Discussiones Mathematicae
Graph Theory 20 (2000) 267–269

A NOTE ON PERIODICITY OF THE 2-DISTANCE OPERATOR

Bohdan Zelinka

Department of Applied Mathematics
Technical University of Liberec
Liberec, Czech Republic

To the memory of Ivan Havel

Abstract

The paper solves one problem by E. Prisner concerning the 2-distance operator T_2. This is an operator on the class C_f of all finite undirected graphs. If G is a graph from C_f, then $T_2(G)$ is the graph with the same vertex set as G in which two vertices are adjacent if and only if their distance in G is 2. E. Prisner asks whether the periodicity ≥ 3 is possible for T_2. In this paper an affirmative answer is given. A result concerning the periodicity 2 is added.

Keywords: 2-distance operator, complement of a graph.

2000 Mathematics Subject Classification: 05C12.

In this paper we consider finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$, its edge set by $E(G)$. The symbol $\overline{G}$ denotes the complement of G, i.e., the graph with the same vertex set as G in which two distinct vertices are adjacent if and only if they are not adjacent in G.

Let ϕ be a graph operator defined on the class C_f of all finite undirected graphs. For every positive integer r we define the power ϕ^r so that $\phi^1 = \phi$ and for $r \geq 2$ the operator ϕ^r is such that $\phi^r(G) = \phi(\phi^{r-1}(G))$ for each $G \in C_f$. A graph $G \in C_f$ is called *ϕ-periodic*, if there exists a positive integer r such that $\phi^r(G) \cong G$. The minimum number r with this property is the *periodicity* of the graph G in the operator ϕ.

For an integer $k \geq 2$ the operator T_k on C_f is defined in such a way that for any graph $G \in C_f$ the graph $T_k(G)$ has the same vertex set as G

and two distinct vertices are adjacent in $T_k(G)$ if and only if their distance in G is k. The operator T_k is called the *k-distance operator*.

In [2], page 170, E. Prisner asks the following problem:

Is period ≥ 3 possible for T_2 ?

An affirmative answer is given by the following theorem.

Theorem. *Let r be an even positive integer. Then there exists a graph G_r whose periodicity in the operator T_2 is r.*

Proof. Let $q = 2^r + 1$. Let $V_0, V_1, \ldots, V_{q-1}$ be pairwise disjoint sets of vertices. Let t be an integer, $t \geq 2$ and let $|V_i| = t^i$ for $i = 0, 1, \ldots, q-1$. The vertex set of G_r is $V(G_r) = \bigcup_{i=0}^{q-1} V_i$. All sets $V_0, V_1, \ldots, V_{q-1}$ are independent in G_r. Let $x \in V_i$, $y \in V_j$ for some i and j from $\{0, 1, \ldots, q-1\}$. These vertices are adjacent in G_r if and only if $j \equiv i + 1 \pmod q$ or $j \equiv i - 1 \pmod q$. This implies that all sets $V_0, V_1, \ldots, V_{q-1}$ induce complete subgraphs in the graph $T_2(G_r)$. If $x \in V_i$, $y \in V_j$, then x, y are adjacent in $T_2(G_r)$ if and only if $j \equiv i + 2 (\text{mod } q)$ or $j \equiv i - 2 (\text{mod } q)$. From these facts by induction we obtain that $T_2^m(G)$ for $m \geq 2$ has the following structure. If m is even, then all sets $V_0, V_1, \ldots, V_{q-1}$ are independent; if m is odd, then they induce complete subgraphs; if $x \in V_i$, $y \in V_j$, then x, y are adjacent if and only if $j \equiv i + 2^m (\text{mod } q)$ or $j \equiv i - 2^m (\text{mod } q)$ in both the cases. This implies that $T_2^r(G_r) \cong G_r$. Now it remains to show that $T_2^m(G)$ is not isomorphic to G_r for $1 \leq m < r$. We do it using the independence number $\alpha(G)$. The greatest independent set in G_r is $\bigcup_{i=1}^{\frac{1}{2}(q-1)} V_{2i}$ and thus $\alpha(G_r) = \sum_{i=1}^{\frac{1}{2}(q-1)} t^{2i} = t^2(t^{q-1} - l)/(t^2 - 1)$. If m is odd, then $\alpha(T_2^m(G)) = \frac{1}{2}(q-1)$. If m is even, $2 \leq m \leq r-2$, then the set $V_0 \cup V_{q-2} \cup V_{q-1}$ is independent in $T_2^m(G_r)$ and thus $\alpha(T_2^m(G_r)) \geq |V_0 \cup V_{q-2} \cup V_{q-1}| = 1 + t^{q-2} + t^{q-1} > t^2(t^{q-1} - 1)/(t^2 - 1)/(t^2 - 1) = \alpha(G_r)$; this inequality may be easily proved. Therefore no graph $T_2^m(G_r)$ for $1 \leq m \leq r-1$ is isomorphic to G_r and thus the periodicity of G_r in T_2 is r. ■

We shall remark also the periodicity 2. In [1] F. Harary, C. Hoede and D. Kadlacek have proved that if a graph G is self-complementary, i.e., $\overline{G} \cong G$, then $T_2(G) \cong G$ and thus the periodicity of G in T_2 is 1. A slight generalization of the result is the following proposition. The diameter of G is denoted by diam G.

Proposition 1. *Let G be a graph such that* diam $G =$ diam $\overline{G} = 2$ *and $\overline{G}$ is not isomorphic to G. Then G is T_2-periodic with the periodicity* 2.

Proof. If two vertices x, y are adjacent in G, then their distance in G is 1 and they are not adjacent in $T_2(G)$. If they are not adjacent in G, then their distance in G is 2 and realizes diam G. Moreover, x and y are adjacent in $T_2(G)$. Hence $T_2(G) = \overline{G}$. As also diam $\overline{G} = 2$, we have $T_2^2(G) = T_2(T_2(G)) = T_2(\overline{G}) = G$. ■

We shall create a class of graph which have the property that diam $G =$ diam $\overline{G} = 2$.

Let H_1, H_2, H_3, H_4, H_5 be pairwise disjoint graphs. The graph $G(H_1, H_2, H_3, H_4, H_5)$ contains mentioned graphs as subgraphs and has new edges xy created in the following way. If $x \in V(H_i)$, $y \in V(H_j)$, then x and y are adjacent in G if and only if $j \equiv i + 1 (\text{mod } 5)$ or $j \equiv i + 4 (\text{mod } 5)$. The simplest is the graph $G(K_1, K_1, K_1, K_1, K_1) = C_5$.

Proposition 2. *For any five graphs H_1, H_2, H_3, H_4, H_5 the graph $G(H_1, H_2, H_3, H_4, H_5)$ has the diameter 2 and so has its complement.*

Proof. Let x, y be two vertices of $G(H_1, H_2, H_3, H_4, H_5)$. Let i, j be such numbers from $\{1, 2, 3, 4, 5\}$ that $x \in V(H_i)$, $y \in V(H_j)$.

If $i = j$, then both x, y are in the graph H_i. If they are adjacent in G, then their distance is 1. If they are not adjacent, then there exists a path of length 2 connecting them; its inner vertex is in $V(H_{j+1}) \cup V(H_{i+4})$, the subscripts being taken modulo 5. If $j \equiv i + 1 (\text{mod } 5)$ or $j \equiv i + 4 (\text{mod } 5)$, then x, y are adjacent in G and their distance is 1.

If $j \equiv i + 2 (\text{mod } 5)$ or $j \equiv i + 3 (\text{mod } 5)$ then x, y are not adjacent, but there exists a path of length 2 connecting them; its inner vertex is in $V(H_{i+1}) \cup V(H_{i+4})$. Therefore diam $G = 2$. The complement of $G(H_1, H_2, H_3, H_4, H_5)$ is isomorphic to $G(\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4, \overline{H}_5)$ and thus also diam $\overline{G} = 2$. ■

References

[1] F. Harary, C. Hoede and D. Kadlacek, *Graph-valued functions related to step graphs*, J. Comb. Ing. Syst. Sci. **7** (1982) 231–246.

[2] E. Prisner, Graph Dynamics (Longman House, Burnt Mill, Harlow, 1995).

Received 18 February 2000
Revised 5 July 2000