_2;\ d_{G_1}(v) = 3\},$$

and observe that

$$D_2 = \{v \in V_2;\ d_{G_2}(v) = 3\},\ \ D_3 = \{v \in V_2;\ d_{G_2}(v) = 2\},\ \text{and } D_2 \cup D_3 = V_2.$$

It follows that for any $v \in V_2$ either $d_{G_1}(v) = 2$ or $d_{G_2}(v) = 2$ (because $d_{K_{m,n}}(u) = 5$). Since G_1 and G_2 play a symmetric role assume without loss of generality that $D_2 \neq \emptyset$. We are going to prove that $D_3 \neq \emptyset$ as well.

From $\mathrm{diam}(G_1) = 3$ we have

$$v_1, v_2 \in D_2 \quad \text{and} \quad v_1 \neq v_2 \Longrightarrow d_{G_1}(v_1, v_2) = 2.$$

This is only possible (because of definition of D_2) if either there is a vertex in V_1 adjacent in G_1 to all vertices of D_2 (case I) or there are three pairwise different vertices in V_1 (let us call them distinguished vertices of V_1) such that each vertex of D_2 is adjacent in G_1 to exactly two of them (case II). We can already conclude that

$$D_3 \neq \emptyset$$

because otherwise (i.e., if $V_2 = D_2$) there would be either a vertex in V_1 adjacent in G_1 to all vertices of V_2 and thus isolated in G_2 or two vertices in V_1, each of them isolated in G_1.

As a next step we show that the case II cannot occur. Assume on the contrary that the case II occurs but the case I does not. Let u_1, u_2, u_3 be the distinguished vertices of V_1. Observe that to any $u_i, 1 \leq i \leq 3$, there is a vertex in D_2 which is not adjacent in G_1 to u_i. Consider arbitrary vertices $v \in D_3$, and $v' \in D_2$. Since $d_{G_1}(v, v') = 2$ and v' is adjacent in G_1 neither to u_4 nor to u_5, we see that v is adjacent in G_1 to at least two of the distinguished vertices and thus it is adjacent in G_1 to at most one of the vertices u_4, u_5. Since

$$d_{G_1}(u_i, u_4) = 2, 1 \leq i \leq 3,$$

and u_4 is adjacent in G_1 to no vertex of D_2, there are two different vertices $v_{41}, v_{42} \in D_3$ such that each of them is adjacent in G_1 to u_4 and to two of the distinguished vertices. The pairs of distinguished vertices adjacent in G_1 to v_{41}, v_{42} are different because their union must be the whole set $\{u_1, u_2, u_3\}$. An analogous assertion holds for u_5, let v_{51}, v_{52} be the corresponding vertices of D_3. Observe that

$$d_{G_2}(v_{41}) = d_{G_2}(v_{42}) = d_{G_2}(v_{51}) = d_{G_2}(v_{52}) = 2.$$

Each of the vertices v_{41}, v_{42} is adjacent in G_2 to u_5, moreover u_{41} is adjacent in G_2 to a distinguished vertex $x_1 \in \{u_1, u_2, u_3\}$ and v_{42} is adjacent in G_2 to a distinguished vertex $x_2 \in \{u_1, u_2, u_3\}$, where $x_1 \neq x_2$. Analogously, each of the vertices v_{51}, v_{52} is adjacent in G_2 to u_4; moreover v_{51} is adjacent in G_2 to $y_1 \in \{u_1, u_2, u_3\}$ and v_{52} is adjacent in G_2 to $y_2 \in \{u_1, u_2, u_3\}$, where $y_1 \neq y_2$. At least one of the vertices y_1, y_2 is different from x_1, suppose

without loss of generality that $y_1 \neq x_1$. Observe that the vertex v_{41} is adjacent in G_2 to u_5 and to x_1, the vertex v_{51} is adjacent in G_2 to u_4 and to y_1. Since $\{u_5, x_1\} \cap \{u_4, y_1\} = \emptyset$ we have

$$d_{G_2}(v_{41}, v_{51}) \geq 4, \text{ hence } \operatorname{diam}(G_2) \geq 4,$$

which is a contradiction. So we have shown that the case II cannot occur.

Hence there is a vertex $x \in V_1$ which is adjacent in G_1 to all the vertices of D_2, and a vertex $y \in V_1$ which is adjacent in G_2 to all the vertices of D_3. The vertex y is adjacent in G_1 only to vertices of D_2. If $x \neq y$ then x is the only vertex in V_1 fulfilling $d_{G_1}(x, y) = 2$; for any $z \in V_1 \setminus \{x, y\}$ we have necessarily

$$d_{G_1}(z, y) \geq 4, \text{ hence } \operatorname{diam}(G_1) \geq 4,$$

a contradiction. If $x = y$ then x is adjacent in G_1 to all the vertices of D_2 and to no vertex of D_3. Consider any vertex $v \in D_2$; it is adjacent in G_1 to exactly one vertex $w \in V_1 \setminus \{x\}$. Any path of length 2 from v to a vertex of D_3 in G_1 must go through w and therefore w is adjacent in G_1 to all the vertices of D_3. On the other hand, each vertex of $V_1 \setminus \{x\}$ must be adjacent in G_1 to a vertex of D_2; otherwise its distance in G_1 from x would be greater than 2. Hence each vertex of $V_1 \setminus \{x\}$ is adjacent in G_1 to all the vertices of D_3. As $V_1 \setminus \{x\}$ has 4 elements, each vertex of D_3 has degree 4, which is a contradiction accomplishing the whole proof. ■

***Proof of Theorem* 2.** To prove the theorem it obviously suffices to exhibit a decomposition of $K_{6,6}$ into two edge-disjoint factors G_1, G_2 fulfilling $\operatorname{diam}(G_1) = \operatorname{diam}(G_2) = 3$ and to apply Lemmas 3 and 4. Let

$$V(K_{6,6}) = \{u_1, \ldots, u_6, v_1, \ldots, v_6\}, \quad E(K_{6,6}) = \{\{u_i, v_j\}; \quad 1 \leq i, j \leq 6\}.$$

Put

$$\begin{aligned} V(G_1) &= V(G_2) = \{u_1, \ldots, u_6, v_1, \ldots, v_6\}, \\ E(G_1) &= \{\{u_i, v_j\};\ 1 \leq i, j \leq 6 \text{ and } j = i \text{ or } j \equiv i + 1 \pmod 6 \text{ or} \\ &\qquad j \equiv i + 3 \pmod 6\}, \end{aligned}$$

$$E(G_2) = E(K_{6,6}) \setminus E(G_1).$$

One verifies easily that G_1 and G_2 posses the required properties. ■

Corollary 2. *For any $m \geq 6$, $n \geq 6$, there is a connected bipartite graph G with vertex classes V_1 and V_2 such that*

(i) $|V_1| = m,\ V_2 = n$,

(ii) $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G)) = G$ *and* $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(\mathrm{Compl}B(G))) = \mathrm{Compl}B(G)$.

3. Generalized Hypercubes and the Operator $\overline{\mathrm{Pow}_2}$

Now we are going to investigate properties of the generalized hypercubes with respect to the operator $\overline{\mathrm{Pow}_2}$.

For $n \geq 1$ we denote the set $\{1, \ldots, n\}$ by $[n]$ and start with the basic definition (cf. [2]):

Definition 1. Let $n \geq 1$ and $S \subseteq [n]$. The generalized hypercube $Q_n(S)$ has as vertices all the $0-1$ vectors of size n. Two vertices are adjacent in $Q_n(S)$ iff their Hamming distance (i.e., the number of coordinates they differ in) belongs to S.

Observe the following facts: $Q_n(\emptyset) \simeq 2^n K_1, Q_n(\{1, \ldots, n\}) \simeq K_{2^n}$, and for the well-known graph of the n-dimensional hypercube Q_n we have $Q_n \simeq Q_n(\{1\})$ (we will omit in this and similar cases parentheses and write simply $Q_n(1)$). For $n \geq 1$ and any S the vertex sets of Q_n and $Q_n(S)$ coincide.

For basic properties of generalized hypercubes (in particular those related to isomorphism) see [2]. Generalized hypercubes (also called distance graphs in [5]) are special case of the cube-like graphs, defined and studied in [4]. It is proved in [5] that the chromatic number of a cube-like graph cannot be 3. Lovász showed that a cube-like graph has an integral spectrum (cf. [4]).

Given a $0-1$ vector $x = (x_1, \ldots, x_n)$ of size $n \geq 1$, denote by $w(x)$ the number of 1-coordinates of x, i.e., $w(x) = \sum_{i=1}^{n} x_i$. (Usually, $w(x)$ is called the Hamming weight of x.) Observe that for any $S \subseteq [n]$, $V(Q_n(S)) = V_o \cup V_e$, where V_o (V_e) is the set of all $0-1$ vectors x with $w(x)$ odd (even, respectively).

Now we are going to determine the complement and square of a generalized hypercube. For $S \subseteq [n]$ put

$$C^{(n)}(S) = [n] \setminus S,$$

$$S^{(n,2)} = \{i \in [n];\ \exists\, j, k \in S \text{ such that } |j-k| \leq i \leq \min(j+k, 2n-j-k) \text{ and } i \equiv j+k \pmod 2\},$$

$$\mathcal{P}_2^{(n)}(S) = S \cup S^{(n,2)}.$$

Remark 2. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two $0-1$ vectors of size n, define their $\oplus$-sum (also called modulo 2 sum) as usual by

$$x \oplus y = (z_1, \ldots, z_n),$$

where $z_i = 1$ if $x_i \neq y_i$ and $z_i = 0$ if $x_i = y_i$, $i = 1, \ldots, n$. One can see that given $S \subseteq [n]$ and $i \in [n]$, $i \in S^{(n,2)}$ if and only if there are $0-1$ vectors x, y of size n such that $w(x) \in S$, $w(y) \in S$, and $i = w(x \oplus y)$.

We have the following

Lemma 5. *Let $n \neq 1$ and $S \subseteq [n]$. Then*

$$\overline{Q_n(S)} = Q_n(\mathcal{C}^{(n)}(S)) \text{ and } \mathrm{Pow}_2(Q_n(S)) = Q_n(\mathcal{P}_2^{(n)}(S)).$$

Proof. The proof follows directly from definitions of the generalized hypercube, complement and the square of a graph. ■

Using Lemma 5 we can study behavior of generalized hypercubes under the operators of square and complement by studying certain subsets of $[n]$ under the operators $\mathcal{P}_2^{(n)}$ and $\mathcal{C}^{(n)}$.

Denote by $\mathcal{R}^{(n)}$ the composed operator $\mathcal{C}^{(n)}\mathcal{P}_2^{(n)}\mathcal{C}^{(n)}\mathcal{P}_2^{(n)}$. We will be interested in the sets $S \subseteq [n]$ fulfilling

$$\mathcal{R}^{(n)}(S) = S$$

because of the following statement, which is an immediate consequence of Lemma 5.

Proposition 1. *Let $n \geq 1$, let $S \subseteq [n]$. Then*

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(Q_n(S))) = Q_n(S)$$

if and only if S fulfils $\mathrm{diam}(Q_n(\sigma)) = 3$.

Hence, if S fulfils $\mathrm{diam}(Q_n(S)) = 3$ then $Q_n(S)$ is 2-periodic with respect to $\overline{\mathrm{Pow}_2}$. It is an open problem whether this equality is also necessary for the 2-periodicity with respect to $\overline{\mathrm{Pow}_2}$. (Proposition 1 only deals with the equality of the graphs $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(Q_n(S)))$ and $Q_n(S)$, not with their isomorphy.) Observe that (4) is trivially fulfilled if $S = \emptyset$ or $S = [n]$. We will be interested in the sets $S \subseteq [n]$ fulfilling $\mathrm{diam}(Q_n(S)) = 3$ and $\emptyset \neq S \neq [n]$:

Definition 2. Let $n \geq 1$ and $S \subseteq [n]$. We call S an $\mathcal{R}^{(n)}$-fixed point if $\emptyset \neq S \neq [n]$ and $\operatorname{diam}(G_n(S)) = 3$ holds. We say that S is a minimal $\mathcal{R}^{(n)}$-fixed point if no proper subset of S is an $\mathcal{R}^{(n)}$-fixed point.

The following statement follows directly from the definition.

Proposition 2. *Let $n \geq 1$, let $S \subseteq [n]$ be an $\mathcal{R}^{(n)}$-fixed point. Then $S' = \mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))$ is also an $\mathcal{R}^{(n)}$-fixed point and*

$$S = \mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S')).$$

Introduce still another denotation: For $n \geq 1$, let

$$[n]_o = \{i \in [n];\ i \text{ is odd}\},$$
$$[n]_e = \{i \in [n],\ i \text{ is even}\}.$$

Observe that $Q_n(S)$ is not connected if $S \subseteq [n]_e$. If $S \subseteq [n]_o$ then $Q_n(S)$ is a bipartite graph with vertex classes V_o and V_e; it is not connected only if $S = \emptyset$ or $S = \{n\}$, otherwise it is connected. If $S = [n]_e$ then $Q_n(S)$ is a disjoint union of two complete subgraphs induced by V_o and V_e. If $[n]_e \subsetneq S$ then $Q_n(S)$ contains the above described graph as a spanning subgraph and moreover it contains at least one matching between V_o and V_e, in this case, $\mathrm{Pow}_2(Q_n(S))$ is a complete graph.

Remark 3. Using the above denotation and definition, we observe that the following facts hold: for any $n \geq 1$,

$$S' \subseteq S \subseteq [n] \Longrightarrow S'^{(n,2)} \subseteq S^{(n,2)},$$
$$S \subseteq [n]_e \text{ or } S \subseteq [n]_o \Longrightarrow S^{(n,2)} \subseteq [n]_e,$$
$$[n]_o^{(n,2)} = [n]_e.$$

In what follows generalized hypercubes $Q_n(S)$ with $S \subseteq [n]_o$ will play an important role. First, we have the following

Lemma 6. *Let $n \geq 1$, let S_1 and S_2 fulfil $S_1 \subseteq [n]_o$, $S_2 \subseteq [n]$, and $S_2 \not\subseteq [n]_o$. Then*

$$Q_n(S_1) \not\simeq Q_n(S_2).$$

Proof. Let n, S_1 and S_2 fulfil the assumptions of Lemma 6. Then either $Q_n(S_1) \simeq 2^{n-1}K_2$ (if n is odd and $S_1 = \{n\}$) or $Q_n(S_1)$ is connected and bipartite (if $S_1 \subseteq [n]_o$ and $S_1 \neq \{n\}$).

Similarly, for $Q_n(S_2)$ we have the following possibilities:

(a) $Q_n(S_2) \simeq 2^{n-1}K_2$, where n is even (if $S_2 = \{n\}$),

(b) $Q_n(S_2)$ consists of two connected components, each of them having 2^{n-1} vertices (if $S_2 \subseteq [n]_e$ and $S_2 \neq \{n\}$),

(c) $Q_n(S_2)$ is connected and non-bipartite (if $S_2 \cap [n]_e \neq \emptyset$ and simultaneously also $S_2 \cap [n]_o \neq \emptyset$).

Hence $Q_n(S_1) \not\simeq Q_n(S_2)$. ■

Theorem 3. *For $n \geq 1$ and $S \subseteq [n]_o$, if*

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(Q_n(S))) \simeq Q_n(S), \tag{1}$$

then

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(Q_n(S))) = Q_n(S). \tag{2}$$

Proof. Let $n \geq 1$ and $S \subseteq [n]_o$, we have to prove that (1) implies (2). If $n = 1$ or $S = \emptyset$ then (2) holds. Let us assume $n \geq 1$. If n is odd and $S = \{n\}$, (1) does not hold (Lemma 1).

Thus we may assume that $\emptyset \neq S \neq \{n\}$. After putting

$$G = Q_n(S)$$

we observe that G is connected, bipartite and has at least 4 vertices. Let $H = \mathrm{Compl}B(G)$. Suppose that (2) does not hold, i.e.,

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G)) \neq G.$$

By Theorem 1, either $\mathrm{diam}(G) \neq 3$ or $\mathrm{diam}(H) \neq 3$. One easily gets a contradiction in cases $\mathrm{diam}(G) = 1$, $\mathrm{diam}(H) = 1$, and $\mathrm{diam}(H) = 2$. If $\mathrm{diam}(G) = 2$ then G is a complete bipartite graph with $|V(G)| \geq 4$ and (1) does not hold. If $\mathrm{diam}(G) = 3$ and $\mathrm{diam}(H) > 3$, then (by (v) of Lemma 2) $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$ contains an edge joining two vertices of the same parity. If $\mathrm{diam}(G) > 3$ then (by (ii) and (iv) of Lemma 2) either $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G))$ is a proper subgraph of G or again it contains an edge joining two vertices of the same parity. Altogether,

$$\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(G)) = Q_n(S'),$$

where either $S' \subsetneq S$ or $S' \cap [n]_e \neq \emptyset$. According to Lemma 6 then (1) does not hold. ■

Theorem 4. *For $n \geq 1$ and $S \subseteq [n]_o$, the following statements are equivalent:*

(i) $S^{(n,2)} = ([n]_o \setminus S)^{(n,2)} = [n]_e$,

(ii) $\overline{\mathrm{Pow}_2}(\overline{\mathrm{Pow}_2}(Q_n(S))) = Q_n(S)$.

Proof. For $n = 1$ and $S = \{1\}$ or $S = \emptyset$ the assertion holds trivially. We may assume therefore $n \geq 1$ and $S \subseteq [n]_o$.

(i) $\Rightarrow$ (ii): As $S \subseteq [n]_o$, $Q_n(S)$ is bipartite. It follows from (i) that both the bipartition classes of $V(Q_n(S))$ induce complete subgraphs in $\mathrm{Pow}_2(Q_n(S))$ and therefore

$$\mathrm{diam}(Q_n(S)) = 3.$$

We observe further that

$$\overline{\mathrm{Pow}_2}(Q_n(S)) = Q_n([n]_o \setminus S)$$

and arguing in a similar way as above we conclude that

$$\mathrm{diam}(Q, ([n]_o \setminus S)) = 3$$

as well. Using Theorem 1, (ii) follows.

(ii) $\Rightarrow$ (i): Using again Theorem 1 we have

$$\mathrm{diam}(Q_n(S)) = \mathrm{diam}(\overline{\mathrm{Pow}_2}(Q_n(5))) = 3$$

and this already implies (i). ■

Proposition 3. *Let k and n be such that k is odd, $k \geq 3$, and $|n - 2k| \geq 1$. Then $\{k\}^{(n,2)} = \{k-2, k+2\}^{(n,2)} = [n]_e$ and both $\{k\}$ and $\{k-2, k+2\}$ are minimal $\mathcal{R}^{(n)}$-fixed points.*

Proof. Let k and n fulfil the assumptions of the statement. Directly from the definition we obtain

$$\{k\}^{(n,2)} = [n]_e \quad \text{and} \quad \{k-2, k+2\}^{(n,2)} = [n]_e.$$

Then, using Remark 3, Theorem 4, and Proposition 1 we conclude that both $\{k\}$ and $\{k-2, k+2\}$ are $\mathcal{R}^{(n)}$-fixed points. It remains to be shown that neither $\{k-2\}$ nor $\{k+2\}$ are $\mathcal{R}^{(n)}$-fixed points. Let us start with $\{k-2\}$. Since $2k - 2 \in [n]_e \setminus \{k-2\}^{(n,2)}$ we get $\mathrm{diam}(Q_n(k-2)) > 3$;

using Theorem 1 and Proposition 1 we conclude that $\{k-2\}$ is not an $\mathcal{R}^{(n)}$-fixed point. We proceed similarly with $\{k+2\}$: since $2n-2k-2 \in [n]_e \setminus \{k+2\}^{(n,2)}$ we see that $\mathrm{diam}(Q_n(k+2)) > 3$ and therefore $\{k+2\}$ is not an $\mathcal{R}^{(n)}$-fixed point. ■

Proposition 4. *Let $n \geq 1$. If $S \subseteq [n]_e$ or $[n]_e \subseteq S$, then S is not an $\mathcal{R}^{(n)}$-fixed point.*

Proof. Assume $n \geq 1$ and $S \in [n]$, let S be an $\mathcal{R}(n)$-fixed point.

a) If $S \subseteq [n]_e$, then $P_2^{(n)}(S) \subseteq [n]_e$, hence

$$[n]_o \subseteq \mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S)), \quad \mathcal{P}_2^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))) = [n],$$

and $\mathcal{R}^{(n)}(S) = \emptyset$, a contradiction.

b) From $[n]_e \subseteq S$ one would have

$$\mathcal{P}_2^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))) \subseteq [n]_o,$$

but this can only hold if

$$\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S)) = \{n\} \text{ and } n \text{ is odd.}$$

Then one has

$$\mathcal{P}_2^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))) = \{n\},$$
$$S = \mathcal{C}^{(n)}\mathcal{P}_2^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))) = [n-1].$$

However, $\mathcal{P}_2^{(n)}([n-1]) = [n]$, which is a contradiction. ■

One verifies directly (using also Proposition 4) that there are no $\mathcal{R}(n)$-fixed points for $1 \leq n \leq 4$ and for $n = 8$. From the Proposition 3 and the following Proposition 5 we conclude that for other n $\mathcal{R}(n)$-fixed points do exist.

Proposition 5. *Let $k \geq 3$, let $n = 4k$. Then both $\{2k-1, 2k+1\}$ and $\{2k-3, 2k+3\}$ are minimal $\mathcal{R}(n)$-fixed points.*

Proof. Let $k \geq 3$ and $n = 4k$. Directly from the definition we have

$$\{2k-1, 2k+1\}^{(n,2)} = \{2k-3, 2k+3\}^{(n,2)} = [n]_e,$$

hence according to Remark 3 and Theorem 4 it follows that both $\{2k-1, 2k+1\}$ and $\{2k-3, 2k+3\}$ are $\mathcal{R}(n)$-fixed points. Similarly as in the proof of Proposition 3 above we show that they are minimal: this follows from

$n \notin \{2k-l\}^{(n,2)}$, $n \notin \{2k+1\}^{(n,2)}$, $n \notin \{2k-3\}^{(n,2)}$ and $n \notin \{2k+3\}^{(n,2)}$

using Theorem 1 and Proposition 1. ■

Proposition 6. *Let $n \geq 1$, let $A, C \subseteq [n]_o$ fulfil*

$$A \cap C = \emptyset \quad \text{and} \quad A^{(n,2)} = C^{(n,2)} = [n]_e.$$

Then every A' fulfilling

$$A \subseteq A' \subseteq [n]_o \quad \text{and} \quad A' \cap C = \emptyset$$

is an $\mathcal{R}(n)$-fixed point.

Proof. From the assumptions we have $A'^{(n,2)} = [n]_e$ and $\mathcal{P}_2^{(n)}(A') = A' \cup [n]_e$. Further,

$$\mathcal{P}^{(n)}(A' \cup [n]_e) = [n]_o \setminus A'.$$

Since $C^{(n,2)} = [n]_e$ and $C \subseteq [n]_o \setminus A'$, it follows that $([n]_o \setminus A')^{(n,2)} = [n]_e$ and therefore

$$\mathcal{P}_2^{(n)}([n]_o \setminus A') = [n]_o \setminus A' \cup [n]_e = \mathcal{C}^{(n)}(A').$$

This already implies $\mathcal{R}_2^{(n)}(A') = A'$. ■

Example. Let $k \geq 1$, let $4k+1 \leq n \leq 4k+3$. It follows from Proposition 3 that

$$\{2k+l\}^{(n,2)} = [n]_e,$$

$\{2k-1, 2k+3\}^{(n,2)} = [n]_e$, and that $\{2k+1\}$ is a minimal $\mathcal{R}(n)$-fixed point. From $(2')$ we obtain (using Remark)

$$([n]_o \setminus \{2k-1, 2k+3\})^{(n,2)} = [n]_e.$$

Hence, using Proposition 2, we claim that $\{2k-1, 2k+3\}$ is an $\mathcal{R}(n)$-fixed point (obviously a minimal one).

4. Concluding Remarks and Open Problems

Our efforts to construct an $\mathcal{R}(n)$-fixed point containing even numbers were unsuccessful — all $\mathcal{R}(n)$-fixed points we know so far consist only of odd numbers. This is the reason we propose the following

Conjecture. *If S is an $\mathcal{R}(n)$-fixed point for certain $n \geq 1$ then $S \subseteq [n]_o$ (i.e., S contains no even number).*

In order to clarify the situation with the Conjecture we argue as follows: let for certain $n \geq 1$ be an $\mathcal{R}(n)$-fixed point. Put

$$B = P_2^{(n)}(A), \quad C = \mathcal{C}^{(n)}(B), \quad \text{and} \quad D = \mathcal{P}_2^{(n)}(C).$$

So we have $A = \mathcal{C}^{(n)}(D)$. Since both A and C are $\mathcal{R}(n)$-fixed points, we obtain (using Proposition 4)

$$B \cap [n]_e \neq \emptyset \neq D \cap [n]_e.$$

From here it follows that exactly one of the following possibilities occurs:

I. $[n]_e \subseteq B$ and $[n]_e \subseteq D$,

II. $[n]_e \subseteq B$ and $[n]_e \not\subseteq D$, or $[n]_e \not\subseteq B$ and $[n]_e \subseteq D$,

III. $[n]_e \not\subseteq B$ and $[n]_e \not\subseteq D$.

The case I is just covered by the Conjecture; from $[n]_e \subseteq D$ one derives $A \cap [n]_e = \emptyset$. All $\mathcal{R}(n)$-fixed points we know so far are of the type I.

Now we show that II cannot hold: assume $n \geq 1$, let $A \subseteq [n]$ be an $\mathcal{R}(n)$-fixed point, let B, C, D be given as above, assume without loss of generality $[n]_e \not\subseteq B$ and $[n]_e \subseteq D$. Then $A \subseteq [n]_o$; by Proposition 1 and Theorem 4, $A^{(n,2)} = [n]_e$, hence $[n]_e \subseteq B$, a contradiction.

Our Conjecture would be settled in positive, if we were able similarly to exclude III.

Coming back from sets of integers that are $\mathcal{R}(n)$-fixed points, to graphs again, we conclude that we succeeded to find graphs that are 2-periodic with respect to the operator $\overline{\mathrm{Pow}_2}$. It might be interesting also to consider periods and powers different from 2; so, in the most general setting we formulate the following

Problem. *Let (i, j) be a pair of integers, $i \geq 1$, $j \geq 2$, $(i, j) \neq (2, 2)$. Do there exist $n \geq 1$ and $S \subseteq [n]$ such that the graph $Q_n(S)$ is i-periodic with respect to the operator $\overline{\mathrm{Pow}_j}$?*

References

[1] J. Akiyama, H. Era and G. Exoo, *Further results on graph equations for line graphs and n'th power graphs*, Discrete Math. **34** (1981) 209–218.

[2] T. Dvořák, I. Havel, J-M. Laborde and P. Liebl, *Generalized hypercubes and graph embedding with dilation*, Rostocker Mathematisches Kolloquium **39** (1990) 13–20.

[3] M. Hall, Jr., Combinatorial Theory (Blaisdell Publishing Company, Waltham (Massachusetts) – Toronto – London 1967).

[4] F. Harary, *Four difficult unsolved problems in graph theory*, in: M. Fiedler ed., Recent Advances in Graph Theory, Proc. Second Czech. Symp. Graph Theory, Prague, 1974 (Academia, Praha, 1975) 249–256.

[5] Ch. Payan, *On the chromatic number of cube-like graphs*, Discrete Math. **103** (1992) 271–277.

[6] E. Prisner, Graph dynamics (Pitman Research Notes in Mathematics Series, 338, 1995).

Received 27 December 1999
Revised 19 October 2000