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SUM LABELLINGS OF CYCLE HYPERGRAPHS

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Abstract

A hypergraph $\mathcal{H}$ is a *sum hypergraph* iff there are a finite $S \subseteq \mathbb{N}^+$ and $\underline{d}, \overline{d} \in \mathbb{N}^+$ with $1 < \underline{d} \leq \overline{d}$ such that $\mathcal{H}$ is isomorphic to the hypergraph $\mathcal{H}_{\underline{d},\overline{d}}(S) = (V, \mathcal{E})$ where $V = S$ and $\mathcal{E} = \{e \subseteq S : \underline{d} \leq |e| \leq \overline{d} \wedge \sum_{v \in e} v \in S\}$. For an arbitrary hypergraph $\mathcal{H}$ the sum number $\sigma = \sigma(\mathcal{H})$ is defined to be the minimum number of isolated vertices $y_1, \ldots, y_\sigma \notin V$ such that $\mathcal{H} \cup \{y_1, \ldots, y_\sigma\}$ is a sum hypergraph.

Generalizing the graph C_n we obtain *d-uniform hypergraphs* where any d consecutive vertices of C_n form an edge. We determine sum numbers and investigate properties of sum labellings for this class of cycle hypergraphs.

Keywords: hypergraphs, sum number, vertex labelling.

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1 Definitions and Introduction

The notion of sum graphs was introduced by Harary [3]. This graph theoretic concept can be generalized to hypergraphs as follows.

All hypergraphs considered here are supposed to be nonempty and finite, without loops and multiple edges. In standard terminology we follow Berge [1]. By $\mathcal{H} = (V, \mathcal{E})$ we denote a hypergraph with vertex set V and edge set $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Further we use the notations $\underline{d} = \underline{d}(\mathcal{H}) = \min\{|e| : e \in \mathcal{E}\}$ and $\overline{d} = \overline{d}(\mathcal{H}) = \max\{|e| : e \in \mathcal{E}\}$; if $\underline{d} = \overline{d} = d$ we say $\mathcal{H}$ is a *d-uniform hypergraph*. A hypergraph is *linear* if no two edges intersect in more than one vertex.

Let $S \subseteq \mathbb{N}^+$ be finite and $\underline{d}, \overline{d} \in \mathbb{N}^+$ such that $1 < \underline{d} \leq \overline{d}$. Then $\mathcal{H}_{\underline{d},\overline{d}}(S) = (V, \mathcal{E})$ is called a $(\underline{d}, \overline{d})$-*sum hypergraph of* S iff $V = S$ and $\mathcal{E} = \{e \subseteq S : \underline{d} \leq |e| \leq \overline{d} \wedge \sum_{v \in e} v \in S\}$. Furthermore, a hypergraph $\mathcal{H}$ is a *sum hypergraph* iff there exist $S \subseteq \mathbb{N}^+$ and $\underline{d}, \overline{d} \in \mathbb{N}^+$ such that $\mathcal{H}$ is isomorphic to $\mathcal{H}_{\underline{d},\overline{d}}(S)$. For $\underline{d} = \overline{d} = 2$ we obtain the known concept of sum graphs. For an arbitrary hypergraph $\mathcal{H}$ the *sum number* $\sigma = \sigma(\mathcal{H})$ is defined to be the minimum number of isolated vertices $y_1, \ldots, y_\sigma \notin V$ such that $\mathcal{H} \cup \{y_1, \ldots, y_\sigma\}$ is a sum hypergraph.

The concept of cycles C_n can be extended to hypergraphs in several ways. One possibility is the consideration of linear hypergraphs $\mathcal{C}_m$ with m vertices and n edges each containing an arbitrary number $d_j \geq 2$ of vertices, $j = 1, \ldots, n$. These *hypercycles* have sum number $\sigma(\mathcal{C}_m) = 1$ if $d_j \geq 3$ for $j = 1, \ldots, n$ (Teichert [9]). Furthermore, in case of $d_j = d$ for $j = 1, \ldots, n$, they represent hamiltonian cycles in the sense of Bermond et al. [2].

Katona and Kierstead [5] explain that this notion of hamiltonian cycles in hypergraphs is not strong enough for many applications. They call a cyclic ordering $(v_1, \ldots, v_n)$ of the vertices of a d-uniform hypergraph a *hamiltonian chain* iff $\{v_i, v_{i+1}, \ldots, v_{i+d-1}\}$ is an edge whenever $1 \leq i \leq n$ (the indices are taken modulo n). This motivates the following definition. Let $d, n \in \mathbb{N}$ with $n \geq 3$ and $2 \leq d \leq n-1$. The d-uniform hypergraph $\hat{\mathcal{C}}_n^d = (V, \mathcal{E})$ is the *strong hypercycle* with n vertices iff

$$V = \{v_1, \ldots, v_n\},\ \mathcal{E} = \{e_i = \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \ : \ i = 1, \ldots, n\}, \tag{1}$$

where indices are taken mod n.

Sonntag [8] proves that every strong hypercycle has an antimagic vertex labelling. In this paper we deal with sum labellings of strong hypercycles. In Section 2 we show for the case $n \geq 2d+1$ that $\sigma(\hat{\mathcal{C}}_n^d) = d$. In Section 3 we investigate properties of strong hypercycles with at most $2d$ vertices; particularly we determine the sum numbers of $\hat{\mathcal{C}}_{d+1}^d, \hat{\mathcal{C}}_5^3$ and $\hat{\mathcal{C}}_6^3$.

2 The Sum Number of $\hat{\mathcal{C}}_n^d$ for the Case $n \geq 2d+1$

Let $Y = \{y_1, \ldots, y_\sigma\}$ with $\sigma = \sigma(\hat{\mathcal{C}}_n^d)$ be a set of isolated vertices such that $\hat{\mathcal{C}}_n^d \cup Y$ is a sum hypergraph. For the edges e_i from (1) we use the notation $e_i = \{v_1^i, \ldots, v_d^i\}$ where $v_1^i = v_i, \ldots, v_d^i = v_{i+d-1}$. All vertices of $\tilde{V} = V \cup Y$ are referenced by their labels. A vertex labelling of $\hat{\mathcal{C}}_n^d \cup Y$ induces the

mapping r^*:

$$\mathcal{P}(\tilde{V}) \ni M \mapsto r^*(M) := \sum_{v \in M} v \in \mathbb{N}^+.$$

A *sum labelling* of $\tilde{V}$ is a vertex labelling such that the set S of the vertex labels defines a (d,d)-sum hypergraph $\mathcal{H}_{d,d}(S)$ of S with $\mathcal{H}_{d,d}(S) \cong \hat{\mathcal{C}}_n^d \cup Y$.

The following three lemmata describe properties of sum labellings for strong hypercycles with $n \geq 2d+1$ vertices; they are needed to prove the main result of this section.

Lemma 1. *Assume $n \geq 2d+1$ and consider a sum labelling of $\tilde{V}$. Then for any two different edges $e_i, e_j \in \mathcal{E}$ holds*

$$e_i \cap e_j \neq \emptyset \quad \Rightarrow \quad r^*(e_i) \neq r^*(e_j). \tag{2}$$

Proof. Let $e_i, e_j \in \mathcal{E}$ with $i < j$ be arbitrarily chosen and suppose $e_i \cap e_j \neq \emptyset$, i.e.,

$$\exists k \in \{2, \ldots, d\} : v_k^i = v_1^j, \ldots, v_d^i = v_{d-k+1}^j. \tag{3}$$

Now assume $r^*(e_i) = r^*(e_j)$. By (3) follows

$$r^*(\{v_{d-k+2}^j, \ldots, v_d^j\}) = r^*(\{v_1^i, \ldots, v_{k-1}^i\}). \tag{4}$$

The structure of $\hat{\mathcal{C}}_n^d$ implies

$$\exists e_p \in \mathcal{E} : e_p \cap e_j = \{v_{d-k+2}^j = v_1^p, \ldots, v_d^j = v_{k-1}^p\}$$

and by (4) we obtain

$$r^*(e_p) = r^*(\{v_{d-k+2}^j, \ldots, v_d^j, v_k^p, \ldots, v_d^p\}) = r^*(\{v_1^i, \ldots, v_{k-1}^i, v_k^p, \ldots, v_d^p\}).$$

The condition $n \geq 2d+1$ provides $e_i \cap e_p = \emptyset$. Moreover v_d^p and v_1^i are not consecutive vertices $v_\mu, v_{\mu+1}$ in (1). Hence $|\{v_1^i, \ldots, v_{k-1}^i, v_k^p, \ldots, v_d^p\}| = d$ and $\{v_1^i, \ldots, v_{k-1}^i, v_k^p; \ldots, v_d^p\} \notin \mathcal{E}$, a contradiction to the sum hypergraph property which proves (2). ■

Lemma 2. *For the sum number of strong hypercycles holds*

$$\sigma(\hat{\mathcal{C}}_n^d) \geq d \quad \text{for } n \geq 2d+1. \tag{5}$$

Proof. Consider in a sum labelling of $\hat{\mathcal{C}}_n^d \cup Y$ the vertex $v_{\max} = \max\{v_1, \ldots, v_n\}$. There are d pairwise distinct edges $e_i' \in \mathcal{E}$ with $v_{\max} \in$

e'_i; $i = 1, \ldots, d$. This yields $R = \{r^*(e'_i) : i = 1, \ldots, d\} \subseteq Y$ and by Lemma 1 follows $|R| = d$. Hence $\sigma(\hat{\mathcal{C}}_n^d) = |Y| \geq d$. ■

Our aim is to show that equality holds in (5). For this purpose we assume $Y = \{y_1, \ldots, y_d\}$ and define the following vertex labelling:

$$v_i = \begin{cases} i, & \text{for} \quad i = 1, \ldots, d, \\ r^*(e_{i-d}), & \text{for} \quad i = d+1, \ldots, n, \end{cases} \tag{6}$$

$$y_{d-k} = r^*(e_{n-k}), \quad \text{for} \quad k = 0, \ldots, d-1.$$

The next step is to show that labelling (6) and the sum hypergraph property generate only edges contained in the set $\mathcal{E}$ of (1).

Lemma 3. *Suppose that the vertex set $V \cup \{y_1, \ldots, y_d\}$ of $\hat{\mathcal{C}}_n^d \cup Y$ is labelled according to* (6) *and let $k \in \{0, \ldots, d-1\}$ be arbitrarily chosen. Then*

$$\forall\ M \subseteq V : r^*(M) = y_{d-k} \ \wedge\ |M| = d \Rightarrow M = e_{n-k}, \tag{7}$$

$$\forall\ M \subseteq V \cup Y : r^*(M) = y_{d-k} \ \wedge\ |M| = d \wedge n \geq 2d+1 \Rightarrow M \cap Y = \emptyset. \tag{8}$$

Proof. 1. To prove (7) we use the notation $e_{n-k} = L \cup F$ with $L = \{v_{n-k}, \ldots, v_n\}$ and $F = \{v_1, \ldots, v_{d-k-1}\}$. First we show

$$L \subseteq M. \tag{9}$$

Assume there is a $v_{n-j} \notin M, j \in \{0, \ldots, k\}$. Then (6) and $|M| = d$ yield

$$r^*(M) \leq r^*(L \setminus \{v_{n-j}\}) + r^*(\{v_{n-k-1}, \ldots, v_{n-d}\}). \tag{10}$$

We define $R := \{v_{n-k-1}, \ldots, v_{n-d}\}$ and consider two cases:

Case 1. If $k \in \{1, \ldots, d-1\}$ it follows $|R| = d - k < d$ and therefore $r^*(R) < v_{n-j}$. Using (10) we obtain

$$r^*(M) < r^*(L) \leq r^*(e_{n-k}),$$

a contradiction to $r^*(M) = y_{d-k} = r^*(e_{n-k})$.

Case 2. If $k = 0$ then $j = 0, L \setminus \{v_{n-j}\} = \emptyset$ and $R = e_{n-d}$. Hence by (10)

$$r^*(M) \leq 0 + r^*(e_{n-d}) = v_n < \min\{y_1, \ldots, y_d\}$$

which contradicts $r^*(M) = y_{d-k} = y_d$.

Thus we have a contradiction for each $k \in \{0, \ldots, d-1\}$ and (9) is true, i.e.,

$$r^*(M \setminus L) = r^*(e_{n-k}) - r^*(L) = r^*(F). \tag{11}$$

Because of $|M \setminus L| = |F|$ it follows with (6) and (11) that only $M \setminus L = F$ is possible. Hence $M = L \cup F = e_{n-k}$ and (7) is shown.

2. To prove (8) we first suppose $|M \cap Y| \geq 2$. This yields

$$r^*(M) \geq 2v_n = v_n + (v_{n-1} + \ldots + v_{n-d}) = y_1 + v_{n-d} > y_1 = \max\{y_1, \ldots, y_d\},$$

a contradiction to $r^*(M) = y_{d-k}$. Hence we know that

$$|M \cap Y| \leq 1 \tag{12}$$

must be fulfilled. Now suppose that (8) is not true, i.e.,

$$\exists\, M \subseteq V \cup Y\ \exists\, i \in \{1, \ldots, d-1\} : \\ r^*(M) = y_{d-i} \wedge |M| = d \ \wedge\ n \geq 2d+1 \ \wedge\ M \cap Y \neq \emptyset.$$

Using (12) this implies

$$\exists\, j \in \{0, \ldots, d-2\}\ \exists\, M' \subseteq V : y_{d-i} = y_{d-j} + r^*(M') \wedge |M'| = d-1. \tag{13}$$

By (6) and (13) follows $r^*(M') = r^*(e_{n-i}) - r^*(e_{n-j})$, i.e.,

$$r^*(M') = r^*(\{v_{n-i}, \ldots, v_{n-j-1}\}) - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}). \tag{14}$$

Hence there must be a vertex $v_{n-k} \in \{v_{n-i}, \ldots, v_{n-j-1}\}$ with $v_{n-k} \notin M'$. Using $|M'| = d-1$ and (14) it follows that the number $v_{n-k} - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\})$ is not greater than the sum of the labels of at most $d-(i-j)$ vertices v_p with $p < n-i$. Observe that $\sum_{k=1}^{d-i+j} v_{n-i-k}$ is the largest sum of this kind. Now consider the number

$$\mu := v_{n-i} - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}) \leq v_{n-k} - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}).$$

In the following we generate a contradiction to (13) by showing that

$$\mu - \sum_{k=1}^{d-i+j} v_{n-i-k} > 0 \tag{15}$$

is fulfilled. With $v_{n-i} = \sum\limits_{k=1}^{d} v_{n-i-k},\ n \geq 2d+1$ and $i > j$ we obtain

$$\begin{aligned}
\mu - \sum_{k=1}^{d-i+j} v_{n-i-k} &= (\sum_{k=d-i+j+1}^{d} v_{n-i-k}) - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}) \\
&\geq (\sum_{k=d-i+j+1}^{d} v_{(2d+1)-i-k}) - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}) \\
&= (\sum_{k=1}^{i-j} v_{d-k-j+1}) - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}) \\
&= r^*(\{v_{d-i+1}, \ldots, v_{d-j}\}) - r^*(\{v_{d-i}, \ldots, v_{d-j-1}\}) \\
&= v_{d-j} - v_{d-i} > 0.
\end{aligned}$$

Hence (15) is true and assertion (8) is proved. ■

Observe that assertion (8) in Lemma 3 cannot be proved if $n < 2d+1$; a simple calculation shows that inequality (15) is not true in this case. Indeed, for instance for $n = 8$ and $d = 4$ the labelling (6) yields $V = \{1, 2, 3, 4, 10, 19, 36, 69\}$ and $Y = \{134, 125, 108, 75\}$ which is not a sum labelling of $\hat{\mathcal{C}}_8^4 \cup Y$ because $2 + 3 + 4 + 125 = 134 \in Y$ but $\{2, 3, 4, 125\} \notin \mathcal{E}$.

Now we can formulate the main result of this section.

Theorem 4. *For $d \geq 2$ and $n \geq 2d+1$ the sum sumber of the strong hypercycle $\hat{\mathcal{C}}_n^d$ is given by*

$$\sigma(\hat{\mathcal{C}}_n^d) = d. \tag{16}$$

Proof. Let $d \geq 2$ and $n \geq 2d+1$. Lemma 2 shows $\sigma(\hat{\mathcal{C}}_n^d) \geq d$. The labelling (6) is a vertex labelling of $\hat{\mathcal{C}}_n^d \cup \{y_1, \ldots, y_d\}$ which generates all edges of the strong hypercycle $\hat{\mathcal{C}}_n^d$. Finally Lemma 3 yields that (6) is a sum labelling of $\tilde{V}$ and this completes the proof. ■

3 Strong Hypercycles With at Most $2d$ Vertices

The sum numbers for cycles $C_n = \hat{\mathcal{C}}_n^2, n \geq 3$ are given in Harary [4] by

$$\sigma(C_n) = \begin{cases} 2, & \text{if} \quad n \neq 4, \\ 3, & \text{if} \quad n = 4. \end{cases} \tag{17}$$

Theorem 4 generalizes (17) for $n \geq 5$ and the next result shows that this generalization is also possible for $n = 3$.

Theorem 5. *For $d \geq 2$ and $n = d + 1$ the sum number of the strong hypercycle $\hat{\mathcal{C}}_n^d$ is given by*

$$\sigma(\hat{\mathcal{C}}_{d+1}^d) = d. \tag{18}$$

Proof. From $n = d+1$ follows that any two different edges $e_i, e_j \in \mathcal{E}$ have exactly $d - 1$ vertices in common. Hence (2) is true in this case too, i.e.,

$$\forall\ e_i, e_j \in \mathcal{E} : i \neq j \Rightarrow r^*(e_i) \neq r^*(e_j),$$

and by analogy with the proof of Lemma 2 we obtain $\sigma(\hat{\mathcal{C}}_{d+1}^d) \geq d$. For $d = 2$ we obtain equality by (17), for instance with the labelling $V = \{1, 10, 11\}$, $Y = \{21, 12\}$. To show equality for $d \geq 3$ we use labelling (6); because $n = d+1$ it suffices to prove, that any edge generated by the sum hypergraph property contains only vertices of V, i.e.,

$$\forall\ M \subseteq V \cup Y : r^*(M) \in V \cup Y \ \wedge\ |M| = d \ \Rightarrow\ M \cap Y = \emptyset. \tag{19}$$

Assuming the contrary we observe that (12) from the proof of Lemma 3 is also true for $n = d + 1$. Hence

$$\exists\ v'_1, \ldots, v'_{d-1} \in V\ \exists\ y' \in Y : |\{v'_1, \ldots, v'_{d-1}\}| = d - 1\ \wedge\ y' + \sum_{j=1}^{d-1} v'_j \in Y.$$

Using $d \geq 3,\ n = d + 1$ and (6) we obtain

$$y' + \sum_{j=1}^{d-1} v'_j \geq y_d + \sum_{j=1}^{d-1} j \geq y_d + \frac{3(d-1)}{2} > y_d + (d-1) = y_1 = \max\{y_1, \ldots, y_d\},$$

a contradiction. Thus (19) holds and the proof is completed. ■

Summarizing the results we see that equalities (16) and (18) generalize the result (17) for cycles C_n with $n \neq 4$. In the following we discuss the remaining cases $d+2 \leq n \leq 2d$ for strong hypercycles $\hat{\mathcal{C}}_n^d$. These cases correspond to the cycle C_4 in (17).

Consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a labelling of $V \cup Y$ such that $\mathcal{H} \cup Y$ is a sum hypergraph. By analogy with Miller et al. [6] a vertex $v \in V \cup Y$ is said to be a *working vertex* iff its label corresponds to an edge $e \in \mathcal{E}$. Hypergraphs which can only be labelled in such a way that all the

working vertices are isolates belonging to Y are called *exclusive*. Sharary [7] shows that the graph C_4 is exclusive; this fact is generalized by the following result.

Theorem 6. *The strong hypercycle $\hat{\mathcal{C}}_n^d$ is exclusive iff $d+2 \leq n \leq 2d$.*

Proof. Theorems 4 and 5 show that $\hat{\mathcal{C}}_n^d$ is not exclusive for $n \geq 2d+1$ and $n = d+1$, respectively. In the following assume that $d+2 \leq n \leq 2d$ and consider an arbitrary sum labelling of $V \cup Y$. It remains to show that

$$\forall\, e \in \mathcal{E} : r^*(e) \in Y. \tag{20}$$

1. Suppose (20) is false; then $M = \{e \in \mathcal{E} : r^*(e) \in V\} \neq \emptyset$. Choose $\hat{e} = \{\hat{v}_1, \ldots, \hat{v}_d\} \in M$ such that

$$r^*(\hat{e}) = \min_{e \in M} r^*(e). \tag{21}$$

Observe that $\hat{e}$ is not necessary uniquely determined. In this case we consider $M' \subseteq M$ with $M' = \{e \in M : r^*(e) = r^*(\hat{e})\}$. If d_{C_n} denotes the distance function in the graph C_n (where the edges of $C_n = \hat{\mathcal{C}}_n^2$ are denoted according to (1)) and $v_{\max} = \max_{v \in V} v$ choose $\hat{e}$ such that additionally to (21) holds

$$\begin{aligned} d_{C_n}(\hat{e}, v_{\max}) &= \min_{e \in M'} d_{C_n}(e, v_{\max}), \\ \text{where} \quad d_{C_n}(e, v_{\max}) &:= \min_{v \in e} d_{C_n}(v, v_{\max}). \end{aligned} \tag{22}$$

Obviously, $v_{\max} \notin \hat{e}$, thus $d_{C_n}(\hat{e}, v_{\max}) \geq 1$.

2. Let $\tilde{v} \in V$ be the vertex with $\tilde{v} = r^*(\hat{e})$. Clearly,$\tilde{v} \notin \hat{e}$ and with $n \leq 2d$ follows the existence of an edge $e' \in \mathcal{E}$, such that $\tilde{v}, v_{\max} \in e'$ and $\tilde{v}$ is a border vertex of e', i.e., $\tilde{v}$ has only one neighbour in C_n that belongs to e' in $\hat{\mathcal{C}}_n^d$. Obviously, $y' := r^*(e') \in Y$.

Now consider an edge $e'' \in \mathcal{E}$ with $|e' \cap e''| = d-1$ and $\tilde{v} \notin e''$. By $n \leq 2d$ we obtain

$$e'' = \{e' \setminus \{\tilde{v}\}\} \cup \{\hat{v}_j\};\ j \in \{1, \ldots, d\}. \tag{23}$$

In part 3 of this proof we will show that

$$y'' := r^*(e'') \in Y \tag{24}$$

is fulfilled. Then, using (23) and (24) it follows for $y', y'' \in Y$:

$$y' = r^*(e') = r^*(e'') + \tilde{v} - \hat{v}_j = y'' + (\sum_{i=1}^{d} \hat{v}_i) - \hat{v}_j = y'' + \sum_{\substack{i=1 \\ i \neq j}}^{d} \hat{v}_i \in Y$$

but $\{y'', \hat{v}_1, \ldots, \hat{v}_{j-1}, \hat{v}_{j+1}, \ldots, \hat{v}_d\} \notin \mathcal{E}$, a contradiction. Hence (20) must be true and it is only left to prove (24).

3. If $\tilde{v} \neq v_{\max}$ then e' is uniquely determined, $v_{\max} \in e''$ and therefore (24) is true.

In the following assume $\tilde{v} = v_{\max}$. Then $n \geq d+2$ implies that e' can be chosen in such a way, that $e'' \neq \hat{e}$. Further by (21) follows

$$\forall\ e \in M : r^*(e) = r^*(\hat{e}) = v_{\max}. \tag{25}$$

Hence (24) is proved by showing that

$$r^*(\hat{e}) \neq r^*(e'') \tag{26}$$

is fulfilled. Clearly, if $d_{C_n}(\hat{e}, v_{\max}) > 1$ it follows by $d_{C_n}(e'', v_{\max}) = 1$ and (22) that (26) is true.

For clarity we summarize the conditions for the remaining case:

$$v_{\max} = \tilde{v} = r^*(\hat{e});\ \hat{e} \neq e'';\ d_{C_n}(\hat{e}, v_{\max}) = d_{C_n}(e'', v_{\max}) = 1.$$

Without loss of generality we can use the following notation (see (1)):

$$v_1 = v_{\max}\ ;\ e'' = \{v_2, \ldots, v_{d+1}\}\ ;\ \hat{e} = \{v_{n-d+1}, \ldots, v_n\}.$$

Observe that $e'' \setminus \hat{e} = \{v_2, \ldots, v_{n-d}\}$ and $\hat{e} \setminus e'' = \{v_{d+2}, \ldots, v_n\}$; further $n \leq 2d$ yields $e'' \cap \hat{e} = \{v_{n-d+1}, \ldots, v_{d+1}\} \neq \emptyset$.

Now assume $r^*(e'') = r^*(\hat{e})$ and consider $e_1 = \{v_1 = v_{\max}, v_2, \ldots, v_d\}$. Obviously $y_1 := r^*(e_1) \in Y$. We distinguish two cases:

Case 1. If $d+2 \leq n \leq 2d-1$ we obtain $n-d+1 \leq d$ and

$$\begin{aligned} y_1 &= v_1 + (v_2 + \ldots + v_{n-d}) + v_{n-d+1} + \ldots + v_d \\ &= v_1 + (v_{d+2} + \ldots + v_n) + v_{n-d+1} + \ldots + v_d \in Y. \end{aligned}$$

This is a contradiction because $|\{v_{n-d+1}, \ldots, v_d, v_{d+2}, \ldots, v_n, v_1\}| = d$ but $\{v_{n-d+1}, \ldots, v_d, v_{d+2}, \ldots, v_n, v_1\} \notin \mathcal{E}$.

Case 2. If $n = 2d$ then $v_1 = v_{\max} = r^*(\hat{e}) = r^*(\{v_{d+1}, \ldots, v_n\})$ yields

$$y_1 = v_1 + \sum_{k=2}^{d} v_k = \sum_{k=d+1}^{n} v_k + \sum_{k=2}^{d} v_k = \sum_{k=2}^{n} v_k \in Y. \tag{27}$$

Observe that $d = 2$ is not possible because $r^*(e'') = r^*(\hat{e})$ would imply $v_2 = v_4$ in that case; hence $d \geq 3$. We consider the edge $e_3 = \{v_3, \ldots, v_{d+2}\} \neq \hat{e}$. By $|e'' \cap e_3| = d - 1$ follows $r^*(e_3) \neq r^*(e'') = v_{\max}$; using (25) this yields $y_3 := r^*(e_3) \in Y$. Furthermore $n \geq d + 2$ implies $v_1 \notin e_3$ and with (27) we obtain

$$y_1 = \sum_{k=2}^{n} v_k = v_2 + (v_3 + \ldots + v_{d+2}) + v_{d+3} + \ldots + v_n = v_2 + y_3 + v_{d+3} + \ldots + v_n \in Y.$$

This is a contradiction because $|\{y_3, v_2, v_{d+3}, \ldots, v_n\}| = d$ but $\{y_3, v_2, v_{d+3}, \ldots, v_n\} \notin \mathcal{E}$.

Summarizing the results of both cases we have shown (26) and the proof is completed. ■

Looking at Theorem 6 one may conjecture that $\sigma(\hat{\mathcal{C}}_n^d) > d$ for $d+2 \leq n \leq 2d$; formula (17) shows that this is true for the graph C_4. On the other hand we observe that Lemma 1 ist not true for $d+2 \leq n \leq 2d$ and this fact could be a reason for decreasing sum numbers. Indeed, for $d = 3$ we obtain the following result.

Theorem 7. *For* 3*-uniform strong hypercycles with* 5 *or* 6 *vertices the sum numbers are given by*

$$\sigma(\hat{\mathcal{C}}_5^3) = 3 \ ; \ \sigma(\hat{\mathcal{C}}_6^3) = 2.$$

Proof. Obviously, if for two different edges $e_i, e_j \in \mathcal{E}$ holds $r^*(e_i) = r^*(e_j)$ then $|e_i \cap e_j| \leq d - 2$. Therefore, if p denotes the maximum number of pairwise distinct edges $e'_1, \ldots, e'_p$ with $r^*(e'_1) = \ldots = r^*(e'_p)$, we have $p \leq \frac{n}{2}$. By $|\mathcal{E}| = n$ follows $\sigma(\hat{\mathcal{C}}_5^3) \geq 3$ and $\sigma(\hat{\mathcal{C}}_6^3) \geq 2$ and we obtain equality by using the sum labellings given below:

$$(v_1, \ldots, v_n; y_1, \ldots, y_\sigma) = \begin{cases} (1, 10, 6, 5, 11; 17, 21, 22), & \text{if} \quad n = 5, \\ (1, 10, 95, 6, 5, 100; 106, 111), & \text{if} \quad n = 6. \end{cases}$$ ■

We do not know the exact values of $\sigma(\hat{\mathcal{C}}_n^d)$ for $d \geq 4$ and $d + 2 \leq n \leq 2d$. For instance the labelling $(v_1, \ldots, v_6; y_1, \ldots, y_3) = (1, 100, 4, 6, 95, 9; 111, 114, 205)$ yields $\sigma(\hat{\mathcal{C}}_6^4) \leq 3$.

Conjecture. $\sigma(\hat{\mathcal{C}}_n^d) < d$ for $d \geq 4$ and $d + 2 \leq n \leq 2d$.

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