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THE DECOMPOSABILITY OF ADDITIVE HEREDITARY PROPERTIES OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. If $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are properties of graphs, then a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-decomposition of a graph G is a partition $E_1, \ldots, E_n$ of $E(G)$ such that $G[E_i]$, the subgraph of G induced by E_i, is in $\mathcal{P}_i$, for $i = 1, \ldots, n$. We define $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ as the property $\{G \in \mathcal{I} : G$ has a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-decomposition$\}$. A property $\mathcal{P}$ is said to be decomposable if there exist non-trivial hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$. We study the decomposability of the well-known properties of graphs $\mathcal{I}_k$, $\mathcal{O}_k$, $\mathcal{W}_k$, $\mathcal{T}_k$, $\mathcal{S}_k$, $\mathcal{D}_k$ and $\mathcal{O}^p$.

Keywords: property of graphs, additive, hereditary, decomposable property of graphs.

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1 Introduction

Following [2] we denote the class of all finite simple graphs by $\mathcal{I}$. A *property* of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. The fact that H is a subgraph of G is denoted by $H \subseteq G$ and the disjoint union of two graphs G and H is denoted by $G \cup H$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. Throughout this paper the term property (of graphs) is used to refer to an additive hereditary property (of graphs).

Example. Some well-known properties are given in the list below.

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,

$\mathcal{I}_k = \{G \in \mathcal{I} : G$ does not contain $K_{k+2}\}$,

$\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k+1$ vertices$\}$,

$\mathcal{W}_k = \{G \in \mathcal{I} :$ the length of any path in G is at most $k\}$,

$\mathcal{T}_k = \{G \in \mathcal{I} : G$ contains no subgraph homeomorphic to K_{k+2} or $K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}$,

$\mathcal{S}_k = \{G \in \mathcal{I} :$ the maximum degree of G is at most $k\}$,

$\mathcal{D}_k = \{G \in \mathcal{I} : G$ is k-degenerate, i.e. every subgraph of G has a vertex of degree at most $k\}$.

We remark that $\mathcal{D}_1 = \mathcal{T}_1$ is the class of all forests so that

$\mathcal{SF} = \mathcal{D}_1 \cap \mathcal{W}_2 = \{G \in \mathcal{I} :$ every component of G is a star $\}$.

The properties $\mathcal{I}$ and $\mathcal{O}$ are defined to be the *trivial properties* and an edgeless graph is called a *trivial graph.* We use the phrase *G has property $\mathcal{P}$* to denote the fact that $G \in \mathcal{P}$.

2 Decomposability

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*decomposition* of a graph G is a partition $E_1, \ldots, E_n$ of $E(G)$ such that $G[E_i]$, the subgraph of G induced by E_i, has property $\mathcal{P}_i$, for $i = 1, \ldots, n$. We denote by $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ the property $\{G \in \mathcal{I} : G$ has a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-decomposition$\}$. If $G \in \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ we also write $G = G_1 \oplus \cdots \oplus G_n$ where $G_i = G[E_i]$ for $i = 1, \ldots, n$. If $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P}$, then the property $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ is also denoted by $n\mathcal{P}$. Note that $\mathcal{O}$ is the identity element for $\oplus$ in the sense that $\mathcal{P} \oplus \mathcal{O} = \mathcal{P}$ for every property $\mathcal{P}$. It is easy to see that if $\mathcal{P}_i$ is additive and hereditary for every i then $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ is also additive and hereditary.

If $\mathbb{L}$ is a set of hereditary properties and $\mathcal{P} \in \mathbb{L}$ then $\mathcal{P}$ is said to be *decomposable in* $\mathbb{L}$ if there exist non-trivial hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{L}$ such that $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$; otherwise $\mathcal{P}$ is said to be *indecomposable in* $\mathbb{L}$. Throughout this paper we use for $\mathbb{L}$ the lattice $\mathbb{L}^a_{\subseteq}$ of all additive hereditary properties of graphs — see [2] for more details on this lattice.

The property $\mathcal{P} \circ \mathcal{Q}$ is the vertex-analoque of $\mathcal{P} \oplus \mathcal{Q}$. For the sake of completeness we give the necessary definitions: For given properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$, a *vertex* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*partition* of a graph G is a partition $V_1, \ldots, V_n$ of $V(G)$ such that for each $i = 1, \ldots, n$ the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$.

(In this context it is convenient to regard the empty set $\emptyset$ as a set inducing a subgraph with every property $\mathcal{P}$.) The *product* $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ is now defined as the set of all graphs having a vertex $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition; each $\mathcal{P}_i$ is called a *factor* of this product. If $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $\mathcal{P}^n = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. As an example we note that $\mathcal{O}^k$ denotes the class of all k-colourable graphs.

These operations come together in the following distributive law which is proved in [3].

Lemma 21. *Let $\mathcal{P}$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be additive and hereditary properties of graphs. Then $\mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2) = (\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2)$.* ■

A property $\mathcal{R}$ is *reducible* if there are properties $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$; otherwise it is *irreducible*. Reducible properties play an important role in the lattice $\mathbb{L}^a_{\subseteq}$ because of following Unique Factorization Theorem of Mihók, Semanišin and Vasky which appears in [5].

Theorem 22. *Every reducible property $\mathcal{P} \neq \mathcal{I}$ is uniquely factorizable into irreducible factors (up to the order of the factors).* ■

In the light of this result it seems imperative to study the decomposability of properties of graphs. It is our aim to study the decomposability of the well-known properties of graphs $\mathcal{I}_k$, $\mathcal{O}_k$, $\mathcal{W}_k$, $\mathcal{T}_k$, $\mathcal{S}_k$, $\mathcal{D}_k$ and $\mathcal{O}^p$.

3 The Indecomposability of $\mathcal{I}_k$, $\mathcal{O}_k$, $\mathcal{W}_k$, and $\mathcal{T}_k$

The indecomposability of $\mathcal{I}_k$ is easy to show using the following well-known result of Nešetřil and Rödl — see [6]. In it we use the notation $\omega(G)$ to denote the clique number of a graph G.

Theorem 31. *For every graph G there is a graph H with $\omega(G) = \omega(H)$ such that for every partition E_1, E_2 of $E(H)$ there is an induced subgraph G' of H with G' isomorphic to G and $E(G') \subseteq E_1$ or $E(G') \subseteq E_2$.* ■

Theorem 32. *For every positive integer k the property $\mathcal{I}_k$ is indecomposable in $\mathbb{L}^a_{\subseteq}$.*

Proof. Suppose to the contrary that $\mathcal{I}_k = \mathcal{P} \oplus \mathcal{Q}$ with both $\mathcal{P}$ and $\mathcal{Q}$ nontrivial. Applying Theorem 31 to $G = K_{k+2} - e \in \mathcal{I}_k$ it follows that there is a graph $H \in \mathcal{I}_k$ so that H has a $(\mathcal{P}, \mathcal{Q})$-decomposition and by considering any $(\mathcal{P},\mathcal{Q})$-decomposition of H we see that $K_{k+2}-e \in \mathcal{P}$ or $K_{k+2}-e \in \mathcal{Q}$. But then $K_{k+2} \in \mathcal{P} \oplus \mathcal{Q} = \mathcal{I}_k$, a contradiction. ■

The following lemma will prove to be useful to show the indecomposability of some properties.

Lemma 33. *If $\mathcal{P}$ is decomposable in $\mathbb{L}^a_{\subseteq}$ then $2\mathcal{S}_1 \subseteq \mathcal{P}$.*

Proof. If $\mathcal{P} = \mathcal{Q} \oplus \mathcal{R}$ with $\mathcal{Q}$ and $\mathcal{R}$ non-trivial then $\mathcal{S}_1 \subseteq \mathcal{Q}$ and $\mathcal{S}_1 \subseteq \mathcal{R}$ since $\mathcal{S}_1$ is the smallest non-trivial property in $\mathbb{L}^a_{\subseteq}$. It follows that $2\mathcal{S}_1 \subseteq \mathcal{Q} \oplus \mathcal{R}$. ■

Corollary 34. *For every positive integer k the properties $\mathcal{O}_k$ and $\mathcal{W}_k$ are indecomposable in $\mathbb{L}^a_{\subseteq}$.*

Proof. The path P_{k+2} of order $k+2$ is in $2\mathcal{S}_1$ but not in $\mathcal{O}_k$, hence $\mathcal{O}_k$ is indecomposable by Lemma 33. Since P_{k+2} is not in $\mathcal{W}_k$ either, $\mathcal{W}_k$ is also indecomposable. ■

Theorem 35. *If $\mathcal{SF} \subseteq \mathcal{P}$ and $\overline{\mathcal{P}} \neq \emptyset$ is closed under taking subdivisions of graphs, then $\mathcal{P}$ is indecomposable in $\mathbb{L}^a_{\subseteq}$.*

Proof. Suppose that $\mathcal{P} = \mathcal{Q} \oplus \mathcal{R}$ with $\mathcal{Q}$ and $\mathcal{R}$ non-trivial. Let G be any graph in $\overline{\mathcal{P}}$ and let G' be the graph obtained from G by subdividing every edge of G twice. Clearly, $G' \in \mathcal{SF} \oplus \mathcal{S}_1$. Since $G' \notin \mathcal{P}$ it follows that $\mathcal{SF} \not\subseteq \mathcal{Q}$ and $\mathcal{SF} \not\subseteq \mathcal{R}$. Therefore there exist positive integers a and b such that $K_{1,a} \notin \mathcal{Q}$ and $K_{1,b} \notin \mathcal{R}$. But then $K_{1,a+b-1} \notin \mathcal{Q} \oplus \mathcal{R}$, contradicting the assumption that $\mathcal{SF} \subseteq \mathcal{P}$. ■

Corollary 36. *For every positive integer k the property $\mathcal{T}_k$ is indecomposable in $\mathbb{L}^a_{\subseteq}$.* ■

4 The Decomposability of $\mathcal{S}_k$

Theorem 41. *Let p and q be any positive integers. Then*
(1) $\mathcal{S}_p \oplus \mathcal{S}_q \subseteq \mathcal{S}_{p+q}$.
(2) $\mathcal{S}_p \subseteq (p+1)\mathcal{S}_1$.
(3) $\mathcal{S}_{p+q} \subseteq \mathcal{S}_{p+1} \oplus \mathcal{S}_q$.
(4) $\mathcal{S}_{2p} = p\mathcal{S}_2$.

Proof. (1) If $G \notin \mathcal{S}_{p+q}$ then G has a vertex of degree at least $p+q+1$, hence $K_{1,p+q+1} \subseteq G$. But since $K_{1,p+q+1} \notin \mathcal{S}_p \oplus \mathcal{S}_q$, it then follows that $G \notin \mathcal{S}_p \oplus \mathcal{S}_q$.

(2) This is Vizing's fundamental result on edge colourings of graphs.

(3) From (2) and (1) it follows that $\mathcal{S}_{p+q} \subseteq (p+q+1)\mathcal{S}_1 \subseteq \mathcal{S}_{p+1} \oplus \mathcal{S}_q$.

(4) The inclusion $p\mathcal{S}_2 \subseteq \mathcal{S}_{2p}$ follows from (1). For the other inclusion, suppose that $G \in \mathcal{S}_{2p}$. Then G is a subgraph of a $2p$-regular graph H. Since H is regular of even degree, H is 2-factorable by Petersen's Theorem. This 2-factorization of H clearly is a $(p\mathcal{S}_2)$-decomposition of H and therefore induces a $(p\mathcal{S}_2)$-decomposition of G. Therefore $G \in p\mathcal{S}_2$. ■

We are now ready to determine exactly when equality holds in part (1) of Theorem 41.

Corollary 42. *Let p and q be any even positive integers. Then $\mathcal{S}_p \oplus \mathcal{S}_q = \mathcal{S}_{p+q}$.*

Proof. An easy calculation using part (4) of Theorem 41. ■

Theorem 43. *Let p and q be positive integers with q odd. Then $\mathcal{S}_{p+q} \nsubseteq \mathcal{S}_p \oplus \mathcal{S}_q$.*

Proof. Suppose first that p is odd too. Since $K_{p+q+1} \in \mathcal{S}_{p+q}$ it is sufficient to show that $K_{p+q+1} \notin \mathcal{S}_p \oplus \mathcal{S}_q$: Suppose, to the contrary, that E_1, E_2 is an $(\mathcal{S}_p, \mathcal{S}_q)$-decomposition of $E(K_{p+q+1})$. Then E_1 has at most $\frac{1}{2}[p(p+q+1)-1]$ edges, since any graph of maximum degree p on n vertices has at most $\frac{1}{2}pn$ edges and if n is odd this inequality is strict. Similarly, E_2 has at most $\frac{1}{2}[q(p+q+1)-1]$ edges. But K_{p+q+1} has $\frac{1}{2}(p+q)(p+q+1) > \frac{1}{2}[p(p+q+1)-1] + \frac{1}{2}[q(p+q+1)-1]$ edges, a contradiction.

Next let $p = 2$. Let $F = K_{q+1,q+1}$ with (vertex) partite sets $V = \{v_1, \ldots, v_{q+1}\}$ and $W = \{w_1, \ldots, w_{q+1}\}$. Let H be the graph obtained from F by adding vertices x and y and edges $\{xv_i : i = 1, \ldots, q+1\}$, $\{w_i w_{i+1} : i = 1, 3, \ldots, q\}$ and xy. Suppose now that E_1, E_2 is any $(\mathcal{S}_2, \mathcal{S}_q)$-decomposition of $E(H)$. Since every vertex other than y has degree $q+2$ in H, every such vertex must have degree 2 in $H[E_1]$. Therefore $xy \notin E_1$, otherwise $H[E_1]$ would have an odd number of odd vertices. Now let G be the graph obtained from $q+1$ copies of H by identifying the vertices of degree one. Then $G \in \mathcal{S}_{2+q}$ but, by the above argument, $G \notin \mathcal{S}_2 \oplus \mathcal{S}_q$. Hence $\mathcal{S}_{2+q} \nsubseteq \mathcal{S}_2 \oplus \mathcal{S}_q$ for every odd positive integer q.

Finally, let $p = 2k$ with $k \geq 2$ and suppose that $\mathcal{S}_{2k+q} \subseteq \mathcal{S}_{2k} \oplus \mathcal{S}_q$. Then $\mathcal{S}_{2+(2(k-1)+q)} = \mathcal{S}_{2k+q} \subseteq \mathcal{S}_{2k} \oplus \mathcal{S}_q = \mathcal{S}_2 \oplus (\mathcal{S}_{2(k-1)} \oplus \mathcal{S}_q) \subseteq \mathcal{S}_2 \oplus \mathcal{S}_{2(k-1)+q}$, contradicting the statement proven in the previous paragraph. ■

Lemma 44. *If $\mathcal{S}_k = \mathcal{P} \oplus \mathcal{Q}$ with $\mathcal{P} \in \mathbb{L}^a_{\subseteq}$ and $\mathcal{Q} \in \mathbb{L}^a_{\subseteq}$ both non-trivial, then $\mathcal{S}_k = \mathcal{S}_p \oplus \mathcal{S}_q$ for some positive integers p and q with $p + q = k$.*

Proof. If $G \in \mathcal{P}$ has a vertex v of degree a and $H \in \mathcal{Q}$ has a vertex w of degree b then there is a graph in $\mathcal{P} \oplus \mathcal{Q}$ with a vertex of degree $a + b$, for example the graph obtained from $G \cup H$ by identifying v and w. Therefore there exist positive integers p and q such that $\mathcal{P} \subseteq \mathcal{S}_p$, $\mathcal{Q} \subseteq \mathcal{S}_q$ and $p+q \leq k$. If $p + q < k$ then $K_{1,k} \in \mathcal{S}_k$ but $K_{1,k} \notin \mathcal{P} \oplus \mathcal{Q}$; therefore $p + q = k$. Now $\mathcal{S}_k = \mathcal{P} \oplus \mathcal{Q} \subseteq \mathcal{S}_p \oplus \mathcal{S}_q \subseteq \mathcal{S}_k$, hence $\mathcal{S}_k = \mathcal{S}_p \oplus \mathcal{S}_q$. ■

We are now ready to prove the main result of this section.

Theorem 45. *$\mathcal{S}_k$ is decomposable in $\mathbb{L}^a_{\subseteq}$ if and only if $k = 2n$ for some $n \geq 2$.*

Proof. Suppose $\mathcal{S}_k$ is decomposable and k is odd or $k = 2$. Then, by Lemma 44, $\mathcal{S}_k = \mathcal{S}_p \oplus \mathcal{S}_q$, where $p + q = k$. In either case p or q is odd and Theorem 43 is contradicted.

The other direction follows from part (4) of Theorem 41. ■

5 The Decomposability of $\mathcal{D}_k$

Theorem 51. *$\mathcal{D}_1$ is indecomposable in $\mathbb{L}^a_{\subseteq}$.*

Proof. The cycle C_4 is in $2\mathcal{S}_1$ but not in $\mathcal{D}_1$, hence $\mathcal{D}_1$ is indecomposable in $\mathbb{L}^a_{\subseteq}$ by Lemma 33. ■

The indecomposability of $\mathcal{D}_2$ is also known — see [4]. We state their result as

Theorem 52. *$\mathcal{D}_2$ is indecomposable.* ■

In order to prove that $\mathcal{D}_k$ is indecomposable for each $k \geq 3$, we first discuss some decomposable bounds for $\mathcal{D}_k$ which are useful. The first bound is due to Borowiecki and Hałuszczak — see [1].

Theorem 53. *For all positive integers a and b the inclusion $\mathcal{D}_{a+b} \subseteq \mathcal{D}_a \oplus \mathcal{D}_b$ holds.* ■

In order to show that $\mathcal{D}_a \oplus \mathcal{D}_b$ is not an upper bound for $\mathcal{D}_{a+b+1}$, we need the following lemma.

Lemma 54. *For all positive integers a, b and n there is a positive integer t such that for every $(\mathcal{D}_a, \mathcal{D}_b)$-decomposition E_1, E_2 of $K_{a+b,t}$, the inclusions $K_{a,n} \subseteq K_{a+b,t}[E_1]$ and $K_{b,n} \subseteq K_{a+b,t}[E_2]$ hold.*

Proof. In order to simplify the arguments we assume throughout that $n > a+b$. For $a = b = 1$ we can take $t = 2n+2$. Assume now inductively that the statement is true for a, b, n and t and consider any $(\mathcal{D}_a, \mathcal{D}_{b+1})$-partition E_1, E_2 of $E(G)$ where $G = K_{a+b+1,2t}$. Let V be the (vertex) partite set of order $a+b+1$ of G, $U = V(G) - V$ and let v be any vertex of V. We consider two cases:

(1) At least t edges incident with v are in E_2: In this case we apply the induction hypothesis to the subgraph of G induced by $(V - v) \cup \{u \in U : vu \in E_2\}$ to obtain $K_{a,n} \subseteq G[E_1]$ and $K_{b,n} \subseteq G[E_2]$. Together with the t edges in E_2 incident with v we then also have $K_{b+1,n} \subseteq G[E_2]$. (Note that the b independent vertices of the $K_{b,n}$ in $G[E_2]$ is necessarily a subset of V since $n > b$.)

(2) At least t edges incident with v are in E_1: By the same arguments as in case (1) we now obtain that $K_{a+1,n} \subseteq G[E_1]$. Since $n > a$ we have a contradiction, hence this case is impossible. ∎

Theorem 55. *Let a and b be positive integers. Then $\mathcal{D}_{a+b+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{D}_b$.*

Proof. Clearly $K_{a+b+1,t} \in \mathcal{D}_{a+b+1}$ for every positive integer t. Further, by Lemma 54, every $(\mathcal{D}_a, \mathcal{D}_{b+1})$-decomposition E_1, E_2 of $K_{a+b+1,t}$ has $K_{b+1,b+1} \subseteq K_{a+b+1,t}[E_2]$ for t large enough. Since a $(\mathcal{D}_a, \mathcal{D}_b)$-decomposition is also a $(\mathcal{D}_a, \mathcal{D}_{b+1})$-decomposition, $K_{a+b+1,t} \notin \mathcal{D}_a \oplus \mathcal{D}_b$. ∎

If $b = 1$ we can demonstrate the sharpness of the inclusion in Theorem 53 in a stronger sense.

Theorem 56. *Let a be a positive integer. If $\mathcal{D}_1 \not\subseteq \mathcal{P}$ then $\mathcal{D}_{a+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{P}$.*

Proof. If $\mathcal{D}_1 \not\subseteq \mathcal{P}$, then there is a tree T with $T \in \mathcal{D}_1$ and $T \notin \mathcal{P}$; suppose T is of size m. We construct a sequence $G_0, G_1, \ldots, G_m$ of graphs in $\mathcal{D}_{a+1}$ as follows: $G_0 = K_1$ and G_{i+1} is obtained from a copies of G_i by adding, for every set V consisting of one vertex from each copy of G_i, a copies of T together with all edges between V and these copies of T.

We now prove by induction on i that for every $(\mathcal{D}_a, \mathcal{P})$-decomposition E_1, E_2 of G_i the graph $G_i[E_2]$ contains every tree of size i. It then follows that $G_m \notin \mathcal{D}_a \oplus \mathcal{P}$.

For $i = 0$ this is trivial. Suppose therefore the statement is true for $i-1$ and consider any $(\mathcal{D}_a, \mathcal{P})$-decomposition E_1, E_2 of $E(G_i)$ and any tree T' of size i. Let v be a leaf of T' and let u be the neighbour of v. Every copy of G_{i-1} contains a copy of $T' - v$ that is in E_2. Let $U = \{u_1, u_2, \ldots, u_a\}$ be the set of vertices corresponding to u in each copy of G_{i-1}. Now consider the subgraph of G_i isomorphic to $aK_1 + aT$ consisting of U together with the corresponding a copies of T added in the construction of G_i. Since $T \notin \mathcal{P}$, every copy of T has an edge in E_1. The end-vertices of these a edges together with U induce a subgraph H isomorphic to $aK_2 + aK_1 \notin \mathcal{D}_a$. Therefore some edge of H incident with an element of U must be in E_2, hence $T' \subseteq G_i[E_2]$. ∎

Corollary 57. *If $\mathcal{P} \subset \mathcal{D}_1$ then $\mathcal{D}_{a+1} \not\subseteq \mathcal{D}_a \oplus \mathcal{P}$.* ∎

The next bound is also due to Borowiecki and Hałuszczak — see [1].

Theorem 58. *Let $n_1, \ldots, n_k$ be positive integers. Then $\mathcal{D}_{n_1} \oplus \cdots \oplus \mathcal{D}_{n_k} \subseteq \mathcal{D}_{2\sum_1^k n_i - 1}$.* ∎

The sharpness of the inclusion of Theorem 58 is demonstrated in Theorem 510.

Lemma 59. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are properties, $G \in \mathcal{P}$ and $H \in \mathcal{Q}$ are graphs and a, b, c, d, k and n are positive integers such that the following holds:*

(1) *G and H both have order n,*

(2) *G has k independent vertices of degree a and the other $n-k$ vertices have degree c,*

(3) *H has $n-k$ independent vertices of degree b and the other k vertices have degree d.*

Then $\mathcal{P} \oplus \mathcal{Q} \not\subseteq \mathcal{D}_m$ where $m = \min\{a+d, b+c\} - 1$.

Proof. Let G_i be a copy of G and H_i be a copy of H and suppose U_i is the set of k vertices of degree a in G_i, V_i is the set of k vertices of degree d in H_i, $W_i = V(G_i) - U_i$ and $X_i = V(H_i) - V_i$, $i = 1, 2$. Let F be the graph obtained from these four graphs by identifying the vertices of U_1 one by one with the vertices of V_2, those of W_2 with X_2, those of V_1 with U_2 and those of X_1 with W_1. Then $F \in \mathcal{P} \oplus \mathcal{Q}$ and, since U_1, U_2, X_1 and X_2 are independent, every vertex in F has degree $a+d$ or $b+c$ so that $F \notin \mathcal{D}_m$. ■

If G is a graph with vertex set $\{v_1, \ldots, v_n\}$ then we denote by $G[k_1, \ldots, k_n]$ the graph with vertex set $\cup_{i=1}^{n}\{(v_i, j) : j = 1, \ldots, k_i\}$ and edge set $\{(v_i, j)(v_m, j) : v_i v_m \in E(G)\}$. If $k_1 = \cdots = k_n = k$ we write $G[k]$ for $G[k_1, \ldots, k_n]$.

Theorem 510. *For all positive integers a and b we have that $\mathcal{D}_a \oplus \mathcal{D}_b \not\subseteq \mathcal{D}_{2a+2b-2}$.*

Proof. Let G_p have vertex set $\{v_0, v_1, \ldots, v_{p+1}, u_1, u_2, \ldots, u_p\}$ and edge set $\{v_i v_{i+1} : i = 0, 1, \ldots, p\} \cup \{u_i v_i : i = 1, 2, \ldots, p\}$. Let $G_p[k, m] = G_p[k, k, \ldots, k, m, m, \ldots, m]$ (with $p+2$ k's and p m's). Note that $G_p[k, m]$ has pk vertices of degree $2k+m$ and $2k+mp$ independent vertices of degree k. Now take $x = 2(a^2+b^2-a) - (a+b-2) > 0$, $y = 2(a^2+b^2-b) - (a+b-2) > 0$ and let $G = G_x[a, b-1] \cup (a+b-2)G_1[a, b-1] \in \mathcal{D}_a$ and $H = G_y[b, a-1] \cup (a+b-2)G_1[b, a-1] \in \mathcal{D}_b$.

G has $a(x+a+b-2) = 2a(a^2+b^2-a)$ vertices of degree $2a+b-1$ and $2a(1+a+b-2)+(b-1)(x+a+b-2) = 2b(a^2+b^2-b)$ independent vertices of degree a. By symmetry, H has $2b(a^2+b^2-b)$ vertices of degree $2b+a-1$ and $2a(a^2+b^2-a)$ independent vertices of degree b. Therefore G and H satisfy the conditions of Lemma 59, with $c = 2a+b-1$, $d = 2b+a-1$, $k = 2b(a^2+b^2-b)$ and $n = 2(a^2+b^2)(a+b-1)$, hence $\mathcal{D}_a \oplus \mathcal{D}_b \not\subseteq \mathcal{D}_m$ where $m = 2a+2b-2$. ■

Theorem 511. *For every positive integer k the property $\mathcal{D}_k$ is indecomposable in $\mathbb{L}^a_{\subseteq}$.*

Proof. Suppose that $\mathcal{D}_k = \mathcal{P} \oplus \mathcal{Q}$ with $\mathcal{P}$ and $\mathcal{Q}$ non-trivial. Note that if $G \in \mathcal{P}$ and $H \in \mathcal{Q}$ then $G \times H \in \mathcal{P} \oplus \mathcal{Q}$ and $\delta(G \times H) = \delta(G) + \delta(H)$. Therefore there exist positive integers a and b with $a+b = k$ such that $\delta(G) \leq a$ and $\delta(H) \leq b$ for all $G \in \mathcal{P}$ and $H \in \mathcal{Q}$. It follows that $\mathcal{P} \subseteq \mathcal{D}_a$ and $\mathcal{Q} \subseteq \mathcal{D}_b$.

Let $P_n[l, m]$ be the graph $P_n[l, m, m, \ldots, m]$ where we take $V(P_n)$ to be $\{v_1, v_2, \ldots, v_n\}$ and $E(P_n)$ to be $\{v_i v_{i+1} : i = 1, 2, \ldots, n-1\}$. We now show that $P_n[a+b, a] \in \mathcal{P}$ and $P_n[a+b, b] \in \mathcal{Q}$ for every n: We do this by finding a graph $G_n \in \mathcal{D}_k$ such that for every $(\mathcal{D}_a, \mathcal{D}_b)$-partition E_1, E_2 of $E(G_n)$, $P_n[a+b, a] \subseteq G_n[E_1]$ and $P_n[a+b, b] \subseteq G_n[E_2]$. For $n = 2$ we can take $G_2 = K_{a+b,t}$ with t large enough, by Lemma 54. Assume therefore that

G_n has been found and consider a $(\mathcal{D}_a, \mathcal{D}_b)$-partition E_1, E_2 of $E(G_{n+1})$, where G_{n+1} is obtained from G_n by adding, for every subset of $a+b$ vertices of $V(G_n)$, t vertices together with all edges connecting the new vertices with these $a+b$ vertices. We know that $P_n[a+b, a] \subseteq G_n[E_1]$. By applying Lemma 54 to the appropriate set of $a+b$ independent vertices and the t new vertices adjacent to them it now follows that $P_{n+1}[a+b, a] \subseteq E_1$. Similarly, $P_{n+1}[a+b, b] \subseteq E_2$.

In particular, $P_4[a] \in \mathcal{P}$ and $P_4[b] \in \mathcal{Q}$. Then $G = bP_4[a]$ and $H = aP_4[b]$ satisfy the conditions of Lemma 59 with $c = 2a$, $d = 2b$, $n = 4ab$ and $k = 2ab$. Therefore $\mathcal{P} \oplus \mathcal{Q} \not\subseteq \mathcal{D}_m$ where $m = \min\{a+2b, b+2a\} - 1 \geq a+b$, a contradiction. ■

6 The Decomposability of $\mathcal{P}^k$

Corollary 61. *If $\mathcal{P} \neq \mathcal{O}$ is a property of graphs and $k \geq 2$ then $\mathcal{P}^k$ is decomposable.*

Proof. It follows easily from Lemma 21 that $\mathcal{P}^k = \mathcal{P} \oplus \mathcal{O}^k$. ■

In the rest of this section we consider the decomposability of the property $\mathcal{O}^k$. First we define two parameters for a property $\mathcal{P}$: Let $\chi(\mathcal{P})$ denote the least k such that $\mathcal{P} \subseteq \mathcal{O}^k$ and let $\chi^*(\mathcal{P})$ denote the largest k such that $\mathcal{O}^k \subseteq \mathcal{P}$. The next lemma is useful in the proof of the main result of this section.

Lemma 62. *Let $\mathcal{P}$ and $\mathcal{Q}$ be properties of graphs. Then $\chi^*(\mathcal{P} \oplus \mathcal{Q}) \leq \chi^*(\mathcal{P})\chi(\mathcal{Q}) \leq \chi(\mathcal{P} \oplus \mathcal{Q})$.*

Proof. Note that the result holds if $\chi(\mathcal{Q})$ is infinite. We may therefore suppose that $a = \chi^*(\mathcal{P})$ and $b = \chi(\mathcal{Q})$ are finite. For the first inequality, let t be such that $K_{a+1}[t] \notin \mathcal{P}$. In order to show that $\chi^*(\mathcal{P} \oplus \mathcal{Q}) \leq \chi^*(\mathcal{P})\chi(\mathcal{Q})$, it is sufficient to show that $G = K_{ab+1}[bt] \notin \mathcal{P} \oplus \mathcal{Q}$. Suppose, to the contrary, that E_1, E_2 is a $(\mathcal{P}, \mathcal{Q})$-decomposition of $E(G)$. Let $U_1, \ldots, U_{ab+1}$ be the partite sets of G, let $V_1, \ldots, V_b$ be a b-colouring of $G[E_2]$ and set $V_{i,j} = V_i \cap U_j$. For every j there must be an i such that $|V_{i,j}| \geq t$ since $|U_j| = bt$ and U_j is a union of only b of the $V_{i,j}$'s. Therefore there are at least $ab+1$ $V_{i,j}$'s with $|V_{i,j}| \geq t$. On the other hand, for every i there are at most a j's such that $|V_{i,j}| \geq t$ since every edge of $G[V_i]$ is in E_1 and $K_{a+1}[t] \notin \mathcal{P}$. Therefore there are at most ab $V_{i,j}$'s with $|V_{i,j}| \geq t$, a contradiction.

For the second inequality, suppose $G \in \mathcal{Q}$ has chromatic number b. Then the graph $H = G + G + \cdots + G$ (a G's) has chromatic number ab and is in $\mathcal{P} \oplus \mathcal{Q}$. ■

We are now ready to discuss the decomposability of $\mathcal{O}^p$.

Theorem 63. *$\mathcal{O}^p$ is indecomposable in $\mathbb{L}^a_{\subseteq}$ if and only if p is prime.*

Proof. If p is not prime, say $p = ab$, then it is easy to see from Lemma 21 that $\mathcal{O}^p = \mathcal{O}^a \oplus \mathcal{O}^b$.

Conversely, suppose that $\mathcal{O}^p = \mathcal{P} \oplus \mathcal{Q}$ with $\mathcal{P}$ and $\mathcal{Q}$ non-trivial properties of graphs. Then $\chi(\mathcal{P})$, $\chi(\mathcal{Q}), p \geq 2$ and, since $\chi(\mathcal{O}^p) = \chi^*(\mathcal{O}^p) = p$, it follows from Lemma 62 that $p = \chi^*(\mathcal{P})\chi(\mathcal{Q}) = \chi(\mathcal{P})\chi^*(\mathcal{Q})$. If $\chi^*(\mathcal{P}) > 1$ or $\chi^*(\mathcal{Q}) > 1$ we are done. Suppose therefore that $\chi^*(\mathcal{P}) = \chi^*(\mathcal{Q}) = 1$. Then $\chi(\mathcal{P}) = \chi(\mathcal{Q}) = p$, hence there exist graphs $F \in \mathcal{P}$ and $H \in \mathcal{Q}$ which both have chromatic number p. Since $F + K_1 \notin \mathcal{O}^p$ and $F + K_1 \in \mathcal{P} \oplus \mathcal{SF}$ it follows that $\mathcal{SF} \nsubseteq \mathcal{Q}$. Similarly, $\mathcal{SF} \nsubseteq \mathcal{P}$. But then $\mathcal{SF} \nsubseteq \mathcal{P} \oplus \mathcal{Q} = \mathcal{O}^p$, a contradiction, since $p \geq 2$. ■

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