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CHROMATIC POLYNOMIALS OF HYPERGRAPHS

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Abstract

In this paper we present some hypergraphs which are chromatically characterized by their chromatic polynomials. It occurs that these hypergraphs are chromatically unique. Moreover we give some equalities for the chromatic polynomials of hypergraphs generalizing known results for graphs and hypergraphs of Read and Dohmen.

Keywords: chromatic polynomial, chromatically unique hypergraphs, chromatic characterization.

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1. Introduction

A *simple hypergraph* $H = (V, \mathcal{E})$ consists of a finite non-empty set V of *vertices* and a family $\mathcal{E}$ of *edges* which are distinct non-empty subsets of V of the cardinality at least 2. An edge of cardinality h is called *h-edge*. H is *h-uniform* if $|e| = h$ for each edge $e \in \mathcal{E}$, i.e., H contains only h-edges. A hypergraph, no edge of which is a subset of another is called *Sperner.*

If $\lambda \in \mathcal{N}$, a λ-*coloring* of H is such a function $f: V(H) \to \{1, 2, \ldots, \lambda\}$ that for each edge e of H there exist x, y in e for which $f(x) \neq f(y)$. The number of λ-colorings of H is given by a polynomial $f(H, \lambda)$ of degree $|V(H)|$ in λ, called the *chromatic polynomial* of H.

A class of hypergraphs is *chromatically characterized* by their chromatic polynomials if for each hypergraph H from this class we have $f(H,\lambda) = f(H',\lambda)$ if and only if H' belongs to this class.

Two hypergraphs H_1 and H_2 are said to be *chromatically equivalent* or *χ-equivalent* if $f(H_1,\lambda) = f(H_2,\lambda)$. A hypergraph H is said to be *chromatically unique* or *χ-unique* if $f(H,\lambda) = f(H',\lambda)$ implies that H' is isomorphic to H. These notions were first introduced and studied only for graphs by Chao and Whitehead [2]. Afterwards many scientists, among them Dohmen, Jones and Tomescu, started to study the chromaticity of hypergraphs. Till now only few chromatically unique hypergraphs are known (see [7]). In order to present our results dealing with this problem we have to recall some theorems giving the methods of calculating the chromatic polynomial of any hypergraph.

Theorem 1 [7]. *Let H be a hypergraph with n vertices. Then $f(H,\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda$, and*

$$a_i = \sum_{j\geq 0}(-1)^j N(i,j) \qquad (1 \leq i \leq n-1),$$

where $N(i,j)$ denote the number of subhypergraphs of H with n vertices, i components and j edges.

Theorem 2 [5]. *Let e and g be two edges of H with $e \subseteq g$. There is an one-to-one correspondence between the λ-colorings of H and those of $H-g$.*

Let $H = (V,\mathcal{E})$ be a Sperner hypergraph, $e \in \mathcal{E}$ and $u,v \in e$. Let H' denote the hypergraph obtained by replacing e by the new 2-edge $\{u,v\}$, and H'' be the hypergraph obtained by identyfying u and v as one vertex and removing all multiple edges and loops if they arise.

Theorem 3 [5]. $f(H,\lambda) = f(H',\lambda) + f(H'',\lambda)$.

Let $H = (V,\mathcal{E})$ be a hypergraph and $A \subseteq \mathcal{E}$. A *rank function* r on $\mathcal{E}$ is defined in the following way:

$$r(A) = |V(H[A])| - c(H[A]),$$

where $H[A]$ is the hypergraph induced by the edges of A and $c(H[A])$ is the number of its connected components.

Theorem 4 [4].

$$f(H,\lambda) = \sum_{A \subseteq \mathcal{E}} (-1)^{|A|} \lambda^{r(\mathcal{E})-r(A)+1}.$$

2. Chromatically Unique Hypergraphs

Let for $q \geq 1$ the symbol $H^n_{q,q+1}$ denotes a $(q+1)$-uniform hypergraph with $n \geq q+1$ vertices in which each two of its edges intersect in exactly q vertices and all of its edges intersect in exactly q vertices. If n is known then we write $H_{q,q+1}$ to denote such a hypergraph. By the definition we have $\left|\mathcal{E}\left(H^n_{q,q+1}\right)\right| = n-q$. We prove that for $q \geq 2$ a hypergraph $H_{q,q+1}$ is chromatically unique. The condition $q \geq 2$ is very important because for $q = 1$ a hypergraph $H^n_{1,2}$ is a graph $K_{1,n-1}$, so it is a tree. However trees are chromatically equivalent but not chromatically unique.

Theorem 5. *A hypergraph H is a hypergraph $H_{q,q+1}$ with $n \geq q+1$ vertices, where $q \geq 2$, if and only if*

$$(1) \qquad f(H,\lambda) = \lambda(\lambda-1)\left[\lambda^{n-2} + \lambda^{n-3} + \ldots + \lambda^{n-q} + (\lambda-1)^{n-q-1}\right].$$

Proof. *Necessity.* The proof uses the induction on q.

According to Theorems 3 and 2 for $q = 2$ the polynomial $f(H_{2,3},\lambda)$ is the sum of the chromatic polynomials of hypergraphs $H'_{2,3}$ and $H''_{2,3}$. If we take two common vertices of all edges of $H_{2,3}$ as u and v then $H'_{2,3} = K_2 \cup (n-2)K_1$ and $H''_{2,3} = K_{1,n-2}$. Therefore

$$\begin{aligned} f(H_{2,3},\lambda) &= \lambda^{n-1}(\lambda-1) + \lambda(\lambda-1)^{n-2} \\ &= \lambda(\lambda-1)\left[\lambda^{n-2} + (\lambda-1)^{n-3}\right]. \end{aligned}$$

Let us now assume that for $2 \leq j \leq q-1$ the chromatic polynomial of the hypergraph $H_{j,j+1}$ with $n \geq j+1$ vertices is given by (1). In order to calculate $f(H_{q,q+1},\lambda)$ we use Theorems 3 and 2 once again taking any two of q common vertices of all edges of $H_{q,q+1}$ as u and v. Then $H'_{q,q+1} = K_2 \cup (n-2)K_1$ and $H''_{q,q+1} = H^{n-1}_{q-1,q}$, and by the induction hypothesis we have

$$
\begin{aligned}
f\left(H_{q,q+1},\lambda\right) &= f\left(K_2\cup(n-2)K_1,\lambda\right)+f\left(H^{n-1}_{q-1,q},\lambda\right)\\
&= \lambda^{n-1}(\lambda-1)+\lambda(\lambda-1)\left[\lambda^{n-3}+\lambda^{n-4}\right.\\
&\quad \left.+\ldots+\lambda^{n-1-(q-1)}+(\lambda-1)^{n-1-(q-1)-1}\right]\\
&= \lambda(\lambda-1)\left[\lambda^{n-2}+\lambda^{n-3}+\ldots+\lambda^{n-q}+(\lambda-1)^{n-q-1}\right].
\end{aligned}
$$

Sufficiency. Let H be a hypergraph with the chromatic polynomial given by (1). Then

$$
\begin{aligned}
f(H,\lambda) &= \lambda^n+\tbinom{n-q}{1}\lambda^{n-q}(-1)^1+\tbinom{n-q}{2}\lambda^{n-q-1}(-1)^2\\
&\quad +\ldots+\tbinom{n-q}{n-q-1}\lambda^2(-1)^{n-q-1}+\tbinom{n-q}{n-q}\lambda^1(-1)^{n-q}\\
&= \lambda^n+\textstyle\sum_{j=1}^{n-q}\tbinom{n-q}{j}\lambda^{n-q+1-j}(-1)^j.
\end{aligned}
$$

We assume that $f(H,\lambda)=a_n\lambda^n+a_{n-1}\lambda^{n-1}+\ldots+a_1\lambda+a_0$, where $a_n=1$, and analyse all its coefficients.

1. Since $a_n\neq 0$ then $f(H,\lambda)$ is of degree n, and so $|V(H)|=n$.
2. We have $a_1\neq 0$ and according to Theorem 1, $a_1=\sum_{j\geq 0}N(1,j)$, so there exists at least one connected spanning subhypergraph of H. Therefore H has to be connected.
3. Using the induction we shall show that H has no j-edges for $2\leq j\leq q$.
   (a) By Theorem 4, $a_{n-1}=\sum_{A\subseteq\mathcal{E}}(-1)^{|A|}$, where each A is a subset of $\mathcal{E}$ such that $r(\mathcal{E})-r(A)+1=n-1$. Then $r(A)=|V(H[A])|-c(H[A])=1$ because $r(\mathcal{E})=n-1$. H is connected so A could only be a set consisting of exactly one 2-edge. However, by (1) we have $a_{n-1}=0$, so H has no 2-edges.
   (b) Since $a_{n-2}=a_{n-3}=\ldots=a_{n-q+1}=0$ then let us assume that H has no j-edges for any $1\leq j\leq t$, where $t<q$. According to Theorem 4 we have $a_{n-t}=\sum_{A\subseteq\mathcal{E}}(-1)^{|A|}$ with $r(A)=t$. By the induction hypothesis the condition $r(A)=t$ holds if and only if all sets A consist of exactly one $(t+1)$-edge. If any A contained an edge with the number of vertices greater than $t+1$ then it would be $r(A)>t$. Since $a_{n-t}=0$ then H has no $(t+1)$-edges.

By the induction we conclude that H has no j-edges for $2 \le j \le q$.

4. According to (1), $a_{n-q} = -(n-q)$, but by Theorem 4 we have $a_{n-q} = \sum_{A \subseteq \mathcal{E}} (-1)^{|A|}$ with $r(A) = q$. H is connected and has no j-edges for all $2 \le j \le q$, so each set A has to consist of exactly one $(q+1)$-edge. Therefore the number of $(q+1)$-edges in H is equal to $n-q$.
5. By Theorem 4 we have $a_{n-q-1} = \sum_{A \subseteq \mathcal{E}} (-1)^{|A|}$ with $r(A) = q+1$. According to the previous conclusions each set A could be of the following form:
   - A consists of i, where $2 \le i \le q+2$, $(q+1)$-edges and $|V(H[A])| = q+2$; a set of this type having i edges will be denoted by X_i;
   - A consists of exactly one $(q+2)$-edge; a set of this type will be denoted by Y.

Then

$$\begin{aligned} a_{n-q-1} &= \textstyle\sum_{X_2 \subseteq \mathcal{E}} (-1)^2 + \sum_{X_3 \subseteq \mathcal{E}} (-1)^3 \\ &\quad + \ldots + \textstyle\sum_{X_{q+2} \subseteq \mathcal{E}} (-1)^{q+2} + \sum_{Y \subseteq \mathcal{E}} (-1)^1 \\ &= x_2 - x_3 + \ldots + (-1)^{q+2} x_{q+2} - y, \end{aligned}$$

where the numbers $x_i = \sum_{X_i \subseteq \mathcal{E}} 1$, for $2 \le i \le q+2$, and $y = \sum_{Y \subseteq \mathcal{E}} 1$ are, of course, non-negative. It is easy to see that $x_i \ge \binom{i+1}{i} x_{i+1}$ for every $i \in \{2, 3, \ldots, q+1\}$ and $x_2 \le \binom{n-q}{2}$ because the number of $(q+1)$-edges in H equals $n-q$. According to (1) we have $a_{n-q-1} = \binom{n-q}{2}$. We prove that $x_2 = \binom{n-q}{2}$ and $x_3 = x_4 = \ldots = x_{q+2} = y = 0$. To this end let us consider two cases.

(a) If q is even then

$$\begin{aligned} &\tbinom{n-q}{2} = a_{n-q-1} \\ &= x_2 - x_3 + x_4 - x_5 + x_6 - \ldots - x_{q+1} + x_{q+2} - y \\ &\le \tbinom{n-q}{2} - \tbinom{4}{3} x_4 + x_4 - \tbinom{6}{5} x_6 + x_6 - \ldots - \tbinom{q+2}{q+1} x_{q+2} + x_{q+2} - y \\ &= \tbinom{n-q}{2} - [3x_4 + 5x_6 + \ldots + (q+1) x_{q+2} + y]. \end{aligned}$$

Since the numbers x_i and y are non-negative we obtain $x_4 = x_6 = \ldots = x_{q+2} = y = 0$. Thus

$$\begin{aligned}\binom{n-q}{2} &= x_2 - (x_3 + x_5 + \ldots + x_{q+1})\\ &\le \binom{n-q}{2} - (x_3 + x_5 + \ldots + x_{q+1}),\end{aligned}$$

so $x_3 = x_5 = \ldots = x_{q+1} = 0$ and $x_2 = \binom{n-q}{2}$.

(b) If q is odd then

$$\begin{aligned}&\binom{n-q}{2} = a_{n-q-1}\\ &= x_2 - x_3 + x_4 - \ldots - x_q + x_{q+1} - x_{q+2} - y\\ &\le \binom{n-q}{2} - \binom{4}{3}x_4 + x_4 - \ldots - \binom{q+1}{q}x_{q+1} + x_{q+1} - x_{q+2} - y\\ &= \binom{n-q}{2} - (3x_4 + 5x_6 + \ldots + qx_{q+1} + x_{q+2} + y)\,.\end{aligned}$$

Similarly as it was above we obtain $x_4 = x_6 = \ldots = x_{q+1} = 0$ and $x_{q+2} = y = 0$. Thus

$$\begin{aligned}\binom{n-q}{2} &= x_2 - (x_3 + x_5 + \ldots + x_q)\\ &\le \binom{n-q}{2} - (x_3 + x_5 + \ldots + x_q),\end{aligned}$$

so $x_3 = x_5 = \ldots = x_q = 0$ and $x_2 = \binom{n-q}{2}$.

In both cases we obtain $\alpha_2 = \binom{n-q}{2}$ and $x_3 = \cdots = x_{q+2} = y = 0$. It implies that H has no $(q+2)$-edges and any number of its $(q+1)$-edges intersect in exactly q vertices.

6. We shall prove by the induction that there are no $(q+j)$-edges in H for $2 \le j \le n-q$.
   (a) The case $j = 2$ was considered above.
   (b) Let us suppose that for a certain $t < n-q$ H has no $(q+j)$-edges for dla $2 \le j \le t$. According to (1) we have $a_{n-q-t} = \binom{n-q}{t+1}(-1)^{t+1}$, while by Theorem 4, $a_{n-q-t} = \sum_{A\subseteq\mathcal{E}}(-1)^{|A|}$ with $r(A) = q+t$. Since any number of $(q+1)$-edges of H intersect in exactly q vertices and by the induction hypothesis we obtain

$$a_{n-q-t} = \sum_{X\subseteq\mathcal{E}}(-1)^{t+1} + \sum_{Y\subseteq\mathcal{E}}(-1)^1,$$

where X consists of $t+1$ $(q+1)$-edges, $|V(H[X])| = q+t+1$ and $c(X) = 1$, while Y consists of exactly one $(q+t+1)$-edge. Since $\sum_{X\subseteq\mathcal{E}}(-1)^{t+1} = \binom{n-q}{t+1}(-1)^{t+1}$, so $\sum_{Y\subseteq\mathcal{E}} 1 = 0$. It means that H has no $(q+t+1)$-edges.

By the induction we conclude that there are no $(q+j)$-edges in H for $2 \le j \le n-q$.

The formula (1) implies the following properties of H:

- H is connected,
- $|V(H)| = n$,
- H is $(q+1)$-uniform,
- each two of its edges intersect in q vertices,
- all of its edges intersect in q vertices.

It means that H is a hypergraph $H^n_{q,q+1}$. ■

Corollary 1. *A hypergraph $H_{q,q+1}$ with $n \ge q+1$ vertices, where $q \ge 2$, is chromatically unique.*

3. Some Generalizations of Known Theorems

The next theorem is a generalization for hypergraphs the corresponding result of Read [6] for graphs.

Theorem 6. *If H is a hypergraph such that $H = H_1 \cup \ldots \cup H_k$, where*

$$H_i \cap H_j = K_p \quad \textit{for} \quad i \ne j \quad \textit{and} \quad \bigcap_{i=1}^{k} H_i = K_p,$$

where K_p is a complete graph with p vertices, then

$$f(H,\lambda) = \left[f(K_p,\lambda)\right]^{1-k} f(H_1,\lambda)\, f(H_2,\lambda)\cdots f(H_k,\lambda)\,.$$

Proof. The number of ways to color a common graph K_p with λ colors is equal to $f(K_p,\lambda)$. If we fix the colors of p vertices of K_p then there exist $f(H_i,\lambda)/f(K_p,\lambda)$ ways of coloring the remaining vertices of each hypergraph H_i. Therefore

$$f(H,\lambda) = f(K_p,\lambda)\,\frac{f(H_1,\lambda)}{f(K_p,\lambda)}\frac{f(H_2,\lambda)}{f(K_p,\lambda)}\cdots\frac{f(H_k,\lambda)}{f(K_p,\lambda)},$$

which completes the proof. ■

Another generalization deals with the chromatic polynomials of hypertrees. A hypergraph is *linear* if any two of its edges do not intersect in more than one vertex. A *hypertree* is a hypergraph which is linear, connected and contains no cycle. (We use the term a *cycle* in a hypergraph in the meaning of Berge [1].) Dohmen gave the explicit formula of the chromatic polynomial of h-uniform hypertree.

Theorem 7 [3]. *If T_m^h is an h-uniform hypertree with m edges, where $h \geq 2$ and $m \geq 0$, then*

$$f\left(T_m^h, \lambda\right) = \lambda\left(\lambda^{h-1} - 1\right)^m.$$

This theorem needs the uniformity of a hypertree. We proved the theorem related to a hypertree with arbitrary edges.

Theorem 8. *If $T_{n_1,n_2,\ldots,n_k}^{i_1,i_2,\ldots,i_k}$ is a hypertree with n_s i_s-edges for $1 \leq s \leq k$, where $i_s \geq 2$ for each s and $n_1 + n_2 + \ldots + n_k \geq 0$, then*

$$f\left(T_{n_1,n_2,\ldots,n_k}^{i_1,i_2,\ldots,i_k}, \lambda\right) = \lambda\left(\lambda^{i_1-1} - 1\right)^{n_1}\left(\lambda^{i_2-1} - 1\right)^{n_2} \cdot \ldots \cdot \left(\lambda^{i_k-1} - 1\right)^{n_k}.$$

Proof. The proof uses the induction on the number of edges of a hypertree. The first step is a consequence of Theorem 7 for $m = 0$. Let us now assume that for a certain number of edges not less than 1 a hypertree $T_{n_1,n_2,\ldots,n_k}^{i_1,i_2,\ldots,i_k}$ has the chromatic polynomial given by Theorem 8. Now we calculate the chromatic polynomial of a hypertree which comes into being by adding one edge to $T_{n_1,n_2,\ldots,n_k}^{i_1,i_2,\ldots,i_k}$ in such a way that this edge and $T_{n_1,n_2,\ldots,n_k}^{i_1,i_2,\ldots,i_k}$ have exactly one vertex in common. Without loss of generality we can assume that we add one i_1-edge. According to Theorem 6 for $p = 1$ we have

$$\begin{aligned}
&f\left(T_{n_1+1,n_2,\ldots,n_k}^{i_1,i_2,\ldots,i_k}, \lambda\right) \\
&= \frac{\lambda\left(\lambda^{i_1-1}-1\right)^{n_1}\left(\lambda^{i_2-1}-1\right)^{n_2}\cdots\left(\lambda^{i_k-1}-1\right)^{n_k}\lambda\left(\lambda^{i_1-1}-1\right)}{\lambda} \\
&= \lambda\left(\lambda^{i_1-1} - 1\right)^{n_1+1}\left(\lambda^{i_2-1} - 1\right)^{n_2}\cdots\left(\lambda^{i_k-1} - 1\right)^{n_k},
\end{aligned}$$

which completes the proof. ∎

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