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DOMINATION AND INDEPENDENCE SUBDIVISION NUMBERS OF GRAPHS

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Abstract

The *domination subdivision number* $sd_\gamma(G)$ of a graph is the minimum number of edges that must be subdivided (where an edge can be subdivided at most once) in order to increase the domination number. Arumugam showed that this number is at most three for any tree, and conjectured that the upper bound of three holds for any graph. Although we do not prove this interesting conjecture, we give an upper bound for the domination subdivision number for any graph G in terms of the minimum degrees of adjacent vertices in G. We then define the *independence subdivision number* $sd_\beta(G)$ to equal the minimum number of edges that must be subdivided (where an edge can be subdivided at most once) in order to increase the independence number. We show that for any graph G of order $n \geq 2$, either $G = K_{1,m}$ and $sd_\beta(G) = m$, or $1 \leq sd_\beta(G) \leq 2$. We also characterize the graphs G for which $sd_\beta(G) = 2$.

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1 Introduction

Let $G = (V, E)$ be a graph of order $|V| = n$. A set of vertices $S \subseteq V$ is said to be *independent* if no two vertices in S are adjacent. The *independence number* $\beta(G)$ is the maximum cardinality of an independent set in G. We call an independent set S of cardinality $\beta(G)$ a $\beta(G)$*-set*. The independence number of a graph has been well studied; discussions of this invariant can be found in any textbook on graph theory (eg. [2, 3, 4, 5, 9]).

In this paper we consider the effect that subdividing an edge has on the independence number of a graph. We say that an edge $uv \in E$ is subdivided if the edge uv is deleted, but a new vertex x (called a *subdivision vertex*) is added, along with two new edges: ux and xv. We only permit an edge to be subdivided once, that is, no edge incident to a subdivision vertex can be subdivided. We define the *independence subdivision number* $sd_\beta(G)$ to equal the minimum number of edges that must be subdivided in order to create a graph G' for which $\beta(G') > \beta(G)$.

The problem of studying this new invariant $sd_\beta(G)$ was suggested to us by a recent result of Arumugam [1], which considers the effect that subdividing an edge has on the domination number of a graph. A set $S \subseteq V$ of vertices is a *dominating set* if every vertex not in S is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set in G, and a dominating set S of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. A thorough study of the concept of domination in graphs can be found in the two books by Haynes, Hedetniemi, and Slater [6] and [7].

The *domination subdivision number* of a graph G, denoted $sd_\gamma(G)$, equals the minimum number of edges that must be subdivided in order to create a graph G' for which $\gamma(G') > \gamma(G)$. We must assume here that the graph G is of order $n \geq 3$, since the domination number of the graph K_2 does not change when its only edge is subdivided.

In a result that follows, we will need the concept of a *private neighbor*. The *closed neighborhood* of a vertex $u \in V$ is the set $N[u] = \{u\} \cup \{v | uv \in E\}$. Given a set $S \subseteq V$ of vertices and a vertex $u \in S$, the *private neighbor set of* u, with respect to S, is the set $pn[u, S] = N[u] - N[S - \{u\}]$. We say that every vertex $v \in pn[u, S]$ is a *private neighbor* of u (with respect to S). Such a vertex v is adjacent to u but is not adjacent to any other vertex of S. Note that if a vertex $u \in S$ is not adjacent to any other vertex of S, then it is an isolated vertex in the subgraph $G[S]$ induced by S. In this case, $u \in pn[u, S]$, and we say that u is its own private neighbor. We note that if a set S is a $\gamma(G)$-set, then for every vertex $u \in S$, $pn[u, S] \neq \emptyset$, i.e., every vertex of S has at least one private neighbor. It can be seen that if S is a $\gamma(G)$-set, and two vertices $u, v \in S$ are adjacent, then each of u and v must have a private neighbor other than itself.

We will also use the following terminology. Let $v \in V$ be a vertex of degree one; v is called a *leaf*. The only vertex adjacent to a leaf, say u, is called a *support vertex*, and the edge uv is called a *pendant edge*.

Results on domination and independence subdivision numbers are given in Sections 2 and 3, respectively.

2 Domination Subdivision Numbers

Arumugam [1] has shown the following.

Theorem 1 [1]. *For any tree T of order $n \geq 3$,*

$$1 \leq sd_\gamma(T) \leq 3.$$

Although Arumugam [1] states that he has not been able to classify the trees for which $sd_\gamma(T) = 1$, $sd_\gamma(T) = 2$, or $sd_\gamma(T) = 3$, he has made the following intriguing conjecture:

Conjecture 2 [1]. *For any connected graph G of order $n \geq 3$,*

$$1 \leq sd_\gamma(G) \leq 3.$$

It appears that this conjecture may be difficult to settle, either to show that it is true, or to find a counterexample, since the conjecture is a statement about the totality of all $\gamma(G)$-sets in a graph G and the effects that edge subdivisions must have on every $\gamma(G)$-set.

Although we have not been able to settle Arumugam's Conjecture, we can provide an upper bound for the domination subdivision number of any graph G.

Theorem 3. *For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v, where $deg(u) \geq 2$ and $deg(v) \geq 2$,*

$$sd_\gamma(G) \leq deg(u) + deg(v) - 1.$$

Proof. Let uv be an edge in G, and let G' be the graph which results from subdividing all edges incident to either u or v. Thus, $deg(u) + deg(v) - 1$ edges will be subdivided. We assume that both $deg(v) \geq 2$ and $deg(u) \geq 2$. We will show that $\gamma(G') > \gamma(G)$ by showing that (I) no $\gamma(G)$-set is also a dominating set of G', and (II) there is no dominating set of G' of cardinality $\gamma(G)$ that contains a subdivision vertex.

(I) Let S be an arbitrary $\gamma(G)$-set. We will show that S is not a dominating set of G'.

Case 1. $u, v \in S$. In this case, both u and v must have private neighbors other than themselves. But then neither u nor v dominate these private neighbors in G'.

Case 2. either $u \notin S$ or $v \notin S$. In this case, S no longer dominates $\{u, v\} \cap (V - S)$ in G'.

(II) Let S be a subset of G' of cardinality $\gamma(G)$ which contains at least one subdivision vertex. We will show that S is not a dominating set of G'.

Assume to the contrary that G' contains a dominating set of cardinality $\gamma(G)$ which contains at least one subdivision vertex. Among all such dominating sets, let S^* be one which contains a minimum number of subdivision vertices. Assume, without loss of generality, that S^* contains a subdivision vertex adjacent to v, call it v', which subdivides the edge vw ($w \neq u$).

It follows that $v \notin S^*$, since if $v \in S^*$, then $S = S^* - \{v'\} \cup \{w\}$ is a dominating set of G' of cardinality $\gamma(G)$ containing fewer subdivision vertices than S^*, contradicting the minimality of S^*.

Clearly, v' can only be used to dominate vertices v, v' and w. It follows that no other subdivision vertex adjacent to v is in S^*, since any such vertices could be exchanged with their neighbors not equal to v, to create a dominating set of the same cardinality with fewer subdivision vertices, again contradicting the minimality of S^*. It follows, therefore, that $u \in S^*$ since S^* is a dominating set and u is the only vertex available to dominate the subdivision vertex, say x, between u and v, and $x \notin S^*$. Then no subdivision vertex adjacent to u is in S^*, since $x \notin S^*$ and any other such vertex can be exchanged with its neighbor, not equal to u, to create a dominating set of the same cardinality with fewer subdivision vertices than S^*, again contradicting the minimality of S^*.

At this point we have established that (i) $v \notin S^*$, (ii) $u, v' \in S^*$, and (iii) every neighbor of v in G other than w is in S^*, since the subdivision vertices adjacent to v are not in S^* and must be dominated. In fact, S^* contains only one subdivision vertex, namely v'.

But if S^* is a dominating set of G' of cardinality $\gamma(G)$, then it follows that $S = S^* - \{u, v'\} \cup \{v\}$ is a dominating set of G of cardinality less than $\gamma(G)$, a contradiction. (This follows from the observation that v' is only needed to dominate vertices v, v' and w in G', and u is only needed to dominate itself and the subdivision vertices adjacent to it in G'.)

Earlier in the proof, we assumed that S^* contains a subdivision vertex adjacent to v, call it v', which subdivides the edge vw $(u \neq w)$. It remains to consider the final case that S^* contains the subdivision vertex x between vertices u and v.

In this case we can assume that S^* contains no other subdivision vertex; otherwise, they could be exchanged, as before, with their neighbors not equal to either u or v, to produce a dominating set of the same cardinality but with fewer subdivision vertices, contradicting the minimality of S^*.

But vertex x can only be used to dominate vertices u, x and v, which means that S^* cannot contain both u and v (else vertex x is not needed). Therefore there are only three remaining cases to consider:

Case 1. $u \in S^*$ and $v \notin S^*$.

Case 2. $u \notin S^*$ and $v \in S^*$.

Case 3. $u \notin S^*$ and $v \notin S^*$.

But in each of the first two cases, it can be seen that the set $S^* - \{x\}$ is a dominating set of G of cardinality less than $\gamma(G)$, a contradiction. In Case 3, since $deg(u) \geq 2$ and $deg(v) \geq 2$, then $S^* - \{x\}$ is a dominating set of G of cardinality less than $\gamma(G)$, since every neighbor of u or v in G, other than u and v, is in S^*, a contradiction. ■

Although the upper bound in Theorem 3 for the subdivision number of an arbitrary graph is not a constant, it provides, perhaps, an incentive to obtain even better upper bounds for $sd_\gamma(G)$, either in special cases where a graph has some structural property, or for certain classes of graphs. Theorem 3 can also be used to obtain constant upper bounds for the domination subdivision numbers of various classes of graphs, such as the following.

Corollary 4. *For any $r \times s$ grid graph $G_{r,s}$, where $2 \leq r \leq s$,*

$$1 \leq sd_\gamma(G_{r,s}) \leq 4.$$

Corollary 4 follows from the simple observation that every such grid graph contains a corner vertex of degree two which is adjacent to a vertex of degree three.

Corollary 5. *For any k-regular graph G where $k \geq 2$,*

$$1 \leq sd_{\gamma}(G) \leq 2k - 1.$$

3 Independence Subdivision Numbers

In this section we will show that except for one class of graphs (the stars $K_{1,m}$, for $m \geq 3$), the independence subdivision number of any graph is either one or two. We then characterize the class of graphs having independence subdivision number two.

If $sd_{\beta}(G) = 1$, for some graph $G = (V, E)$, then, by definition, there must exist an edge $uv \in E$, which when subdivided into edges ux and xv results in a graph G' for which $\beta(G') = \beta(G) + 1$. This can happen in only one of two ways: either G has a $\beta(G)$-set which does not contain either u or v; or uv is a pendant edge and G has a $\beta(G)$-set S which contains the support vertex u but not the leaf v; in which case $S \cup \{v\}$ becomes a larger independent set when the edge uv is subdivided into ux and xv.

The following results are all straightforward; their proofs are therefore omitted.

Proposition 6. *For every graph G having a $\beta(G)$-set S, where the subgraph $G[V - S]$ induced by $V - S$ has at least one edge, $sd_{\beta}(G) = 1$.*

Corollary 7. *For every graph G having an odd cycle, $sd_{\beta}(G) = 1$.*

Proposition 8. *For every graph G having a $\beta(G)$-set S and a pendant edge uv, where S contains the support vertex u (and not the leaf v), $sd_{\beta}(G) = 1$.*

Proposition 9. *For every graph G having a $\beta(G)$-set S and a vertex $u \in S$ which is adjacent to at least two vertices in $V - S$, $sd_{\beta}(G) \leq 2$.*

Corollary 10. *For every graph G having an even cycle, $sd_{\beta}(G) \leq 2$.*

Proposition 11. *For any star $K_{1,m}$, $sd_{\beta}(K_{1,m}) = m$.*

Theorem 12. *For any connected graph G of order $n \geq 3$, either*

(i) $G = K_{1,m}$ *and* $sd_\beta(G) = m$*, or*

(ii) $1 \leq sd_\beta(G) \leq 2$.

Proof. Assume first that G is connected and contains a cycle. By Corollary 7, if G contains an odd cycle, then $sd_\beta(G) = 1$.

If G has no odd cycle, then it must have a even cycle. By Corollary 10, we can conclude that $sd_\beta(G) \leq 2$.

Assume therefore that G is connected and contains no cycles, i.e., G is a tree T. If $T = K_{1,m}$, then by Proposition 11, $sd_\beta(T) = m$, and, in particular, if $T = K_{1,2}$, then $sd_\beta(T) = 2$.

Assume therefore that $T \neq K_{1,m}$, for $m \geq 3$, and hence that the diameter of T is at least three.

Case 1. If T has a $\beta(T)$-set S, for which $G[V - S]$ has an edge, then by Proposition 6, $sd_\beta(G) = 1$.

Case 2. For every $\beta(T)$-set S, $V - S$ is an independent set. Let S be any $\beta(T)$-set. Since T is connected, and has diameter at least three, there must be at least one vertex in S which is adjacent to two or more vertices in $V - S$. By Proposition 9 it then follows that $sd_\beta(G) \leq 2$. ■

It follows from the previous theorem that every connected graph of order $n \geq 3$ can be placed into one of three classes, according to their independence subdivision number:

Class I: Graphs G for which $sd_\beta(G) = 1$.
Class II: Graphs G for which $sd_\beta(G) = 2$.
Class III: Graphs $G = K_{1,m}$ for $m \geq 3$.

It follows from Corollary 7 that Class I contains all graphs which are not bipartite. Class I also contains some bipartite graphs G, i.e., those having a $\beta(G)$-set S, for which the induced subgraph $G[V - S]$ contains at least one edge (cf. Proposition 6), or those having a $\beta(G)$-set which includes at least one support vertex (cf. Proposition 8).

Class II, which consists of all graphs G for which $sd_\beta(G) = 2$, contains only bipartite graphs, eg., C_4, for every $\beta(G)$-set S of which, $V - S$ is an independent set. This class includes, for example, all even cycles C_{2k}, all odd paths P_{2k+1}, and all complete bipartite graphs $K_{r,s}$, $2 \leq r \leq s$.

We next characterize the graphs in Class II.

Theorem 13. *A connected graph G is in Class II if and only if either $G = K_{1,2}$ or G is bipartite with partite sets V_1 and V_2 such that either*

(a) $2 \leq |V_1| = |V_2| = \beta(G)$, *and* V_1 *and* V_2 *are the only* $\beta(G)$*-sets, or*

(b) $2 \leq |V_1| < |V_2| = \beta(G)$ *and* V_2 *is the unique* $\beta(G)$*-set.*

Proof. If $G = K_{1,2}$, then the theorem holds. First assume that $G \neq K_{1,2}$ is bipartite with partite sets V_1 and V_2 such that either (a) or (b) holds. Since G is connected and not a star, it follows from Theorem 12 that $1 \leq sd_\beta(G) \leq 2$. We show that $sd_\beta(G) \neq 1$. Assume to the contrary that subdividing the edge v_1v_2 yielding v_1vv_2 for some $v_1 \in V_1$ and $v_2 \in V_2$ increases the independence number, and let G' be the graph obtained from G by subdividing edge v_1v_2. If condition (a) holds, then $2 \leq |V_1| = |V_2| = \beta(G)$ and V_1 and V_2 are the unique $\beta(G)$-sets. If $x_1 \in V_1$ is an endvertex with support $y_2 \in V_2$, then $V_2 - \{y_2\} \cup \{x_1\}$ is another $\beta(G)$-set, contradicting our assumption that V_1 and V_2 are the unique $\beta(G)$-sets. Similarly, V_2 has no endvertices. Thus, $\delta(G) \geq 2$. But since v_1 (respectively, v_2) has at least two neighbors in V_2 (respectively, V_1), it follows that $\beta(G') = \beta(G)$, contradicting our assumption. If condition (b) holds, then $2 \leq |V_1| < |V_2| = \beta(G)$ and V_2 is the unique $\beta(G)$-set. Hence, for every vertex $u \in V_1$, $deg(u) \geq 2$, and the result follows as before. Thus, $G \in$ Class II.

For the converse, assume that connected graph $G \neq K_{1,2} \in$ Class II, i.e. $sd_\beta(G) = 2$. Let S be a $\beta(G)$-set. Proposition 6 implies that $V - S$ is independent and hence, G is bipartite. Since G is connected and not a star, $2 \leq |V - S| \leq |S| = \beta(G)$. If any vertex, say v, in $V - S$ has exactly one neighbor, say u, in S, then subdividing the edge uv forming uxv increases the independence number since $S \cup \{v\}$ is an independent set, contradicting that $G \in$ Class II. Thus, every vertex in $V - S$ has at least two neighbors in S. Note that if $|S| = |V - S|$, then both S and $V - S$ are $\beta(G)$-sets implying that $\delta(G) \geq 2$. Suppose S' is a $\beta(G)$-set that is not a partite set of G, that is, $S' \cap S = A \neq \emptyset$ and $S' \cap (V - S) = B \neq \emptyset$. Let $C = S - A$ and $D = V - S - B$. Note that $S' = A \cup B$ and $V - S' = (S - A) \cup (V - S - B) = C \cup D$.

If $C \cup D$ contains an edge, then by Proposition 6, $sd_\beta(G) = 1$. But since $G \in$ Class II, $C \cup D$ must be an independent set. But in this case there are no edges between $A \cup D$ and $C \cup B$, implying that G is not a connected graph, a contradiction. Hence, either condition (a) or (b) holds. ∎

A graph is a *strong unique independence graph* if G is bipartite and has a unique $\beta(G)$-set. Hopkins and Staton [8] characterized strong unique independence graphs as follows:

Theorem 14 [8]**.** *A tree is a strong unique independence tree if and only if the distance between any pair of its leaves is even.*

Theorem 15 [8]. *A connected graph G is a strong unique independence graph if and only if G is bipartite and has a spanning tree which is a strong unique independence tree.*

We can now restate Theorem 13.

Theorem 16. *A connected graph $G \in$ Class II if and only if G is bipartite and either*

(a) *G has equal partite sets, i.e., $|V_1| = |V_2|$, and $\delta(G) \geq 2$, or*

(b) *$G \neq K_{1,m}$, $m \geq 3$, has a spanning tree T such that the distance between any pair of leaves in T is even.*

At this point it remains an interesting open question whether there exists a polynomial algorithm for deciding whether a graph $G \neq K_{1,m}$ belongs to either Class I or Class II.

INDEPENDENCE SUBDIVISION NUMBER
INSTANCE: Graph $G = (V, E) \neq K_{1,m}$.
QUESTION: Is $sd_\beta(G) = 1$ or $sd_\beta(G) = 2$?

Denote the class of graphs satisfying condition (a) of Theorem 16 as Class IIa and those satisfying condition (b) as Class IIb. Certainly one can tell whether a graph is bipartite in polynomial time. Therefore, one can determine if a graph is in Class IIa in polynomial time. However, it is not immediately obvious whether one can determine in polynomial time whether V_2 is the unique $\beta(G)$-set, that is, whether G has a spanning tree T such that the distance between any pair of leaves in T is even.

References

[1] S. Arumugam, private communication, June 2000.

[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (American Elsevier, New York, 1977).

[4] G. Chartrand and L. Lesniak, Graphs & Digraphs (Wadsworth and Brooks/Cole, Monterey, CA, third edition, 1996).

[5] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[7] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs, Advanced Topics (Marcel Dekker, Inc., New York, 1998).

[8] G. Hopkins and W. Staton, *Graphs with unique maximum independent sets*, Discrete Math. **57** (1985) 245–251.

[9] D.B. West, Introduction to Graph Theory (Prentice Hall, New Jersey, 1996).

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