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ON THE RANK OF RANDOM SUBSETS OF FINITE AFFINE GEOMETRY

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Abstract

The aim of the paper is to give an effective formula for the calculation of the probability that a random subset of an affine geometry $AG(r-1,q)$ has rank r. Tables for the probabilities are given for small ranks. The expected time to the first moment at which a random subset of an affine geometry achieves the rank r is derived.

Keywords: finite affine geometry, random matroids, hitting time.

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1 Introduction

Random subsets of finite projective geometries are most interested and extensively investigated objects from the wide class of random matroids. Already several papers have been published in this area. On the other hand, random subsets of finite affine geometries — very similar to projective geometries — have only a few references. Voigt [7] considers mainly the case in which the rank of the geometry tends to infinity. In this paper we are concerned with the finite case, i.e., when the rank is fixed.

The main aim of this paper is to provide a formula, which enables a simple calculation of the probability that a random subset of an affine finite geometry $AG(r-1,q)$ of rank r has the same rank. To derive this formula we use a more general result published in [3] and [4]. To make the paper self-contained in Section 2 we give some indispensible tools from those papers. Next, in Section 3 we derive the result stated in Theorem 1. This result is the basis of calculating these probabilities, using routines written in Turbo Pascal and *Mathematica*. The probabilities are given in tables 1 – 8. In Section 4 we consider a hitting time, a moment at which the random subset has the same rank as the geometry.

2 Preliminaries

One can find the following basic definitions from the theory of matroids in the books of Oxley [6] and Welsh [8].

Let $M = (E, \mathcal{F})$ be a matroid of flats on the ground set E. The *rank* of A is denoted by $\rho(A)$ and the *span* of A is denoted by $\sigma(A)$. Suppose A, B and C are flats. Denote $A \lessdot B$, if $A \subseteq C \subseteq B$ implies that either $C = A$ or $C = B$. Assume that $\sigma(\emptyset) = \emptyset$.

Let Λ be a measure on E, $\lambda_e = \Lambda(E)$ and

$$\Lambda(A) = \sum_{e \in E} \lambda_e \,.$$

We define the *continuous random-M-process* $\{\omega(M, \Lambda, t), t \in [0, \infty)\}$ as the process which starts from the empty set at $t = 0$ such element e arises before t with probability $1 - \mathrm{e}^{-\lambda_e t}$ independently of each other elements. The set of elements arising before the moment t, forms the random set R. If t is fixed then $\omega(M, \Lambda, t)$ is a random matroid $\omega(M)$ with $p_e = 1 - \mathrm{e}^{-\lambda_e t}$ for all $e \in E$, (see Kordecki [3]).

Let us consider a random process $\overline{\omega}(M, \Lambda, t)$ with values in the family of flats $\mathcal{F}$. The process $\overline{\omega}(M, \Lambda, t)$ will be in state $A \in \mathcal{F}$ if $\sigma(R) = A$. Assume that our process starts at the moment $t = 0$ from the state $\emptyset$. Therefore, $\overline{\omega}(M, \Lambda, t)$ is a Markov process. The transitional rates are given by

$$\mu_{AB} = \begin{cases} \Lambda(B \setminus A), & \text{if } B \gtrdot A, \\ 0, & \text{otherwise,} \end{cases}$$

where $A, B \in \mathcal{F}$ and $A \neq B$. Let $\mu_A = \Lambda\,(E \setminus A)$.

Let $P_A(t)$ denote the probability that $\overline{\omega}(M, \Lambda, t)$ is in state A at time t. Then Kolmogorov's equations are of the form

$$\frac{d}{dt} P_B(t) = -P_B(t) \sum_{C \gtrdot B} \mu_{BC} + \sum_{A \lessdot B} \mu_{AB} P_A(t) \,, \tag{1}$$

if $B \neq \emptyset$ and $B \neq E$, and

$$\frac{d}{dt} P_E(t) = \sum_{A \lessdot E} \mu_A P_A(t), \tag{2}$$

$$\frac{d}{dt} P_\emptyset(t) = -P_\emptyset(t)\Lambda(E). \tag{3}$$

From the last equation and the initial condition we have

$$P_\emptyset(t) = \mathrm{e}^{-\Lambda(E)t}.$$

In [5] the following lemma was proved, (see [3] for simplified proof).

Lemma 1. *Let $F_i(t)$ are exponential distribution functions, $F_i(t) = 1 - \mathrm{e}^{-\lambda_i t}$ for $i = 1, 2, \ldots, n$, $t > 0$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Then*

$$F(t) = \sum_{k=1}^{n} \left(1 - \mathrm{e}^{-\lambda_k t}\right) \prod_{j \neq k} \frac{\lambda_j}{\lambda_j - \lambda_k}. \tag{4}$$

3 Rank of Random Subsets

Let $GF(q)$ be a Galois field, where q is a prime power and let $V(r, q)$ be an r-dimensional vector space on $GF(q)$. Let $\mathcal{L}$ be the lattice of subspaces of V. Atoms of $\mathcal{L}$ are *points* and 2-dimensional subspaces, (2-flats) are *lines* in a *projective geometry* $PG(r-1, q)$ of dimension $r-1$. Projective geometries can be defined in an axiomatic way (see [8], p. 193), but every (finite) projective geometry of dimension $n > 2$ is isomorphic to the geometry defined above. A *hyperplane* of $PG(r-1, q)$ is subspace of rank $r-1$. The affine geometry $AG(r-1, q)$ is obtained from $PG(r-1, q)$ by deleting from the latter all the points contained in fixed hyperplane. One can find concise, but detailed information about affine geometries in the chapter *Classical Geometries* in [1], written by Beutelspacher, p. 694.

Let $q \neq 1$, k be natural numbers and $n \geq 0$. Define some q-analogs in the following way, where q is assumed to be fixed: *q-numbers*

$$[n] = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1}$$

and *q-factorials*

$$[k]! = \prod_{j=1}^{k} [x - j + 1], \quad [0]! = 1.$$

Gaussian coefficients are defined as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \prod_{j=1}^{k} \frac{q^{n-j+1}-1}{q^j-1} \quad \text{for} \quad 0 \le k \le n\,.$$

It is well-known that $PG(r-1,q)$ has $[r]$ elements and has $\left[\begin{smallmatrix} r \\ k \end{smallmatrix}\right]$ rank-k subspaces. Similarly, $AG(r-1,q)$ has q^{r-1} elements and $q^{r-k}\left[\begin{smallmatrix} r-1 \\ k-1 \end{smallmatrix}\right]$ rank-k subspaces.

Let $P^{(r)}$ denote the probability that the random subset of $AG(r-1,q)$ has rank r. First we derive the recurrence formula $P^{(r)}$. Assuming $P^{(0)}=1$ and using the easily obtained formula

$$(1-p)^{q^{r-1}} + \sum_{k=1}^{r} q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} (1-p)^{q^{r-1}-q^{k-1}}\, P^{(k)} = 1\,,$$

we have

$$P^{(r)} = 1 - (1-p)^{q^{r-1}} - \sum_{k=1}^{r-1} q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} (1-p)^{q^{r-1}-q^{k-1}}\, P^{(k)}\,. \tag{5}$$

Recall that subspaces of $PG(r-1,q)$ and $AG(r-1,q)$ form a modular geometric lattice. Hence if $A, B \in \mathcal{J}$ then $\rho(A \cup B) = \rho(A) + \rho(B) - \rho(A \cap B)$, (see Welsh [8], p. 195). Let M_r be a matroid of flats $AG(r-1,q)$.

The process $\omega(M,\Lambda,t)$ is now denoted by $\omega(t)$ and $\overline{\omega}(M,\lambda,t)$ by $\overline{\omega}(t)$.

Theorem 1. *Let $P(r,t)$ be the probability that the rank of the random subset of M_r is equal to r, where $p = 1 - \mathrm{e}^{-t}$. Then*

$$\begin{aligned} P(r,t) &= (-1)^{r-1}(q-1)^{r-1}[r-1]!\left(1-\mathrm{e}^{-tq^{r-1}}\right) \\ &\quad + \sum_{j=1}^{r-1} (-1)^{r-j-1} q^{\binom{r-j+1}{2}} \begin{bmatrix} r-1 \\ j-1 \end{bmatrix} \left(1-\mathrm{e}^{-t\left(q^{r-1}-q^{j-1}\right)}\right). \end{aligned} \tag{6}$$

Proof. Let us consider the process $\overline{\omega}_r(t)$ defined as $\overline{\omega}(M_r,t)$. Denote

$$l\,(r,k) = \prod_{i \ne k} \frac{[r]-[i]}{[k]-[i]}. \tag{7}$$

Using Lemma 1, and denoting $m_i = \mu(A)$, where $\rho(A) = i$, we have

$$m_i = \begin{cases} q^{r-1} - q^{i-1} & \text{for } i \ge 1, \\ q^{r-1} & \text{for } i = 0. \end{cases}$$

Hence

$$m_i - m_j = \begin{cases} -q^{i-1} & \text{for } i \neq 0,\ j = 0, \\ q^{j-1} & \text{for } i = 0,\ j \neq 0, \\ q^{j-1} - q^{i-1} & \text{for } i \neq 0,\ j \neq 0. \end{cases}$$

For $l(r,j)$ defined by (7), we have two separate cases.

$$l(r,0) = \prod_{i=1}^{r-1} \frac{q^{i-1}\left(q^{r-i}-1\right)}{-q^{i-1}} = (-1)^{r-1}(q-1)^{r-1}[r-1]!$$

and for $j \geq 1$

$$\begin{aligned} l(r,j) &= q^{r-j} \prod_{\substack{1\leq i\leq r-1 \\ i\neq j}} \frac{q^{i-1}\left(q^{r-i}-1\right)}{q^{i-1}\left(q^{j-i}-1\right)} \\ &= q^{r-j} \frac{\prod_{i=1}^{j-1}\left(q^{r-i}-1\right)\prod_{i=j+1}^{r-1}\left(q^{r-i}-1\right)}{\prod_{i=1}^{j-1}\left(q^{j-i}-1\right)\prod_{i=j+1}^{r-1}\left(q^{j-i}-1\right)} \\ &= (-1)^{r-j-1} q^{r-j} \prod_{i=1}^{r-j-1} q^i \frac{\prod_{i=1}^{j-1}\left(q^{r-i}-1\right)\prod_{i=1}^{r-j-1}\left(q^{r-j-i}-1\right)}{\prod_{i=1}^{j-1}\left(q^{j-i}-1\right)\prod_{i=1}^{r-j-1}\left(q^{i}-1\right)} \\ &= (-1)^{r-j-1} q^{\binom{r-j}{2}} q^{r-j} \frac{\prod_{i=1}^{j-1}\left(q^{r-i}-1\right)\prod_{i=1}^{r-j-1}\left(q^{r-j-i}-1\right)}{\prod_{i=1}^{j-1}\left(q^{i}-1\right)\prod_{i=1}^{r-j-1}\left(q^{i}-1\right)} \\ &= (-1)^{r-j-1} q^{\binom{r-j+1}{2}} \begin{bmatrix} r-1 \\ j-1 \end{bmatrix}. \end{aligned}$$

Therefore we have obtained the assertion for $AG(r-1, q-1, q)$. ■

If $r = 3$ (the affine plane), we obtain for $q = 2$

$$P_E(t) = 1 - 6\mathrm{e}^{-2t} + 8\mathrm{e}^{-3t} - 3\mathrm{e}^{-4t}$$

and for $q = 3$

$$P_E(t) = 1 - 12\mathrm{e}^{-6t} + 27\mathrm{e}^{-8t} - 15\mathrm{e}^{9t}\,.$$

Tables 1 to 5 give values of $P_E(t)$ for $q = 2$, $q = 3$, $q = 4$ and $q = 5$, $r = 2, 3, \ldots, 6$ and $t = 0.1, \ldots, t = 0.5$.

Table 1. Values of $P_E(t)$ for $AG(r-1,2)$

r	$t=0.1$	0.2	0.3	0.4	0.5
2	0.009056	0.032859	0.067175	0.108689	0.154818
3	0.003201	0.020586	0.056105	0.107890	0.171759
4	0.003420	0.033892	0.107884	0.217816	0.345378
5	0.009306	0.108335	0.316592	0.546366	0.729862
6	0.048213	0.405054	0.767941	0.932362	0.983301

Table 2. Values of $P_E(t)$ for $AG(r-1,3)$

r	$t=0.1$	0.2	0.3	0.4	0.5
2	0.025444	0.086663	0.166704	0.254402	0.342622
3	0.041028	0.192093	0.390510	0.574785	0.719333
4	0.207400	0.680338	0.913480	0.980957	0.996266
5	0.837605	0.998192	0.999990	1.000000	
6	0.999971	1.000000			

Table 3. Values of $P_E(t)$ for $AG(r-1,4)$

r	$t=0.1$	0.2	0.3	0.4	0.5
2	0.047687	0.152740	0.277304	0.400913	0.513485
3	0.171103	0.537714	0.794163	0.919275	0.970727
4	0.829155	0.996584	0.999961	1.000000	
5	0.999999	1.000000			

Table 4. Values of $P_E(t)$ for $AG(r-1,5)$

r	$t=0.1$	0.2	0.3	0.4	0.5
2	0.074522	0.224873	0.386550	0.531859	0.651664
3	0.399526	0.832406	0.965865	0.994044	0.999048
4	0.996781	1.000000			

Table 5. Values of $P_E(t)$ for $AG(r-1,7)$

r	$t=0.1$	0.2	0.3	0.4	0.5
2	0.137830	0.371222	0.577646	0.729835	0.832675
3	0.838435	0.994668	0.999883	0.999998	1.000000

Tables 6 to 8 give values of $P_E(t)$ for $q=7$, $q=8$ and $q=9$, $r=2,3,\ldots,4$ and $t=0.01,\ldots,t=0.05$

Table 6. Values of $P_E(t)$ for $AG(r-1,8)$

r	$t=0.01$	0.02	0.03	0.04	0.05
2	0.002664	0.010141	0.021721	0.036773	0.054736
3	0.023991	0.124267	0.276170	0.438916	0.585832
4	0.723917	0.988288	0.999734	0.999995	1.000000

Table 7. Values of $P_E(t)$ for $AG(r-1,9)$

r	$t=0.01$	0.02	0.03	0.04	0.05
2	0.003402	0.012868	0.027385	0.046069	0.068145
3	0.043949	0.203896	0.410554	0.598625	0.742646
4	0.920386	0.999547	0.999999	1.000000	

Table 8. Values of $P_E(t)$ for $AG(r-1,11)$

r	$t=0.01$	0.02	0.03	0.04	0.05
2	0.005130	0.019150	0.040237	0.066844	0.097661
3	0.113778	0.413638	0.679875	0.844720	0.930331
4	0.998898	1.000000			

4 Hitting Time

Let τ denotes the hitting time, i.e., the first moment at which $\rho(\omega(t)) = r$. Therefore

$$\tau = \min\{t : \rho(\omega(t) = r\}$$

and

$$\Pr(\tau < t) = P(r,t)\,.$$

The considerations below are based on similar ones, given in Kordecki [2] for the case of $PG(r-1,q)$, (see also [3]).

The Laplace transform of τ is of the form

$$\overline{P}(r,s) = \int_0^\infty \mathrm{e}^{-st} dP(r,t) = \mathrm{E}\mathrm{e}^{-s\tau}\,.$$

Formula (2) is of the form

$$\frac{dP(r,t)}{dt} = q^{r-1}[r-1]q^{r-2}(q-1)P(r-1,t)\mathrm{e}^{-t(q-1)q^{r-2}}\,, \tag{8}$$

where $q^{r-1}[r-1]$ is the number of hyperplanes, $q^{r-2}(q-1)$ is the number of elements outside some fixed hyperplane and $P(r-1,t)\mathrm{e}^{-t(q-1)q^{r-2}}$ is the probability that $\rho(H)=r-1$ for some fixed hyperplane.

Lemma 2. *For $r \geq 1$ we have*

$$\overline{P}(r,t) = q^{\binom{r}{2}}(q-1)^{r-1}[r-1]! \prod_{k=1} r(s+q^{r-k}[k-1])^{-1} . \tag{9}$$

Proof. Taking the Laplace transforms of both sides of equation 8 we obtain

$$s\overline{P}(r,s) = q^{r-1}q^{r-2}(q^{r-1}-1)\overline{P}\left(s+(q-1)q^{r-2}\right).$$

If $r=1$, then $\overline{P}(r,s) = 1/\left(1+s\right)$ and (9) is satisfied. If $r \geq 2$, we assume that (9) is true for some r and prove (9) for $r+1$, which is an easy task and we obtain the assertion. ∎

Now we give the formulae for $\mathrm{E}\tau$ and $\mathrm{D}^2\tau$.

Theorem 2.

$$\mathrm{E}\tau = q^{-r}(q-1)^{r-1}\sum_{k=1}^{r}\frac{q^k}{[k-1]}, \tag{10}$$

$$\mathrm{D}^2\tau = q^{-2r}(q-1)^{r-1}\sum_{k=1}^{r}\frac{q^2k}{[k-1]^2}. \tag{11}$$

Proof. Using Lemma 2 we calculate the two first derivatives of $\overline{P}(r,s)$. First we calculate

$$\frac{d}{ds}\prod_{k=1} r\left(s+q^{r-k}[k-1]\right) = \sum_{k=1}^{r}\prod_{k\neq j}\left(s+q^{r-j}[j-1]\right).$$

Hence

$$\frac{d\overline{P}(r,s)}{ds} = -q^{\binom{r}{2}}(q-1)^{r-1}[r-1]!\frac{\sum_{k=1}^{r}\left(s+q^{r-k}[k-1]\right)^{-1}}{\prod_{k=1}^{r}\left(s+q^{r-k}[k-1]\right)}. \tag{12}$$

Substituting $s=0$ into (12) we obtain (10). In a similar manner we can calculate the second derivative of $\overline{P}(r,s)$ and (11). Then we obtain the assertion. ∎

The following theorem states the asymptotic behaviour of $\mathrm{E}\tau(r)$ and $\mathrm{D}^2\tau(r)$ where p and q are fixed but $r\to\infty$.

Theorem 3.

$$\mathrm{E}\tau = r\left(\frac{q-1}{q}\right)^{r}(1+o(1))\,, \tag{13}$$

$$\mathrm{D}^2\tau = r\left(\frac{q-1}{q}\right)^{2r}(1+o(1))\,. \tag{14}$$

Proof. Since

$$\sum_{k=1}^{r}\frac{q^k}{q^k-1} = \sum_{k=1}^{r}\left(1+\frac{1}{q^k-1}\right) = r(1+o(1)),$$

then from (10) we obtain Formula (13). In the same manner, from (11) we obtain (14). ■

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