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# ON 2-PERIODIC GRAPHS OF A CERTAIN GRAPH OPERATOR

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## Abstract

We deal with the graph operator  $\overline{\text{Pow}_2}$  defined to be the complement of the square of a graph:  $\overline{\text{Pow}_2}(G) = \overline{\text{Pow}_2(G)}$ . Motivated by one of many open problems formulated in [6] we look for graphs that are 2-periodic with respect to this operator. We describe a class  $\mathcal{G}$  of bipartite graphs possessing the above mentioned property and prove that for any  $m, n \geq 6$ , the complete bipartite graph  $K_{m,n}$  can be decomposed in two edge-disjoint factors from  $\mathcal{G}$ . We further show that all the incidence graphs of Desarguesian finite projective geometries belong to  $\mathcal{G}$  and find infinitely many graphs also belonging to  $\mathcal{G}$  among generalized hypercubes.

**Keywords:** graph operator, power and complement of a graph, Desarguesian finite projective geometry, decomposition of a complete bipartite graph, generalized hypercube.

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# 1. INTRODUCTION AND NOTATION

The aim of this paper is to investigate the graph operator  $\overline{\text{Pow}_2}$  defined and studied (among many other graph operators) in [6].

We start by definitions. We give some fundamental definitions of the graph theory and some special definition from [6] concerning graph operators.

Our graphs are finite, undirected, having neither loops nor multiple edges. If G is a graph, then V(G) (E(G)) denotes the vertex (edge) set of G. We denote by  $d_G(u)$  the degree of the vertex u in G, by  $d_G(u, v)$  the distance of the vertices u and v in G and by diam(G) the diameter of G. If  $G_1$ , and  $G_2$  are graphs, w shall write  $G_1 = G_2$  if  $V(G_1) = V(G_2)$  and  $E(G_1) = E(G_2)$ ; we shall write  $G_1 \cong G_2$  if  $G_1$  and  $G_2$  are isomorphic. By i, j, k, l, m and n we denote integers.

We take from a general theory of graph operators the following concepts (cf. [6]). Let  $\phi$  be an operator and G a graph such that  $\phi^n(G)$  is defined for every  $n \geq 1$ . We say that G is *convergent* under  $\phi$  if  $\{\phi^n(G) : n \geq 1 \text{ is finite}\}$ . We say that G is *periodic* if there is an integer n such that  $G \cong \phi^n(G)$ . (Observe that here only an isomorphy, not equality of G and  $\phi^n(G)$  is required.) The smallest n with this property is called the *period* of G in  $\phi$  and G is called n-periodic in  $\phi$ . A 1-periodic graph G is called  $\phi$ -fixed or a fixed point of  $\phi$ . A *circuit* is any sequence of the form  $(G, \phi(G), \ldots, \phi^{n-1}(G), \phi^n(G))$ , where  $G \cong \phi^n(G)$ . Notice that a subsequence of a circuit may also be a circuit.

For  $k \geq 2$ , the k-th power  $\operatorname{Pow}_k(G)$  of a graph G is defined as follows:  $V(\operatorname{Pow}, (G)) = V(G), E(\operatorname{Pow}_k(G)) = \{\{u, v\} : u \in V(G), v \in V(G), 0 < d_G(u, v) \leq k\}$ . The second power of G is also called the square of G. The complement  $\overline{G}$  of a graph G is defined by  $V(\overline{G}) = V(G), E(\overline{G}) = \{\{u, v\} : u \in V(G), v \in V(G), d_G(u, v) > 1\}$ . The graph operator  $\overline{\operatorname{Pow}}_k$  is defined by  $\overline{\operatorname{Pow}}_k(G) = \overline{\operatorname{Pow}}_k(G)$ .

Observe that for any  $k \geq 2$  and G,  $V(\operatorname{Pow}_k(G)) = V(G)$ , hence G is convergent under  $\overline{\operatorname{Pow}_k}$ . The trivial circuits are  $(nK_1, K_r, nK_1)$  (where  $K_m$  denotes the complete graph on m vertices) and the main question is what are further circuits.

From the literature certain  $\overline{\text{Pow}_2}$ -fixed graphs are known. First, for general  $k \geq 2$  the graphs  $\{G_k^{(l)} : l \geq 0\}$  defined below are  $\overline{\text{Pow}_k}$ -fixed.  $G_k^{(0)}$ is the cycle  $C_{2k+3}$ , and for  $l \geq 1$ ,  $G_k^{(l)}$  is constructed as follows: let u and v be two vertices of  $G_k^{(b)} = C_{2k+3}$  at distance 2; add l new vertices to  $C_k^{(0)}$ and join each to u and v. (The graphs  $G_k^{(l)}$  are constructed in [1],  $G_k^{(0)}$  also in [6].)

For k = 2 have two more graphs that are  $\overline{\text{Pow}_2}$ -fixed. Both of them are given in [1], the second one also in [6]. Apparently we are still far from the exhaustive solution of the problem of  $\overline{\text{Pow}_2}$ -fixed graphs.

The open problem No. 36 [6] (p. 194) asks to determine  $\overline{\text{Pow}_2}$ -fixed graphs and to say something about periods (under this operator) greater than 1.

Below we will be dealing with the second part of this problem. We are going to describe by simple means (using only the concept of a diameter of a graph) a class  $\mathcal{G}$  of bipartite graphs such that

$$G \in \mathcal{G} \Rightarrow \overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G)) = G.$$

Hence the graphs from  $\mathcal{G}$  are 2-periodic with respect to  $\overline{\text{Pow}_2}$ . Looking for examples of graphs  $\mathcal{G}$  we show that

- for any  $m, n \ge 6$ , the complete bipartite graph  $K_{m,n}$  can be decomposed into two edge-disjoint factors from  $\mathcal{G}$ ,
- $\mathcal{G}$  contains all incidence graphs of Desarguesian projective geometries (cf. e.g. [3]),
- in the class of the s.c. generalized hypercubes (cf. [2]) there are infinitely many graphs from  $\mathcal{G}$ .

# 2. Bipartite Graphs that are 2-Periodic with Respect to $\overline{Pow_2}$

We start our search for graphs that are 2-periodic with respect to  $\overline{Pow_2}$  with the following statement:

**Lemma 1.** If a graph G is not connected, then  $\overline{\text{Pow}_2}(\overline{\text{Pow}_2}(G))$  consists of isolated vertices.

The proof is straightforward and we omit it.

Because of Lemma 1 we can limit ourselves, in what follows, only to connected graphs. (We will, however, formulate the condition of connectedness explicitly any time we need it.)

The concept of a *bipartite graph* will be used in its usual sense; we will always assume that a bipartite graph has at least 2 vertices. Observe that for a connected bipartite graph G the vertex set V(G) partitions into the independent subsets in a unique way. We will say that vertices  $u, v \in V(G)$  are of the same (different) parity in G if  $d_G(u, v)$  is even (odd). One can define the s.c. complementary bipartite graph of G (we denote it by ComplB(G)) as follows:

$$V(\text{Compl}B(G)) = V(G),$$
  

$$E(\text{Compl}B(G)) = \{\{u, v\}; u, v \in V(\text{Compl}B(G)) \text{ and } \{u, v\} \notin E(G),$$
  
and  $u, v$  are of different parity}.

Observe that ComplB(ComplB(G)) = G.

**Lemma 2.** Let G be connected and bipartite, let H = ComplB(G). Then

- (i)  $E(H) \subseteq E(\overline{\text{Pow}_2}(G)),$
- (ii)  $E(H) \cap E(\overline{\text{Pow}_2}(\overline{\text{Pow}_2}(G))) = \emptyset$ ,
- (iii) if diam(G) = 3, then  $\overline{\text{Pow}_2}(G) = H$ ,
- (iv) if diam(G) > 3, then  $E(G) \setminus E(\overline{\text{Pow}_2}(\overline{\text{Pow}}(G))) \neq \emptyset$ ,
- (v) if diam(G) = 3 and diam(H) > 3, then there is an edge in  $E(\overline{Pow_2}(Pow_2(G)))$  joining two vertices of the same parity in G.

**Proof.** Let G be connected and bipartite, let  $V_1$ ,  $V_2$  be the bipartition classes of V(G), let H = ComplB(G).

- (i) If  $\{u, v\} \in E(H)$  then  $d_G(u, v) \ge 3$  and therefore  $\{u, v\} \in E(\operatorname{Pow}_2(G))$ .
- (ii) Following (i) we have  $E(H) \subseteq E(\overline{\text{Pow}_2}(G))$  and obviously also  $E(\overline{\text{Pow}_2}(G)) \subseteq E(\text{Pow}_2(\overline{\text{Pow}_2}(G))$ , hence

$$E(H) \subseteq E(\operatorname{Pow}_2(\overline{\operatorname{Pow}_2}(G)))$$

and this is what we need.

(iii) Observe that diam(G) = 3 yields

$$u, v \in V_j$$
 and  $u \neq v \Rightarrow d_G(u, v) = 2$ ,  $j = 1, 2$ ,  
 $u \in V_1$  and  $u \in V_2 \Rightarrow d_G(u, v) = 1$  or  $d_G(u, v) = 3$ .

Hence  $Pow_2(G)$  is the union of G and of the complete graphs on the vertex sets  $V_1$  and  $V_2$ . This implies

$$\overline{\operatorname{Pow}_2}(G) = H.$$

(iv) Let diam(G) > 3. Then there are vertices u, v of the same parity in G such that  $d_G(u, v) > 2$ . We have  $\{u, v\} \notin \operatorname{Pow}_2(G)$  and thus  $\{u, v\} \in \overline{\operatorname{Pow}_2}(G)$ . The graph  $\operatorname{Pow}_2(G)$  is the union of G and of the graphs  $L_1, L_2$  such that  $V(L_1) = V_1, V(L_2) = V_2$  and at least one of  $L_1, L_2$  is a non-complete graph. The complement  $\overline{\operatorname{Pow}_2}(G)$  is then the union of H and of the complements of  $L_1$  and  $L_2$  with respect to complete graphs on  $V_1$  and  $V_2$ . We have  $\{u, v\} \notin \operatorname{Pow}_2(G)$  and thus  $\{u, v\} \in \overline{\operatorname{Pow}_2}(G)$ . Evidently then  $\{u, v\} \in \operatorname{Pow}_2(\overline{\operatorname{Pow}_2}(G))$  and  $\{u, v\} \notin \overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G))$ . Finally,  $\{u, v\} \in E(G) \setminus E(\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G))$ and this yields the assertion.

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(v) Let diam(G) = 3 and diam(H) > 3. As in (iii), we have  $\overline{\text{Pow}_2}(G) = H$ . Analogously as in (iii) there exists vertices u, v of the same parity such that  $d_H(u, v) > 3$ . We have  $\{u, v\} \in E(\text{Pow}_2(H))$  and thus  $\{u, v\} \in E(\overline{\text{Pow}_2}(H)) = E(\overline{\text{Pow}_2}(\overline{\text{Pow}_2}(G))$  and the assertion is true.

**Remark 1.** For a bipartite graph G, diam(G) < 3 if and only if G is a complete bipartite graph.

**Theorem 1.** Let G be a connected bipartite graph with at least 3 vertices. Then the following two assertions are equivalent:

- (i)  $\operatorname{diam}(G) = \operatorname{diam}(\operatorname{Compl}B(G)) = 3$ ,
- (ii)  $\overline{Pow_2}(\overline{Pow_2}(G)) = G.$

**Proof.** Let G be connected and bipartite, let  $|V(G)| \ge 3$ , put

$$H = \operatorname{Compl}B(G).$$

We first prove (i)  $\Rightarrow$  (ii), so assume diam(G) = diam(H) = 3. Using (iii) of Lemma 2 we get  $\overline{\text{Pow}_2}(G) = H$  (from diam(G) = 3) and also  $\overline{\text{Pow}_2}(H) = G$  (from diam(H) = 3). Hence

$$\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G)) = \overline{\operatorname{Pow}_2}(H) = G.$$

To prove  $\neg(i) \Rightarrow \neg(ii)$  assume that (i) does not hold. We may assume (because of Remark 1) that either diam(G) > 3 or diam(H) > 3.

(a) Let first diam(G) > 3. We use (iv) of Lemma 2 and get

$$E(G) \setminus E(\overline{\text{Pow}_2}(\text{Pow}_2)(G))) \neq \emptyset.$$

Hence the proof is finished in the case a).

(b) Assuming  $\operatorname{diam}(G) = 3$  and  $\operatorname{diam}(H) > 3$  we use (v) of Lemma 2 and get

$$G \neq \overline{\text{Pow}_2}(\overline{\text{Pow}_2}(G))$$

as well.

An infinite number of examples of bipartite graphs satisfying (i) of Theorem 1 are yielded by the incidence graphs of Desarguesian finite projective geometries (see e.g. [3]).

An incidence graph of a given finite projective geometry  $\mathcal{G}$  is the graph whose vertex set is the union of the point set P and the line set L of  $\mathcal{G}$  and in which a point and a line are adjacent if and only if they are incident in  $\mathcal{G}$ . Let G be such a graph. Obviously, G is a bipartite graph with vertex classes P and L. The axioms of the projective geometry guarantee that to any two distinct points there exists exactly one line incident to both of them and to any two distinct lines there exists exactly one point incident to both of them. This implies that diam(G) = 3. On the other hand, in every Desarguesian projective geometry  $\mathcal{G}$  each line is incident with at least three points and each point is incident with at least three lines. Let H = ComplB(G). In H, a point and a line are adjacent if and only if they are not incident in  $\mathcal{G}$ . To any two distinct points  $p_1, p_2$  there exists at least one line l incident with none of them: it suffices to take the line l' joining  $p_1, p_2$ , to choose a point p' on l' distinct from both  $p_1$  and  $p_2$  and to choose as l another line incident with none of them. Hence diam(H) = 3, (i) of Theorem 1 is fulfilled and we have the following

**Corollary 1.** Let G be the incidence graph of a Desarguesian finite projective geometry. Then

 $\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G)) = G.$ 

Observe that if G is a connected bipartite graph with vertex classes  $V_1$  and  $V_2$ , where  $|V_1| = m$  and  $|V_2| = n$ , then G and ComplB(G) are edge-disjoint factors of the complete bipartite graph  $K_{m,n}$  whose union is  $K_{m,n}$ . It is natural to ask, in connection with Theorem 1, for which integers m, n there is a decomposition of  $K_{m,n}$  into two edge-disjoint factors of diameter 3.

The answer is given by the following

**Theorem 2.** A complete bipartite graph  $K_{m,n}$  can be decomposed into two edge-disjoint factors of diameter 3 if and only if  $\min(m, n) \ge 6$ .

First we prove the following lemmas:

**Lemma 3.** Let  $m, n \ge 1$ . If the complete bipartite graph  $K_{m,n}$  can be decomposed into two edge-disjoint factors of diameter 3 then so can be decomposed both  $K_{m+1,n}$  and  $K_{m,n+1}$ .

**Proof.** Let  $m, n \geq 1$ , assume that  $K_{m,n}$  can be decomposed into two edge-disjoint factors of diameter 3. Obviously, it suffices to show that such a decomposition exists also for  $K_{m+1,n}$ . Assume that  $K_{m+1,n}$  has vertex classes  $V_1$  and  $V_2$ , where  $|V_1| = m + 1$  and  $|V_2| = n$ . Choose two different

vertices  $x, y \in V_1$  and consider the complete bipartite graph  $K_{m,n}$  with vertex classes  $V_1 \setminus \{y\}$  and  $V_2$ . Use the assumption of the statement and consider  $G_1, G_2$ , two edge-disjoint factors of  $K_{m,n}$  fulfilling diam $(G_1) =$ diam $(G_2) = 3$ . Let  $G'_1 (G'_2)$  be the graph obtained from  $G_1 (G_2)$  by adding the vertex y and joining it by new edges with exactly those vertices of  $V_2$  which are adjacent in  $G_1 (G_2$ , respectively) to x. Then, for any two vertices  $u, v \in V(G_1), d_{G'_1}(u, v) = d_{G'_1}(u, u)$ . If  $u \in V(G_1), u \neq x$ , then  $d_{G'_1}(u, y) = d_{G'_1}(u, x)$ . Obviously,  $d_{G'_1}(x, y) = 2$ . Since diam $(G_1) = 3$ , diam $(G'_1) = 3$  as well. One shows analogously also diam $(G'_2) = 3$ .

**Lemma 4.** Let  $n \ge 1$ . The complete bipartite graph  $K_{5,n}$  cannot be decomposed into two edge-disjoint factors of diameter 3.

**Proof.** The statement obviously holds for n = 1, 2. Assume that for certain  $n \geq 3$  there is a decomposition of  $K_{5,n}$  into two edge-disjoint factors  $G_1, G_2$  such that

$$\operatorname{diam}(G_1) = \operatorname{diam}(G_2) = 3.$$

We will show that this assumption leads to a contradiction. Let  $V_1$  and  $V_2$  be the vertex classes of  $K_{5,n}$ , assume

$$V_1 = \{u_1, u_2, u_3, u_4, u_5\}, \qquad |V_2| = n.$$

First we show that for every  $v \in V_2$ ,  $2 \leq d_{G_1}(u) \leq 3$  and  $2 \leq d_{G_2}(u) \leq 3$ . In fact,  $d_{G_1}(v) = 0$  means that  $G_1$  is not connected,  $d_{G_1}(u) = 1$  implies that the only neighbor of v in  $G_1$  is adjacent in  $G_1$  to all the vertices of  $V_2$  (since diam $(G_1) = 3$ ) hence it is an isolated vertex of  $G_2$ . Finally,  $d_{G_1}(u) \in \{4, 5\}$  is equivalent to  $d_{G_2}(v) \in \{0, 1\}$  and we exclude it in a similar way. Put

$$D_2 = \{ v \in V_2; \ d_{G_1}(v) = 2 \}, \quad D_3 = \{ v \in V_2; \ d_{G_1}(v) = 3 \},$$

and observe that

$$D_2 = \{v \in V_2; \ d_{G_2}(v) = 3\}, \ D_3 = \{v \in V_2; \ d_{G_2}(v) = 2\}, \ \text{and} \ D_2 \cup D_3 = V_2.$$

It follows that for any  $v \in V_2$  either  $d_{G_1}(v) = 2$  or  $d_{G_2}(v) = 2$  (because  $d_{K_{m,n}}(u) = 5$ ). Since  $G_1$  and  $G_2$  play a symmetric role assume without loss of generality that  $D_2 \neq \emptyset$ . We are going to prove that  $D_3 \neq \emptyset$  as well.

From  $diam(G_1) = 3$  we have

$$v_1, v_2 \in D_2$$
 and  $v_1 \neq v_2 \Longrightarrow d_{G_1}(v_1, v_2) = 2.$ 

This is only possible (because of definition of  $D_2$ ) if either there is a vertex in  $V_1$  adjacent in  $G_1$  to all vertices of  $D_2$  (case I) or there are three pairwise different vertices in  $V_1$  (let us call them distinguished vertices of  $V_1$ ) such that each vertex of  $D_2$  is adjacent in  $G_1$  to exactly two of them (case II). We can already conclude that

$$D_3 \neq \emptyset$$

because otherwise (i.e., if  $V_2 = D_2$ ) there would be either a vertex in  $V_1$  adjacent in  $G_1$  to all vertices of  $V_2$  and thus isolated in  $G_2$  or two vertices in  $V_1$ , each of them isolated in  $G_1$ .

As a next step we show that the case II cannot occur. Assume on the contrary that the case II occurs but the case I does not. Let  $u_1, u_2, u_3$  be the distinguished vertices of  $V_1$ . Observe that to any  $u_i, 1 \le i \le 3$ , there is a vertex in  $D_2$  which is not adjacent in  $G_1$  to  $u_i$ . Consider arbitrary vertices  $v \in D_3$ , and  $v' \in D_2$ . Since  $d_{G_1}(v, v') = 2$  and v' is adjacent in  $G_1$  neither to  $u_4$  nor to  $u_5$ , we see that v is adjacent in  $G_1$  to at least two of the distinguished vertices and thus it is adjacent in  $G_1$  to at most one of the vertices  $u_4, u_5$ . Since

$$d_{G_1}(u_i, u_4) = 2, 1 \le i \le 3,$$

and  $u_4$  is adjacent in  $G_1$  to no vertex of  $D_2$ , there are two different vertices  $v_{41}, v_{42} \in D_3$  such that each of them is adjacent in  $G_1$  to  $u_4$  and to two of the distinguished vertices. The pairs of distinguished vertices adjacent in  $G_1$  to  $v_{41}, v_{42}$  are different because their union must be the whole set  $\{u_1, u_2, u_3\}$ . An analogous assertion holds for  $u_5$ , let  $v_{51}, v_{52}$  be the corresponding vertices of  $D_3$ . Observe that

$$d_{G_2}(v_{41}) = d_{G_2}(v_{42}) = d_{G_2}(v_{51}) = d_{G_2}(v_{52}) = 2.$$

Each of the vertices  $v_{41}, v_{42}$  is adjacent in  $G_2$  to  $u_5$ , moreover  $u_{41}$  is adjacent in  $G_2$  to a distinguished vertex  $x_1 \in \{u_1, u_2, u_3\}$  and  $v_{42}$  is adjacent in  $G_2$  to a distinguished vertex  $x_2 \in \{u_1, u_2, u_3\}$ , where  $x_1 \neq x_2$ . Analogously, each of the vertices  $v_{51}, v_{52}$  is adjacent in  $G_2$  to  $u_4$ ; moreover  $v_{51}$  is adjacent in  $G_2$  to  $y_1 \in \{u_1, u_2, u_3\}$  and  $v_{52}$  is adjacent in  $G_2$  to  $y_2 \in \{u_1, u_2, u_3\}$ , where  $y_1 \neq y_2$ . At least one of the vertices  $y_1, y_2$  is different from  $x_1$ , suppose without loss of generality that  $y_1 \neq x_1$ . Observe that the vertex  $v_{41}$  is adjacent in  $G_2$  to  $u_5$  and to  $x_1$ , the vertex  $v_{51}$  is adjacent in  $G_2$  to  $u_4$  and to  $y_1$ . Since  $\{u_5, x_1\} \cap \{u_4, y_1\} = \emptyset$  we have

$$d_{G_2}(v_{41}, v_{51}) \ge 4$$
, hence diam $(G_2) \ge 4$ ,

which is a contradiction. So we have shown that the case II cannot occur.

Hence there is a vertex  $x \in V_1$  which is adjacent in  $G_1$  to all the vertices of  $D_2$ , and a vertex  $y \in V_1$  which is adjacent in  $G_2$  to all the vertices of  $D_3$ . The vertex y is adjacent in  $G_1$  only to vertices of  $D_2$ . If  $x \neq y$  then x is the only vertex in  $V_1$  fulfilling  $d_{G_1}(x, y) = 2$ ; for any  $z \in V_1 \setminus \{x, y\}$  we have necessarily

$$d_{G_1}(z, y) \ge 4$$
, hence diam $(G_1) \ge 4$ ,

a contradiction. If x = y then x is adjacent in  $G_1$  to all the vertices of  $D_2$ and to no vertex of  $D_3$ . Consider any vertex  $v \in D_2$ ; it is adjacent in  $G_1$  to exactly one vertex  $w \in V_1 \setminus \{x\}$ . Any path of length 2 from v to a vertex of  $D_3$  in  $G_1$  must go through w and therefore w is adjacent in  $G_1$  to all the vertices of  $D_3$ . On the other hand, each vertex of  $V_1 \setminus \{x\}$  must be adjacent in  $G_1$  to a vertex of  $D_2$ ; otherwise its distance in  $G_1$  from x would be greater than 2. Hence each vertex of  $V_1 \setminus \{x\}$  is adjacent in  $G_1$  to all the vertices of  $D_3$ . As  $V_1 \setminus \{x\}$  has 4 elements, each vertex of  $D_3$  has degree 4, which is a contradiction accomplishing the whole proof.

**Proof of Theorem 2.** To prove the theorem it obviously suffices to exhibit a decomposition of  $K_{6,6}$  into two edge-disjoint factors  $G_1, G_2$  fulfilling diam $(G_1) = \text{diam}(G_2) = 3$  and to apply Lemmas 3 and 4. Let

$$V(K_{6,6}) = \{u_1, \dots, u_6, v_1, \dots, v_6\}, \quad E(K_{6,6}) = \{\{u_i, v_j\}; \quad 1 \le i, j \le 6\}.$$

Put

$$V(G_1) = V(G_2) = \{u_1, \dots, u_6, v_1, \dots, v_6\},\$$
  

$$E(G_1) = \{\{u_i, v_j\}; \ 1 \le i, j \le 6 \text{ and } j = i \text{ or } j \equiv i+1 \pmod{6} \text{ or } j \equiv i+3 \pmod{6}\},\$$

$$E(G_2) = E(K_{6,6}) \setminus E(G_1).$$

One verifies easily that  $G_1$  and  $G_2$  posses the required properties.

**Corollary 2.** For any  $m \ge 6$ ,  $n \ge 6$ , there is a connected bipartite graph G with vertex classes  $V_1$  and  $V_2$  such that

- (i)  $|V_1| = m, V_2 = n,$
- (ii)  $\overline{\text{Pow}_2}(\overline{\text{Pow}_2}(G)) = G \text{ and } \overline{\text{Pow}_2}(\overline{\text{Pow}_2}(\text{Compl}B(G))) = \text{Compl}B(G).$

## 3. Generalized Hypercubes and the Operator $\overline{Pow_2}$

Now we are going to investigate properties of the generalized hypercubes with respect to the operator  $\overline{\text{Pow}_2}$ .

For  $n \ge 1$  we denote the set  $\{1, \ldots, n\}$  by [n] and start with the basic definition (cf. [2]):

**Definition 1.** Let  $n \ge 1$  and  $S \subseteq [n]$ . The generalized hypercube  $Q_n(S)$  has as vertices all the 0-1 vectors of size n. Two vertices are adjacent in  $Q_n(S)$  iff their Hamming distance (i.e., the number of coordinates they differ in) belongs to S.

Observe the following facts:  $Q_n(\emptyset) \simeq 2^n K_1, Q_n(\{1, \ldots, n\}) \simeq K_{2^n}$ , and for the well-known graph of the *n*-dimensional hypercube  $Q_n$  we have  $Q_n \simeq Q_n(\{1\})$  (we will omit in this and similar cases parentheses and write simply  $Q_n(1)$ ). For  $n \ge 1$  and any S the vertex sets of  $Q_n$  and  $Q_n(S)$  coincide.

For basic properties of generalized hypercubes (in particular those related to isomorphism) see [2]. Generalized hypercubes (also called distance graphs in [5]) are special case of the cube-like graphs, defined and studied in [4]. It is proved in [5] that the chromatic number of a cube-like graph cannot be 3. Lovász showed that a cube-like graph has an integral spectrum (cf. [4]).

Given a 0-1 vector  $x = (x_1, \ldots, x_n)$  of size  $n \ge 1$ , denote by w(x) the number of 1-coordinates of x, i.e.,  $w(x) = \sum_{i=1}^n x_i$ . (Usually, w(x) is called the Hamming weight of x.) Observe that for any  $S \subseteq [n]$ ,  $V(Q_n(S)) = V_o \cup V_e$ , where  $V_o(V_e)$  is the set of all 0-1 vectors x with w(x) odd (even, respectively).

Now we are going to determine the complement and square of a generalized hypercube. For  $S \subseteq [n]$  put

$$C^{(n)}(S) = [n] \setminus S,$$
  

$$S^{(n,2)} = \{ i \in [n]; \exists j, k \in S \text{ such that } |j-k| \le i \le \min(j+k, 2n-j-k) \text{ and } i \equiv j+k \pmod{2} \},$$

 $\mathcal{P}_2^{(n)}(S) = S \cup S^{(n,2)}.$ 

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**Remark 2.** If  $x = (x_1, \ldots, x_n)$  and  $y = (y_1, \ldots, y_n)$  are two 0 - 1 vectors of size n, define their  $\oplus$ -sum (also called modulo 2 sum) as usual by

$$x \oplus y = (z_1, \ldots, z_n),$$

where  $z_i = 1$  if  $x_i \neq y_i$  and  $z_i = 0$  if  $x_i = y_i$ , i = 1, ..., n. One can see that given  $S \subseteq [n]$  and  $i \in [n]$ ,  $i \in S^{(n,2)}$  if and only if there are 0-1 vectors x, y of size n such that  $w(x) \in S$ ,  $w(y) \in S$ , and  $i = w(x \oplus y)$ .

We have the following

**Lemma 5.** Let  $n \neq 1$  and  $S \subseteq [n]$ . Then

$$\overline{Q_n(S)} = Q_n(\mathcal{C}^{(n)}(S))$$
 and  $\operatorname{Pow}_2(Q_n(S)) = Q_n(\mathcal{P}_2^{(n)}(S)).$ 

**Proof.** The proof follows directly from definitions of the generalized hypercube, complement and the square of a graph.

Using Lemma 5 we can study behavior of generalized hypercubes under the operators of square and complement by studying certain subsets of [n] under the operators  $\mathcal{P}_2^{(n)}$  and  $\mathcal{C}^{(n)}$ .

Denote by  $\mathcal{R}^{(n)}$  the composed operator  $\mathcal{C}^{(n)}\mathcal{P}_2^{(n)}\mathcal{C}^{(n)}\mathcal{P}_2^{(n)}$ . We will be interested in the sets  $S \subseteq [n]$  fulfilling

$$\mathcal{R}^{(n)}(S) = S$$

because of the following statement, which is an immediate consequence of Lemma 5.

**Proposition 1.** Let  $n \ge 1$ , let  $S \subseteq [n]$ . Then

$$\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(Q_n(S))) = Q_n(S)$$

if and only if S fulfils  $\operatorname{diam}(Q_n(\sigma)) = 3$ .

Hence, if S fulfils diam $(Q_n(S)) = 3$  then  $Q_n(S)$  is 2-periodic with respect to  $\overline{\text{Pow}_2}$ . It is an open problem whether this equality is also necessary for the 2-periodicity with respect to  $\overline{\text{Pow}_2}$ . (Proposition 1 only deals with the equality of the graphs  $\overline{\text{Pow}_2}(\overline{\text{Pow}_2}(Q_n(S)))$  and  $Q_n(S)$ , not with their isomorphy.) Observe that (4) is trivially fulfilled if  $S = \emptyset$  or S = [n]. We will be interested in the sets  $S \subseteq [n]$  fulfilling diam $(Q_n(S)) = 3$  and  $\emptyset \neq S \neq [n]$ : **Definition 2.** Let  $n \ge 1$  and  $S \subseteq [n]$ . We call S an  $\mathcal{R}^{(n)}$ -fixed point if  $\emptyset \ne S \ne [n]$  and diam $(G_n(S)) = 3$  holds. We say that S is a minimal  $\mathcal{R}^{(n)}$ -fixed point if no proper subset of S is an  $\mathcal{R}^{(n)}$ -fixed point.

The following statement follows directly from the definition.

**Proposition 2.** Let  $n \ge 1$ , let  $S \subseteq [n]$  be an  $\mathcal{R}^{(n)}$ -fixed point. Then  $S' = \mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))$  is also an  $\mathcal{R}^{(n)}$ -fixed point and

$$S = \mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S')).$$

Introduce still another denotation: For  $n \ge 1$ , let

$$[n]_o = \{i \in [n]; i \text{ is odd}\},$$
$$[n]_e = \{i \in [n], i \text{ is even}\}.$$

Observe that  $Q_n(S)$  is not connected if  $S \subseteq [n]_e$ . If  $S \subseteq [n]_o$  then  $Q_n(S)$  is a bipartite graph with vertex classes  $V_o$  and  $V_e$ ; it is not connected only if  $S = \emptyset$  or  $S = \{n\}$ , otherwise it is connected. If  $S = [n]_e$  then  $Q_n(S)$  is a disjoint union of two complete subgraphs induced by  $V_o$  and  $V_e$ . If  $[n]_e \subsetneq S$ then  $Q_n(S)$  contains the above described graph as a spanning subgraph and moreover it contains at least one matching between  $V_o$  and  $V_e$ , in this case,  $\operatorname{Pow}_2(Q_n(S))$  is a complete graph.

**Remark 3.** Using the above denotation and definition, we observe that the following facts hold: for any  $n \ge 1$ ,

$$S' \subseteq S \subseteq [n] \Longrightarrow S'^{(n,2)} \subseteq S^{(n,2)},$$
  

$$S \subseteq [n]_e \text{ or } S \subseteq [n]_o \Longrightarrow S^{(n,2)} \subseteq [n]_e,$$
  

$$[n]_o^{(n,2)} = [n]_e.$$

In what follows generalized hypercubes  $Q_n(S)$  with  $S \subseteq [n]_o$  will play an important role. First, we have the following

**Lemma 6.** Let  $n \ge 1$ , let  $S_1$  and  $S_2$  fulfil  $S_1 \subseteq [n]_o$ ,  $S_2 \subseteq [n]$ , and  $S_2 \not\subseteq [n]_o$ . Then

$$Q_n(S_1) \not\simeq Q_n(S_2).$$

**Proof.** Let  $n, S_1$  and  $S_2$  fulfil the assumptions of Lemma 6. Then either  $Q_n(S_1) \simeq 2^{n-1}K_2$  (if n is odd and  $S_1 = \{n\}$ ) or  $Q_n(S_1)$  is connected and bipartite (if  $S_1 \subseteq [n]_o$  and  $S_1 \neq \{n\}$ ).

Similarly, for  $Q_n(S_2)$  we have the following possibilities:

- (a)  $Q_n(S_2) \simeq 2^{n-1} K_2$ , where *n* is even (if  $S_2 = \{n\}$ ),
- (b)  $Q_n(S_2)$  consists of two connected components, each of them having  $2^{n-1}$  vertices (if  $S_2 \subseteq [n]_e$  and  $S_2 \neq \{n\}$ ),
- (c)  $Q_n(S_2)$  is connected and non-bipartite (if  $S_2 \cap [n]_e \neq \emptyset$  and simultaneously also  $S_2 \cap [n]_o \neq \emptyset$ ).

Hence  $Q_n(S_1) \not\simeq Q_n(S_2)$ .

**Theorem 3.** For  $n \ge 1$  and  $S \subseteq [n]_o$ , if

(1) 
$$\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(Q_n(S))) \simeq Q_n(S),$$

then

(2) 
$$\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(Q_n(S))) = Q_n(S).$$

**Proof.** Let  $n \ge 1$  and  $S \subseteq [n]_o$ , we have to prove that (1) implies (2). If n = 1 or  $S = \emptyset$  then (2) holds. Let us assume  $n \ge 1$ . If n is odd and  $S = \{n\}$ , (1) does not hold (Lemma 1).

Thus we may assume that  $\emptyset \neq S \neq \{n\}$ . After putting

$$G = Q_n(S)$$

we observe that G is connected, bipartite and has at least 4 vertices. Let H = ComplB(G). Suppose that (2) does not hold, i.e.,

$$\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G)) \neq G.$$

By Theorem 1, either  $\operatorname{diam}(G) \neq 3$  or  $\operatorname{diam}(H) \neq 3$ . One easily gets a contradiction in cases  $\operatorname{diam}(G) = 1$ ,  $\operatorname{diam}(H) = 1$ , and  $\operatorname{diam}(H) = 2$ . If  $\operatorname{diam}(G) = 2$  then G is a complete bipartite graph with  $|V(G)| \geq 4$  and (1) does not hold. If  $\operatorname{diam}(G) = 3$  and  $\operatorname{diam}(H) > 3$ , then (by (v) of Lemma 2)  $\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G))$  contains an edge joining two vertices of the same parity. If  $\operatorname{diam}(G) > 3$  then (by (ii) and (iv) of Lemma 2) either  $\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G))$  is a proper subgraph of G or again it contains an edge joining two vertices of the same parity. Altogether,

$$\overline{\operatorname{Pow}_2}(\overline{\operatorname{Pow}_2}(G)) = Q_n(S'),$$

where either  $S' \subsetneq S$  or  $S' \cap [n]_e \neq \emptyset$ . According to Lemma 6 then (1) does not hold.

**Theorem 4.** For  $n \ge 1$  and  $S \subseteq [n]_o$ , the following statements are equivalent:

- (i)  $S^{(n,2)} = ([n]_o \setminus S)^{(n,2)} = [n]_e$ ,
- (ii)  $\overline{\text{Pow}_2}(\overline{\text{Pow}_2}(Q_n(S))) = Q_n(S).$

**Proof.** For n = 1 and  $S = \{1\}$  or  $S = \emptyset$  the assertion holds trivially. We may assume therefore  $n \ge 1$  and  $S \subseteq [n]_o$ .

(i)  $\Rightarrow$  (ii): As  $S \subseteq [n]_o$ ,  $Q_n(S)$  is bipartite. It follows from (i) that both the bipartition classes of  $V(Q_n(S))$  induce complete subgraphs in  $Pow_2(Q_n(S))$  and therefore

$$\operatorname{diam}(Q_n(S)) = 3.$$

We observe further that

$$\overline{\operatorname{Pow}_2}(Q_n(S)) = Q_n([n]_o \setminus S)$$

and arguing in a similar way as above we conclude that

 $\operatorname{diam}(Q, ([n]_o \setminus S)) = 3$ 

as well. Using Theorem 1, (ii) follows.

(ii)  $\Rightarrow$  (i): Using again Theorem 1 we have

$$\operatorname{diam}(Q_n(S)) = \operatorname{diam}(\operatorname{Pow}_2(Q_n(5))) = 3$$

and this already implies (i).

**Proposition 3.** Let k and n be such that k is odd,  $k \ge 3$ , and  $|n-2k| \ge 1$ . Then  $\{k\}^{(n,2)} = \{k-2, k+2\}^{(n,2)} = [n]_e$  and both  $\{k\}$  and  $\{k-2, k+2\}$  are minimal  $\mathcal{R}^{(n)}$ -fixed points.

**Proof.** Let k and n fulfil the assumptions of the statement. Directly from the definition we obtain

$$\{k\}^{(n,2)} = [n]_e$$
 and  $\{k-2, k+2\}^{(n,2)} = [n]_e$ .

Then, using Remark 3, Theorem 4, and Proposition 1 we conclude that both  $\{k\}$  and  $\{k-2, k+2\}$  are  $\mathcal{R}^{(n)}$ -fixed points. It remains to be shown that neither  $\{k-2\}$  nor  $\{k+2\}$  are  $\mathcal{R}^{(n)}$ -fixed points. Let us start with  $\{k-2\}$ . Since  $2k-2 \in [n]_e \setminus \{k-2\}^{(n,2)}$  we get diam $(Q_n(k-2)) > 3$ ;

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using Theorem 1 and Proposition 1 we conclude that  $\{k-2\}$  is not an  $\mathcal{R}^{(n)}$ -fixed point. We proceed similarly with  $\{k+2\}$ : since  $2n - 2k - 2 \in [n]_e \setminus \{k+2\}^{(n,2)}$  we see that diam $(Q_n(k+2)) > 3$  and therefore  $\{k+2\}$  is not an  $\mathcal{R}^{(n)}$ -fixed point.

**Proposition 4.** Let  $n \geq 1$ . If  $S \subseteq [n]_e$  or  $[n]_e \subseteq S$ , then S is not an  $\mathcal{R}^{(n)}$ -fixed point.

**Proof.** Assume  $n \ge 1$  and  $S \in [n]$ , let S be an  $\mathcal{R}(n)$ -fixed point.

a) If  $S \subseteq [n]_e$ , then  $P_2^{(n)}(S) \subseteq [n]_e$ , hence

$$[n]_o \subseteq \mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S)), \quad \mathcal{P}_2^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))) = [n],$$

and  $\mathcal{R}^{(n)}(S) = \emptyset$ , a contradiction.

b) From  $[n]_e \subseteq S$  one would have

$$\mathcal{P}_2^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_2^{(n)}(S))) \subseteq [n]_o,$$

but this can only hold if

$$C^{(n)}(\mathcal{P}_2^{(n)}(S)) = \{n\} \text{ and } n \text{ is odd.}$$

Then one has

$$\mathcal{P}_{2}^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_{2}^{(n)}(S))) = \{n\},\$$
  
$$S = \mathcal{C}^{(n)}\mathcal{P}_{2}^{(n)}(\mathcal{C}^{(n)}(\mathcal{P}_{2}^{(n)}(S))) = [n-1].$$

However,  $\mathcal{P}_2^{(n)}([n-1]) = [n]$ , which is a contradiction.

One verifies directly (using also Proposition 4) that there are no  $\mathcal{R}(n)$ -fixed points for  $1 \leq n \leq 4$  and for n = 8. From the Proposition 3 and the following Proposition 5 we conclude that for other  $n \mathcal{R}(n)$ -fixed points do exist.

**Proposition 5.** Let  $k \ge 3$ , let n = 4k. Then both  $\{2k - 1, 2k + 1\}$  and  $\{2k - 3, 2k + 3\}$  are minimal  $\mathcal{R}(n)$ -fixed points.

**Proof.** Let  $k \geq 3$  and n = 4k. Directly from the definition we have

$$\{2k-1, 2k+1\}^{(n,2)} = \{2k-3, 2k+3\}^{(n,2)} = [n]_e,$$

hence according to Remark 3 and Theorem 4 it follows that both  $\{2k - 1, 2k + 1\}$  and  $\{2k - 3, 2k + 3\}$  are  $\mathcal{R}(n)$ -fixed points. Similarly as in the proof of Proposition 3 above we show that they are minimal: this follows from

$$n \notin \{2k-l\}^{(n,2)}, n \notin \{2k+1\}^{(n,2)}, n \notin \{2k-3\}^{(n,2)} \text{ and } n \notin \{2k+3\}^{(n,2)}$$
  
using Theorem 1 and Proposition 1.

using Theorem 1 and Proposition 1.

**Proposition 6.** Let  $n \ge 1$ , let  $A, C \subseteq [n]_o$  fulfil

$$A \cap C = \emptyset$$
 and  $A^{(n,2)} = C^{(n,2)} = [n]_e.$ 

Then every A' fulfilling

$$A \subseteq A' \subseteq [n]_o$$
 and  $A' \cap C = \emptyset$ 

is an  $\mathcal{R}(n)$ -fixed point.

**Proof.** From the assumptions we have  $A'^{(n,2)} = [n]_e$  and  $\mathcal{P}_2^{(n)}(A') = A' \cup [n]_e$ . Further,

$$\mathcal{P}^{(n)}(A' \cup [n]_e) = [n]_o \setminus A'.$$

Since  $C^{(n,2)} = [n]_e$  and  $C \subseteq [n]_o \setminus A'$ , it follows that  $([n]_o \setminus A')^{(n,2)} = [n]_e$ and therefore

$$\mathcal{P}_2^{(n)}([n]_o \setminus A') = [n]_o \setminus A' \cup [n]_e = \mathcal{C}^{(n)}(A').$$

This already implies  $\mathcal{R}_2^{(n)}(A') = A'$ .

**Example.** Let  $k \ge 1$ , let  $4k + 1 \le n \le 4k + 3$ . It follows from Proposition 3 that

$$\{2k+l\}^{(n,2)} = [n]_e,$$

 $\{2k-1, 2k+3\}^{(n,2)} = [n]_e$ , and that  $\{2k+1\}$  is a minimal  $\mathcal{R}(n)$ -fixed point. From (2') we obtain (using Remark)

$$([n]_o \setminus \{2k-1, 2k+3\})^{(n,2)} = [n]_e.$$

Hence, using Proposition 2, we claim that  $\{2k - 1, 2k + 3\}$  is an  $\mathcal{R}(n)$ -fixed point (obviously a minimal one).

### 4. Concluding Remarks and Open Problems

Our efforts to construct an  $\mathcal{R}(n)$ -fixed point containing even numbers were unsuccessful — all  $\mathcal{R}(n)$ -fixed points we know so far consist only of odd numbers. This is the reason we propose the following

**Conjecture.** If S is an  $\mathcal{R}(n)$ -fixed point for certain  $n \ge 1$  then  $S \subseteq [n]_o$  (i.e., S contains no even number).

In order to clarify the situation with the Conjecture we argue as follows: let for certain  $n \ge 1$  be an  $\mathcal{R}(n)$ -fixed point. Put

$$B = P_2^{(n)}(A), \quad C = \mathcal{C}^{(n)}(B), \text{ and } D = \mathcal{P}_2^{(n)}(C).$$

So we have  $A = \mathcal{C}^{(n)}(D)$ . Since both A and C are  $\mathcal{R}(n)$ -fixed points, we obtain (using Proposition 4)

$$B \cap [n]_e \neq \emptyset \neq D \cap [n]_e$$

From here it follows that exactly one of the following possibilities occurs:

I.  $[n]_e \subseteq B$  and  $[n]_e \subseteq D$ , II.  $[n]_e \subseteq B$  and  $[n]_e \not\subseteq D$ , or  $[n]_e \not\subseteq B$  and  $[n]_e \subseteq D$ , III.  $[n]_e \not\subseteq B$  and  $[n]_e \not\subseteq D$ .

The case I is just covered by the Conjecture; from  $[n]_e \subseteq D$  one derives  $A \cap [n]_e = \emptyset$ . All  $\mathcal{R}(n)$ -fixed points we know so far are of the type I.

Now we show that II cannot hold: assume  $n \ge 1$ , let  $A \subseteq [n]$  be an  $\mathcal{R}(n)$ -fixed point, let B, C, D be given as above, assume without loss of generality  $[n]_e \not\subseteq B$  and  $[n]_e \subseteq D$ . Then  $A \subseteq [n]_o$ ; by Proposition 1 and Theorem 4,  $A^{(n,2)} = [n]_e$ , hence  $[n]_e \subseteq B$ , a contradiction.

Our Conjecture would be settled in positive, if we were able similarly to exclude III.

Coming back from sets of integers that are  $\mathcal{R}(n)$ -fixed points, to graphs again, we conclude that we succeeded to find graphs that are 2-periodic with respect to the operator  $\overline{\text{Pow}_2}$ . It might be interesting also to consider periods and powers different from 2; so, in the most general setting we formulate the following

**Problem.** Let (i, j) be a pair of integers,  $i \ge 1$ ,  $j \ge 2$ ,  $(i, j) \ne (2, 2)$ . Do there exist  $n \ge 1$  and  $S \subseteq [n]$  such that the graph  $Q_n(S)$  is *i*-periodic with respect to the operator  $\overline{\text{Pow}_j}$ ?

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