

## ON VIZING'S CONJECTURE

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### Abstract

A dominating set  $D$  for a graph  $G$  is a subset of  $V(G)$  such that any vertex in  $V(G) - D$  has a neighbor in  $D$ , and a domination number  $\gamma(G)$  is the size of a minimum dominating set for  $G$ . For the Cartesian product  $G \square H$  Vizing's conjecture [10] states that  $\gamma(G \square H) \geq \gamma(G)\gamma(H)$  for every pair of graphs  $G, H$ . In this paper we introduce a new concept which extends the ordinary domination of graphs, and prove that the conjecture holds when  $\gamma(G) = \gamma(H) = 3$ .

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## 1 Introduction

A conjecture proposed by Vizing [10] has been a challenge for several authors [1–9]. So far only partial solutions are known, which show that

$$(1) \quad \gamma(G \square H) \geq \gamma(G)\gamma(H).$$

holds for graphs  $G, H$ , which belong to certain classes of graphs.

We shall consider finite, undirected, connected graphs without loops or multiple edges. Let us recall that the *Cartesian product*  $G \square H$  of graphs  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  has a vertex set  $V(G) \times V(H)$ , and vertices  $(u, v), (x, y)$  are adjacent whenever  $u = x$  and  $vy \in E(H)$  or  $ux \in E(G)$  and  $v = y$ . For a fixed vertex  $u \in V(G)$ , a *H-layer*  $H_u$  is a subgraph induced by the set of vertices  $\{(u, v_i), v_i \in V(H)\}$ , and analogously we define *G-layers*. A *2-packing number*  $P_2(G)$  of a graph  $G$  is defined as

the maximum cardinality of a set  $S \subset V(G)$  such that any two vertices in  $S$  are on distance at least three. As usual, the *distance*  $d_G(u, v)$  in  $G$  between vertices  $u, v$  is the length of a shortest path between  $u$  and  $v$ . A *diameter*  $\text{diam}(G)$  of a graph  $G$  is  $\max\{d(u, v) : u, v \in V(G)\}$ . For a vertex  $v$  in  $G$ , a *neighborhood*  $N(v)$  is defined as a set of neighbors of  $v$ , while a *closed neighborhood*  $N[v]$  is  $N(v) \cup \{v\}$ .

We say that a graph  $G$  satisfies Vizing's conjecture if the inequality (1) holds for any graph  $H$ . It is trivial that if  $\gamma(G) = P_2(G)$  then  $G$  satisfies Vizing's conjecture. Several authors have proved in different ways that graphs with domination number at most two satisfy the conjecture. Thus the smallest unsolved case in this direction are graphs with domination number three, in particular, the conjecture has been opened for  $\gamma(G) = \gamma(H) = 3$ .

In the next section we introduce a so-called graph-domination of graphs which generalizes the usual domination of graphs. The idea for this concept is obtained from observing dominating sets of a Cartesian product of two graphs. This concept proves to be useful in the proof of Fisher's Vizing-like result in which one of the dominating numbers of factors is changed with a fractional domination number. Furthermore, it enables us to give quite a brief proof of (1) for the case where both factors have domination number three, which is done in the third section.

## 2 Graph-Domination

The hard task of (dis)proving Vizing's conjecture has led several authors to approach it from different angles and study problems which are closely related to the original one. We shall present yet another approach, and define a so-called graph-domination which extends the ordinary domination of graphs.

First, let us recall a *fractional domination* of graphs. Let  $G$  be a graph and let  $f : V(G) \rightarrow \mathbb{R}_0^+$  be a map that assigns nonnegative weights to vertices. An additional condition is required for each  $v \in V(G)$ :

$$(2) \quad \sum_{u \in N[v]} f(u) \geq 1.$$

A *fractional domination number* of a graph  $G$  is

$$\gamma_f(G) = \min_f \left\{ \sum_{v \in V(G)} f(v) \right\},$$

where  $f$  obeys (2). If in the range of  $f$  we allow only weights 0 or 1 then this coincides with the ordinary domination of  $G$ .

For an arbitrary graph  $H$  we introduce a *graph-domination* of a graph  $G$  with respect to  $H$ . Let  $f : V(G) \rightarrow \mathcal{P}(V(H))$  be a map which assigns to each vertex of  $G$  a subset of vertices of  $V(H)$ . In addition, for each  $v \in V(G)$  let

$$(3) \quad \left[ \bigcup_{u \in f(v)} N_H[u] \right] \bigcup \left[ \bigcup_{z \in N_G(v)} f(z) \right] = V(H).$$

A *graph-domination number* of a graph  $G$  with respect to  $H$  is

$$\gamma_H(G) = \min_f \left\{ \sum_{v \in V(G)} |f(v)| \right\}.$$

where  $f$  obeys (3). If for  $H$  we take a trivial graph  $K_1$  then this definition coincides with the ordinary domination of  $G$ .

The definition of graph-domination is closely connected with the domination of a Cartesian product of graphs since we clearly have  $\gamma_H(G) = \gamma(G \square H)$  for any pair of graphs  $G, H$  (note that any dominating set of a Cartesian product implies an appropriate map  $f$ , and vice-versa). Hence we can reformulate (1) by saying that a graph  $G$  satisfies the conjecture if for any graph  $H$  we have

$$\gamma_H(G) \geq \gamma(G)\gamma(H).$$

This approach will be used in the proof of Fisher's result:

$$\gamma(G \square H) \geq \gamma_f(G)\gamma(H),$$

which holds for all pairs of graphs  $G, H$  [5]. Fisher's proof of this result follows from a Vizing-like result on a strong product of graphs, and is quite difficult. The proof that we are about to present is self-contained and rather straightforward.

Let  $G$  and  $H$  be arbitrary connected graphs and let  $f : V(G) \rightarrow \mathcal{P}(V(H))$  be a map which obeys condition (3) of a graph-domination of  $G$  with respect to  $H$ , and also assume that in  $f$  the minimum  $\gamma_H(G)$  is achieved. Now, define a map  $g : V(G) \rightarrow \mathbb{R}_0^+$ , with

$$v \mapsto \frac{|f(v)|}{\gamma(H)}.$$

Using condition (3) we deduce that for any  $v \in V(G)$  we have  $\sum_{u \in N[v]} |f(u)| \geq \gamma(H)$ , since vertices from  $f(v)$  together with vertices from  $\bigcup_{u \in N(v)} f(u)$  must dominate  $H$  (in fact, the condition is even stronger). Hence we have  $\sum_{u \in N[v]} \frac{|f(u)|}{\gamma(H)} \geq 1$ , and we infer that  $g$  obeys condition (2) of the fractional domination of  $G$ . We easily see that

$$\begin{aligned} \sum_{v \in V(G)} g(v) &\geq \gamma_f(G), \\ \sum_{v \in V(G)} \frac{|f(v)|}{\gamma(H)} &\geq \gamma_f(G), \\ \sum_{v \in V(G)} |f(v)| &\geq \gamma_f(G)\gamma(H), \\ \gamma_H(G) &\geq \gamma_f(G)\gamma(H), \end{aligned}$$

so we have proved

**Theorem 1** (Fisher, 1994). *For connected graphs  $G, H$  we have  $\gamma(G \square H) \geq \gamma_f(G)\gamma(H)$ .*

### 3 The $3 \times 3$ Case

Following the approach of Barcalkin and German [1], Hartnell and Rall [6] provided, so far, the largest class of graphs which satisfy the conjecture. As a by-product they obtained the following improvement considering the 2-packing number which we will use in the sequel.

**Proposition 2** [6]. *If for a graph  $G$  we have  $\gamma(G) \leq P_2(G) + 1$ , then  $G$  satisfies Vizing's conjecture.*

From Proposition 2 we immediately deduce

**Corollary 3.** *If  $\gamma(G) = 3$  and  $\text{diam}(G) \geq 3$ , then  $G$  satisfies Vizing's conjecture.*

Therefore, to prove Vizing's conjecture for graphs with domination number 3, one must solve it for graphs with diameter 2. It is intuitively clear that by limiting the diameter of a graph  $G$  more vertices are needed to ensure that  $\gamma(G) = 3$ . First, we observe the following

**Lemma 4.** *If  $\gamma(G) = 3$  and  $\text{diam}(G) = 2$ , then the smallest vertex degree  $\delta(G)$  in  $G$  is at least 3.*

**Proof.** Observe that in graphs with diameter 2 every neighborhood of a vertex is a dominating set for  $G$ . ■

It is obvious in general graphs that the largest degree of a vertex in  $G$ ,  $\Delta(G)$ , is at most  $n - \gamma(G) + 1$ . We can improve this bound for graphs with diameter 2.

**Lemma 5.** *If  $G$  is a graph with  $\text{diam}(G) = 2$ , then  $\Delta(G) \leq n - 2\gamma(G) + 2$ .*

**Proof.** Let  $v$  be the vertex of largest degree in  $G$  and consider a subset  $A = V(G) - N[v]$  of vertices outside the closed neighborhood of  $v$ . Obviously, we need at least  $\gamma(G) - 1$  vertices to dominate  $A$  with vertices from  $G$ . Since  $\text{diam}(G) = 2$  every pair of vertices in  $A$  is either adjacent or has a common neighbor in  $V(G) - \{v\}$ . For a fixed  $n$  and  $\gamma(G)$ ,  $A$  has the smallest cardinality if it is an odd number, such that  $2(\gamma(G) - 2)$  vertices in  $A$  are dominated as pairs, and a vertex in  $A$  remains undominated. Hence together we have at least  $2\gamma(G) - 3$  vertices in  $A$  and from that the desired bound is obtained. ■

As mentioned, the lower bound for  $|V(G)|$  increases when  $\text{diam}(G)$  is 2.

**Lemma 6.** *If  $G$  is a graph with  $\text{diam}(G) = 2$ ,  $\gamma(G) = 3$ , then  $G$  has at least 8 vertices.*

**Proof.** We know that  $\delta(G) \geq 3$ , and suppose that  $G$  has 7 vertices. Then from Lemma 5 we deduce that  $\Delta(G) \leq 3$ , thus  $G$  would be 3-regular, but this is impossible by the handshaking lemma. If  $G$  had fewer than 7 vertices then, in view of Lemma 5, we have  $\Delta(G) \leq 2$ , a contradiction to  $\delta(G) \geq 3$ . ■

El-Zahar and Pareek have established the following lower bound for a domination number of a Cartesian product of two graphs [3],

$$\gamma(G \square H) \geq \min\{|V(G)|, |V(H)|\}.$$

We can improve this bound for graphs with domination number at least three.

**Proposition 7.** *Let  $G$  and  $H$  be graphs with domination numbers at least 3, such that  $|V(G)| \neq |V(H)|$ . Then*

$$\gamma(G \square H) \geq \min\{|V(G)|, |V(H)|\} + 1.$$

**Proof.** Suppose that  $H$  has fewer vertices than  $G$  and that the lemma is false. Let  $D$  be any minimum dominating set for  $G \square H$  with  $|V(H)|$  vertices (this number of vertices is obtained from the bound of El-Zahar and Pareek). It is clear that in every layer  $G_u$  ( $u \in V(H)$ ) we have exactly one vertex of  $D$  (otherwise exists a vertex  $(v', u') \in V(G \square H)$  such that  $V(G_{u'}) \cap D = \emptyset$  and  $V(H_{v'}) \cap D = \emptyset$ ). Also it is obvious that there exists a layer  $H_v$  (for  $v \in V(G)$ ) which does not have any vertex from  $D$ . Let  $v_1, \dots, v_k$  be all vertices in  $G$  such that  $H_{v_i} \cap D \neq \emptyset$ , and let  $w_1, \dots, w_l$  be the rest of  $G$ . Then every vertex  $(w_i, u)$  must be dominated by a certain  $(v_j, u)$ , and each  $(v_j, u)$  must dominate all  $(w_i, u)$  where  $u$  is an arbitrary vertex of  $H$ . But then any pair of vertices  $v_i, w_j$  in  $G$  (e.g.,  $v_1, w_1$ ) form a dominating set for  $G$ , thereby  $\gamma(G) \leq 2$  which is a contradiction. ■

We are now prepared for our main result.

**Theorem 8.** *Vizing's conjecture holds for any graphs  $G, H$  with  $\gamma(G) = \gamma(H) = 3$ .*

**Proof.** Let  $G$  and  $H$  be graphs with domination number 3, and assume that  $\gamma(G \square H) \leq 8$ . By Lemma 6,  $G$  and  $H$  must have at least 8 vertices, and by the bound of El-Zahar and Pareek, one of these graphs must have exactly 8 vertices, so that  $\gamma(G \square H) = 8$ . Finally, Proposition 7 tells us that both  $G$  and  $H$  must have 8 vertices, so let  $G$  and  $H$  be such graphs.

Let  $D$  be a minimum dominating set of  $G \square H$ , which has 8 vertices. With the same argument as in the proof of Proposition 7 we deduce that in each layer  $G_u$  and  $H_v$  ( $u \in V(H), v \in V(G)$ ) we must have at least one vertex from  $D$  (otherwise one of the graphs would have a domination number less than 3). In terms of the graph-domination, we must dominate  $G$  with respect to  $H$  in such a way that to each vertex of  $G$  exactly one vertex of  $H$  is assigned. Let  $f : V(G) \rightarrow \mathcal{P}(V(H))$  be a corresponding map (note that  $f$  maps vertices to singleton subsets of  $V(H)$ , and that it may be viewed as a bijective mapping from  $V(G)$  to  $V(H)$ ).

Let  $a, b \in V(H)$  where  $ab$  is not an edge of  $H$ . Clearly, there exist vertices  $r, s \in V(G)$  such that  $f(r) = \{a\}$  and  $f(s) = \{b\}$ , where  $(r, a) \in D$  and  $(s, b) \in D$ . Noting that  $(r, b)$  is not dominated by  $D \cap H_r$  (that is, by  $(r, a)$ ), it follows that  $(r, b)$  must be dominated by  $D \cap G_b$ . But  $D \cap G_b = \{(s, b)\}$  implies that  $sr$  is an edge of  $G$ . Therefore  $f^{-1}(\{a\})f^{-1}(\{b\})$  is an edge of  $G$ . Thus the complement of  $H$ , graph  $\overline{H}$ , is a spanning subgraph of  $G$ , so  $\gamma(G) \leq \gamma(\overline{H})$ . Using a well-known and easy fact that

$$\gamma(\overline{H})\gamma(H) \leq |V(H)|$$

which is true for any graph  $H$ , we infer in our case that

$$\gamma(G)\gamma(H) \leq 8,$$

which is a contradiction with  $\gamma(G) = \gamma(H) = 3$ . The proof is complete. ■

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