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DICHROMATIC NUMBER, CIRCULANT TOURNAMENTS AND ZYKOV SUMS OF DIGRAPHS

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Abstract

The *dichromatic number* $dc(D)$ of a digraph D is the smallest number of colours needed to colour the vertices of D so that no monochromatic directed cycle is created. In this paper the problem of computing the dichromatic number of a Zykov-sum of digraphs over a digraph D is reduced to that of computing a multicovering number of an hypergraph $H_1(D)$ associated to D in a natural way. This result allows us to construct an infinite family of pairwise non isomorphic vertex-critical k-dichromatic circulant tournaments for every $k \geq 3$, $k \neq 7$.

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1 Introduction

The *dichromatic number* $dc(D)$ of a digraph D is the least number of colours needed to colour the vertices of D in such a way that each chromatic class is acyclic ([3, 9, 10]). It is apparent that this invariant measures in some way the complexity of the cyclic structure of digraphs. The importance of studying this invariant, introduced in [10], comes from the following fact: If G is a graph and G^* denotes the digraph obtained from G by orienting each one of the edges in both directions, then $\chi(G) = dc(G^*)$; so the dichromatic number is a natural extension of the chromatic number to the class of all digraphs.

The structure of arc-critical k-dichromatic digraphs was investigated in [10] and consequently new remarkable properties of k-chromatic graphs were obtained there.

We continue here the study of vertex-critical k-dichromatic tournaments initiated in [15]. Related topics have been considered in [4, 5, 11].

Let H be an hypergraph without isolated vertices and suppose a positive integer ξ_u has been assigned to each vertex u of H; the *covering number* of H corresponding to that assignment of weights is defined to be the minimum cardinality of a family of not necessarily different edges of H such that each vertex u belongs to at least ξ_u edges of the family.

Let D be a digraph and let $H_1(D)$ be the hypergraph whose vertex set is $V(D)$ and has the maximal acyclic subsets of $V(D)$ as hyperedges. In this paper the problem of computing the dichromatic number of a Zykov-sum of digraphs over a digraph D is reduced to that of computing the covering number of $H_1(D)$ with respect to an adequate assignment of weights (Theorem 4.2). We apply this result to construct an infinite family of pairwise non isomorphic vertex-critical k-dichromatic circulant tournaments for every $k \geq 3$, $k \neq 7$. This improves previous results included in [15]. Other related results are also presented.

2 Preliminary Results and Terminology

For general concepts we refer the reader to [2].

Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively, $o(D) = |V(D)|$ is the *order* of D; D is *acyclic* provided no directed cycle is contained in D. The subdigraph of D induced by a subset S of $V(D)$ will be denoted by $D[S]$; S is said to be *acyclic* iff $D[S]$ is acyclic. The maximal cardinality of an acyclic set of vertices of D will be denoted by $\beta^{\rightarrow}(D)$. A colouring of $V(D)$ is *acyclic* if all the chromatic classes are acyclic. So the dichromatic number $dc(D)$ of a digraph D is the minimum number of colours in an acyclic colouring of $V(D)$. Clearly $dc(D^{\mathrm{op}}) = dc(D)$ where D^{op} is obtained from D by reversing each one of its arcs.

D is called *r-dichromatic* if $dc(D) = r$ and *vertex-critical r-dichromatic* if $dc(D) = r$ and $dc(D - u) < r$ for every $u \in V(D)$.

$\mathbb{N}$ will denote the set of nonnegative integers, $I_n = \{1, \ldots, n\}$ and Z_n is the set of integers mod n. For any nonempty subset J of $Z_n - \{0\}$, the *circulant digraph* $\vec{C}_n(J)$ is defined by $V(\vec{C}_n(J)) = Z_n$ and $A(\vec{C}_n(J)) = \{(i,j) \colon i, j \in Z_n \text{ and } j - i \in J\}$. In particular, $\vec{C}_n(\{1\})$ is the directed cycle

$\vec{C}_n$; $\vec{C}_{2m+1}(J)$ is a circulant tournament whenever $|\{j, -j\} \cap J| = 1$ for every $j \in Z_{2m+1} - \{0\}$. If $i, j \in V(\vec{C}_n)$, $A_{i,j}$ will denote the directed ij-path in $\vec{C}_n$. For $j \in I_m, I_{m,j}$ will denote the set $I_m \cup \{2m+1-j\} - \{j\}$ considered as a subset of Z_{2m+1}.

In [12] it was proved that there is only one 4-dichromatic oriented graph of order at most 11, namely $\vec{C}_{11}(I_{5,2})$; this tournament is not only vertex-critical but also arc-critical. In [13] it was proved that $\vec{C}_{6m+1}(I_{3m,2m})$ is a vertex-critical 4-dichromatic circulant tournament for $m \geq 2$. In a previous paper [15] an infinite family of vertex-critical r-dichromatic regular tournaments was constructed for each $r \geq 3$, $r \neq 4$. However these tournaments were circulants only for $r = 3, 5, 8$.

We will need the following

Lemma 2.1 [13]. *For any two integers r, s such that $1 \leq s < r$ holds $\beta^{\rightarrow}(H_{r,s}) = r$ where $H_{r,s}$ is the tournament defined by $V(H_{r,s}) = \{1, 2, \ldots, r+s\}$ and $A(H_{r,s}) = \{(i,j) : (i < j \text{ and } j - i \neq r)\} \cup \{(i+r, i) : i \leq s\}$.*

3 Multicoverings of Hypergraphs

If $H = (V(H), E(H))$ is an hypergraph, the *rank* $\rho(H)$ of H is defined to be the maximum cardinality of an edge of H; H is an *r-graph* if each one of its edges has cardinality r.

Let H be a finite hypergraph without isolated points. A function ξ: $V(H) \rightarrow \mathbb{N}$ will be called a *weight function* (w.f.) *on* H; ξ will be said to be *degenerate* if $\xi^{-1}(0) \neq \emptyset$. We define $\|\xi\| = \sum_{w \in V(H)} \xi(w)$ and denote by $\mathbf{k}$ the w.f. on H, which has constant value k. Let $(\alpha_j)_{j \in J}$ be a family of edges of H and $u \in V(H)$; define $J_u = \{j \in J : u \in \alpha_j\}$. We will say that $(\alpha_j)_{j \in J}$ is a *ξ-covering* of H whenever $|J_u| \geq \xi(u)$ for every $u \in V(H)$. Finally, we define the *ξ-covering number* $\tilde{n}(H, \xi)$ of H by $\tilde{n}(H, \xi) = \min\{|J| : (\alpha_j)_{j \in J}$ is a ξ-covering of $H\}$. So the $\mathbf{k}$-covering number of H is the usual (multi)covering number which has been studied in many articles (see [1]).

Remark 31. Note that if H' is the spanning subhypergraph of H whose edges are the maximal edges of H, then $\tilde{n}(H, \xi) = \tilde{n}(H', \xi)$.

Proposition 32.

(i) $\tilde{n}(H, \xi + \xi') \leq \tilde{n}(H, \xi) + \tilde{n}(H, \xi')$ *and* $\tilde{n}(H, k\xi) \leq k\tilde{n}(H, \xi)$ *for every positive integer k.*

(ii) $\tilde{n}(H,\xi) \le \tilde{n}(H,\xi')$ *whenever* $\xi \le \xi'$.

(iii) $\tilde{n}(H,\xi) \ge \lceil \|\xi\|/\rho(H) \rceil$.

(iv) *If* H_0 *is a spanning subhypergraph of* H *then* $\tilde{n}(H,\xi) \le \tilde{n}(H_0,\xi)$.

Proof. Properties (i), (ii) and (iv) are obvious, Property (iii) follows from the inequality $\rho(H)\tilde{n}(H,\xi) \ge \|\xi\|$. ■

An hypergraph H is called *circulant* if it has an automorphism which is a cyclic permutation of $V(H)$. If $r \le m$, the circulant r-graph $\Lambda_{m,r}$ is defined by $V(\Lambda_{m,r}) = \mathbb{Z}_m$ and $E(\Lambda_{m,r}) = \{\alpha_j : j \in \mathbb{Z}_m\}$ where $\alpha_j = \{j, j+1, \dots, j+r-1\}$ for $j \in \mathbb{Z}_m$. For every positive integer s, we define the w.f. $\xi^{(s)}$ on $\Lambda_{m,r}$ as follows: If $sr = qm + t$ where t is the residue of sr mod m, then $\xi^{(s)}(j) = q$ or $q+1$ depending on whether j belongs or not to $A_{t,m-1}$. In particular, $\xi^{(s)} = \mathbf{q}$ when $t = 0$. Notice that $\|\xi^{(s)}\| = sr$.

Proposition 33. *If* H *contains* $\Lambda_{m,r}$ *as a spanning subhypergraph and* $\rho(H) = r$ *then* $\tilde{n}(H,\xi^{(s)}) = s$ *and* $\tilde{n}(H,\xi') > s$ *whenever* $\|\xi'\| > \|\xi^{(s)}\|$.

Proof. The family $\{\alpha_j : j = rj', j' = 0, 1, \dots, s\}$ is a $\xi^{(s)}$-covering of H and so $\tilde{n}(H,\xi^{(s)}) \le s$. The equality and the second inequality follow from Proposition 3.2 (iii) and the fact that $\|\xi'\| > \|\xi^{(s)}\| = sr$. ■

Proposition 34. *Let* k *be a positive integer. If* $\rho(H) = r$ *and* H *contains an isomorphic copy of* $\Lambda_{m,r}$ *as a spanning subhypergraph, then* $\tilde{n}(H,\mathbf{k}) = \tilde{n}(\Lambda_{m,r},\mathbf{k}) = \lceil km/r \rceil$.

Proof. We may assume that $\Lambda_{m,r}$ is a spanning subhypergraph of H. The inequality $\tilde{n}(H,\mathbf{k}) \ge \lceil km/r \rceil$ follows from Proposition 3.2 (iii). Since $\xi^{(s)} \ge \mathbf{k}$ for $s = \lceil km/r \rceil$, the equality is obtained by applying Propositions 3.2 and 3.3. ■

Proposition 3.4 applies in particular to $K_m^{(r)}$, the complete r-graph of order m.

4 Zykov Sums and Dichromatic Number

Let D be a digraph and $\alpha = (\alpha_i)_{i \in V(D)}$ a family of nonempty mutually disjoint digraphs. The Zykov sum $\sigma(\alpha, D)$ of α over D is defined by $V(\sigma(\alpha,D)) = \bigcup_{i \in V(D)} V(\alpha_i)$; $A(\sigma(\alpha,D)) = \bigcup_{i \in V(D)} A(\alpha_i) \cup \{uw : u \in V(\alpha_i), w \in V(\alpha_j), ij \in A(D)\}$.

If the members of the family α are not mutually disjoint we replace each of them by one isomorphic copy so that the new family α' becomes one of mutually disjoint digraphs; nevertheless $\sigma(\alpha, D)$ will still denote the resulting digraph $\sigma(\alpha', D)$ which is defined up to isomorphism. The function $p: \sigma(\alpha, D) \to D$ whose value is constant in each α_u and equal to u, is a reflexive epimorphism which will be called the *natural projection* from $\sigma(\alpha, D)$ onto D. If $\alpha_i \cong W$ for every $i \in V(D)$ we will write $D[W]$ instead of $\sigma(\alpha, D)$.

In [10] it was proved that $dc(D[W]) \geq dc(D) + dc(W) - 1$. In [15], $t(\alpha_1, \alpha_2, \alpha_3)$ denoted the same as $\sigma(\alpha, D)$ for $D = \vec{C}_3$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2)$.

Now, if D is a digraph, the hypergraph $H_1(D)$ is defined by $V(H_1(D)) = V(D)$, $E(H_1(D)) = \{S \subseteq V(D): S \text{ is a maximal acyclic set}\}$.

Proposition 41.

(i) $H_1(\vec{C}_{2m+1}(I_m)) \supseteq \Lambda_{2m+1,m+1}$, $\beta^{\rightarrow}(\vec{C}_{2m+1}(I_m)) = m + 1$.

(ii) $H_1(\vec{C}_{2m+1}(I_{m,m})) \supseteq \Lambda_{2m+1,m}$, $\beta^{\rightarrow}(\vec{C}_{2m+1}(I_{m,m})) = m$.

(iii) $H_1(\vec{C}_{6m+1}(I_{3m,2m})) \supseteq \Lambda_{6m+1,2m}$, $\beta^{\rightarrow}(\vec{C}_{6m+1}(I_{3m,2m})) = 2m$.

(iv) $H_1(\vec{C}_{17}(I_{8,5})) \supseteq \Lambda_{17,5}$, $\beta^{\rightarrow}(\vec{C}_{17}(I_{8,5})) = 5$.

(v) $H_1(\vec{C}_{17}(I_{8,7})) \supseteq \Lambda_{17,7}$, $\beta^{\rightarrow}(\vec{C}_{17}(I_{8,7})) = 7$ *and*

(vi) $H_1(\vec{C}_{17}(I_{8,6})) \supseteq \Lambda_{17,6}$, $\beta^{\rightarrow}(\vec{C}_{17}(I_{8,6})) = 6$.

Proof. (i) is trivial, (ii) and (iii) were proved in [15] and [13] respectively. The inclusions of (iv), (v) and (vi) are obvious. Let $T^j = \vec{C}_{17}(I_{8,j})$, $j = 5, 6, 7$ and notice that $A_{i(i+j-1)}$ is an acyclic set of cardinality j. Let S_j be an acyclic set of T^j. We will prove that $|S_j| \leq j$. We may assume that 0 is the source of $T^j[S_j]$. Let N_j be the out neighbourhood of 0 in T^j. So $S_j - \{0\} \subseteq N_j$. Notice that $T^j[N_j - \{17-j\}] \cong H_{j-1,8-j}$ (the correspondence $i \to i$ for $0 < i \leq j - 1$ and $i \to i + 1$ for $j \leq i \leq 7$ is an isomorphism from $H_{j-1,8-j}$ onto $T^j[N_j - \{17 - j\}]$) and $j - 1 > 8 - j$. So by Lemma 2.1, $|S_j| \leq j$ whenever $17 - j \notin S_j$. We assume that $17 - j \in S_j$.

Case $j = 5$. We have $12 \in S_5$. If $4 \in S_5$ then $S_5 \cap \{1, 2, 3\} = \emptyset$ and since $|S_5 \cap \{7, 8\}| \leq 1$ we obtain $|S_5| \leq 5$. If $4 \notin S_5$ and $8 \in S_5$ then $S_5 \cap \{1, 2, 7\} = \emptyset$ and so $|S_5| \leq 5$. Finally if $S_5 \cap \{4, 8\} = \emptyset$, then since $T^5[N_5] - \{0, 4, 8, 12\} \cong H_{3,2}$, we conclude by Lemma 2.1 that $|S_5| \leq 5$. So the proof of (iv) is complete.

Case $j = 7$. We have $10 \in S_7$. If $\{1, 3\} \cap S_7 \neq \emptyset$ then $\{4, 5, 6\} \cap S_7 = \emptyset$ and again by Lemma 2.1, $|S_7| \leq 5$. In the remaining case $|S_7| \leq 7$. So (v) holds.

Case $j = 6$. Since $11 \in S_6$, either $\{1,2\} \cap S_6 = \emptyset$ or $\{3,4\} \cap S_6 = \emptyset$. Moreover $|S_6 \cap \{5,7\}| \leq 1$, therefore $|S_6| \leq 6$ and the proof of (vi) ends. ■

Let D be a digraph and $Q = (Q_u)_{u \in V(D)}$ a family of digraphs. Define the w.f. $\xi_Q: V(D) \to \mathbb{N}$ by $\xi_Q(u) = dc(Q_u)$.

Theorem 42. *$dc(\sigma(D,Q)) = \tilde{n}(H_1(D), \xi_Q)$.*

Proof. We may assume that Q is formed with mutually disjoint digraphs and so $Q_u \subseteq \sigma(D,Q)$. Let $p: \sigma(D,Q) \to D$ be the natural projection, so $p(Q_u) = u$ for every $u \in V(D)$. Let $(\alpha_j)_{j \in J}$ be an optimal ξ_Q-covering of $H_1(D)$; then $|J| = \tilde{n}(H_1(D), \xi_Q)$. Define a colouring f of $\sigma(D,Q)$ with J as set of colours, as follows: For each $u \in V(D)$, take an acyclic colouring of Q_u with colours in J_u (this is possible because Q_u is $\xi_Q(u)$-dichromatic and $|J_u| \geq \xi_Q(u)$). Let C be a directed cycle of $\sigma(D,Q)$. If $C \subseteq Q_u$ for some u, C is not monochromatic. Otherwise, $p(C)$ contains a directed cycle C_0. If C were monochromatic of colour $j, \alpha_j \supseteq V(p(C)) \supseteq V(C_0)$ which is impossible since α_j is acyclic. Then C is not monochromatic and f is an acyclic colouring. Therefore $dc(\sigma(D,Q)) \leq \tilde{n}(H_1(D), \xi_Q)$. Let J be a set of cardinality $dc(\sigma(D,Q))$ and $f: \sigma(D,Q) \to J$ an optimal acyclic colouring of $\sigma(D,Q)$. Denote by R_j the chromatic class of colour j. Then $\alpha_j = p(R_j)$ is an acyclic subset of $V(D)$ since R_j is acyclic and so $\alpha_j \in E(H_1(D))$. Since $J_u = \{j : u \in \alpha_j\}$, $j \in J_u$ if and only if $R_j \cap V(Q_u)$ is nonempty, then $|J_u| \geq dc(Q_u) = \xi_Q(u)$ and $(\alpha_j)_{j \in J}$ is a ξ_Q-covering of $H_1(D)$. Therefore $\tilde{n}(H_1(D), \xi_Q) \leq dc(\sigma(D,Q))$ and the proof is complete. ■

From here on, we will write $\tilde{n}_1(D,\xi)$ instead of $\tilde{n}(H_1(D),\xi)$. Note that $\tilde{n}_1(D, \mathbf{1}) = dc(D)$.

Corollary 43. *If $dc(\alpha) = k$ then $dc(D[\alpha])) = \tilde{n}_1(D, \mathbf{k})$.*

Let ξ be a w.f. on $\vec{C}_3$ such that $\xi_0 \geq \xi_1 \geq \xi_2$ where $\xi(j) = \xi_j$. In [15], the following result was proved.

Proposition 44. *$\tilde{n}_1(\vec{C}_3, \xi) = \lceil (\xi_0 + \xi_1 + \xi_2)/2 \rceil$ or ξ_0 depending on whether $\xi_0 \leq \xi_1 + \xi_2$ or $\xi_1 + \xi_2 \leq \xi_0$. In particular $\tilde{n}_1(\vec{C}_3, \mathbf{k}) = \lceil 3k/2 \rceil$.*

Proposition 45.

(i) $\tilde{n}_1(\vec{C}_{2m+1}(I_m), \mathbf{k}) = \lceil k(2m+1)/(m+1) \rceil$ *for* $m \geq 2$.

(ii) $\tilde{n}_1(\vec{C}_{2m+1}(I_{m,m}), \mathbf{k}) = \lceil k(2m+1)/m \rceil$ *for* $m \geq 3$.

(iii) $\tilde{n}_1(\vec{C}_{6m+1}(I_{3m,2m}), \mathbf{k}) = \lceil k(6m+1)/2m \rceil$ *for* $m \geq 2$.

(iv) $\tilde{n}_1(\vec{C}_{17}(I_{8,5}), \mathbf{k}) = \lceil 17k/5 \rceil$, $\tilde{n}_1(\vec{C}_{17}(I_{8,7}), \mathbf{k}) = \lceil 17k/7 \rceil$ *and* $\tilde{n}_1(\vec{C}_{17}(I_{8,6}), \mathbf{k}) = \lceil 17k/6 \rceil$.

Proof. The equalities follow directly from Proposition 4.1 and Proposition 3.4. ■

4.6. An application. If $\xi \colon Z_7 \to \mathbb{N}$ is defined by $\xi(j) = 2$ for $j \neq 0$ and $\xi(0) = 1$, then for $T = \vec{C}_7(1, 2, 4), \tilde{n}_1(T, \xi) \geq \lceil 13/3 \rceil = 5$ by Proposition 3.2 (iii) since $\beta^{\rightarrow}(\vec{C}_7(1, 2, 4)) = 3$. By Proposition 3.3, $\tilde{n}_1(T, \xi^{(5)}) = 5$ and since $\xi \leq \xi^{(5)}$ it follows from Proposition 3.2 that $\tilde{n}_1(T, \xi) = 5$. Define $Q_0 = T_1$ and $Q_j = \vec{C}_3$ for $j \in Z_7 - \{0\}$. Because of Theorem 4.2, $\sigma(T, (Q_u)_{u \in V(T)})$ is a 5-dichromatic tournament of order 19. The minimum order of a 5-dichromatic tournament is not known, this example shows that it is not bigger than 19. It can be proved that it is at least 17.

Let G^* be the digraph obtained from a graph G by orienting each one of the edges in both directions. Some properties and the behaviour of the function $\tilde{n}_1(G^*, \mathbf{k})$ have been studied in several papers [6, 7, 8, 17].

5 Subcritical and Upcritical Weight Functions

A weight function ξ on H is said to be *H-subcritical* if for every w.f. ξ' such that $\xi' \leq \xi$ and $\|\xi'\| = \|\xi\| - 1$, we have $\tilde{n}(H, \xi') < \tilde{n}(H, \xi)$ (and therefore $\tilde{n}(H, \xi') = \tilde{n}(H, \xi) - 1$). For brevity we will write *D-subcritical* instead of $H_1(D)$-subcritical.

Notice that the w.f. ξ considered in Proposition 4.4 is $\vec{C}_3$-subcritical iff $\xi_0 \leq \xi_1 + \xi_2$ and $\xi_0 + \xi_1 + \xi_2$ is odd [15].

Theorem 51. *If for every $u \in V(D)$, Q_u is a vertex-critical $\xi_Q(u)$-dichromatic digraph and ξ_Q is D-subcritical, then $\sigma(D, Q)$ is vertex-critical $\tilde{n}_1(D, Q)$-dichromatic.*

Proof. This follows directly from Theorem 4.2. ■

It is not difficult to prove that the w.f. ξ defined in 4.6 is $\vec{C}_7(1, 2, 4)$-subcritical. Therefore the tournament $\sigma(T, (Q_u)_{u \in V(T)})$ constructed there is vertex-critical.

Theorem 52.

(i) $\mathbf{k}$ *is* $\vec{C}_{2m+1}(I_m)$*-subcritical iff* $k \equiv m \bmod (m + 1)$ *and* $m \geq 2$.

(ii) $\mathbf{k}$ *is* $\vec{C}_{2m+1}(I_{m,m})$*-subcritical iff* $k \equiv 1 \bmod m$ *and* $m \geq 3$.

(iii) $\mathbf{k}$ *is* $\vec{C}_{6m+1}(I_{3m,2m})$*-subcritical iff* $k \equiv 1 \bmod 2m$ *and* $m \geq 2$.

(iv) $\mathbf{k}$ *is*
$\vec{C}_3$*-subcritical iff* k *is odd,*
$\vec{C}_{17}(I_{8,5})$*-subcritical iff* $k \equiv 3 \bmod 5$,
$\vec{C}_{17}(I_{8,7})$*-subcritical iff* $k \equiv 5 \bmod 7$,
$\vec{C}_{17}(I_{8,6})$*-subcritical iff* $k \equiv 5 \bmod 6$.

Proof. It follows from Proposition 4.4 that $\mathbf{k}$ is $\vec{C}_3$-subcritical iff k is odd. Let T any of the tournaments of (i), (ii) or (iii) and let ξ be a w.f. such that $\xi \leq \mathbf{k}$, $\|\xi\| = \|\mathbf{k}\| - 1$. From Proposition 4.5 it follows immediately that $\tilde{n}(T, \xi) = \tilde{n}(T, \mathbf{k})$ unless $k \equiv m \bmod (2m+1)$ in case (i), $k \equiv 1 \bmod (2m+1)$ in case (ii) or $k \equiv 1 \mathrm{mod}\ 2m$ in case (iii). In these last cases $r = \beta^{\rightarrow}(T)$ divides $\|\xi\|$. Since $\mathrm{Aut}(T)$ is vertex transitive, we may assume that $\xi = \xi^{(s)}$ for $s = \|\xi\|/r$ and the assertion follows from Proposition 3.3. The remaining cases can be proved in a similar way. ∎

5.3. Another application. Let $\xi\colon Z_7 - \{0\} \to \mathbb{N}$ be defined by $\xi(j) = 1$ for $j \in \{1, 2, 3, 4, 5\}$ and $\xi(6) = 2$. It is easy to see that ξ is ST_6-subcritical where $ST_6 = \vec{C}_7(1, 2, 4) - \{0\}$ and $\tilde{n}_1(ST_6, \xi) = 3$. Proceeding as in the example of 4.6, a vertex-critical 3-dichromatic tournament $T^{(3)}$ of order 8 is obtained. Let $T^{(m)}$ (resp: $W^{(m)}$) denote a generic vertex-critical m-dichromatic tournament of even (resp: odd) order. Recall that $t(T^{(m)}, W^{(m)}, T_1)$ is a vertex-critical $(m + 1)$-dichromatic tournament of even order and that there are infinitely many pairwise non isomorphic tournaments $W^{(3)}$ [15]. Using induction, it follows that an infinite family of pairwise non isomorphic vertex-critical r- dichromatic tournaments of even order can be constructed for every integer $r \geq 4$. This solves a question of [15].

After considering subcritical w.f., we define in a similar way a w.f. ξ on H to be *H-upcritical* if for every w.f. ξ' such that $\xi \leq \xi'$ and $\|\xi'\| = \|\xi\| + 1$, we have $\tilde{n}(H, \xi) < \tilde{n}(H, \xi')$ (and therefore $\tilde{n}(H, \xi') = \tilde{n}(H, \xi) + 1$). For brevity we will write *D-upcritical* instead of $H_1(D)$-upcritical.

As an example, Proposition 3.3 asserts that the w.f. $\xi^{(s)}$ is H-upcritical. Notice that the w.f. ξ considered in Proposition 4.4, is $\vec{C}_3$-upcritical iff $\xi_0 \leq \xi_1 + \xi_2$ and $\xi_0 + \xi_1 + \xi_2$ is even [16, Lemma 2]. Lemma 3 in [16] can be easily generalized as follows.

Theorem 53. *If* ξ_Q *is D-upcritical then every acyclic* $\tilde{n}_1(D, Q)$*-colouring of* $\sigma(D, Q)$ *induces in each* Q_u *an optimal acyclic colouring.* ∎

6 Vertex-Critical r-Dichromatic Circulant Tournaments

In this section we will prove the existence of vertex-critical k-dichromatic circulant tournaments for every $k \geq 3$, $k \neq 7$. We will use the fact that the composition of two circulant tournaments is a circulant tournament [14, Proposition 3.3].
Let f_0, f'_0, f_1 and f'_1 be the functions with codomain $\mathbb{N}^2$ defined by:

(1) $f_0(r,m) = r(2m+1) - 1$, $\quad f'_0(r,m) = r(m+1) - 1$
for $r \geq 1$, $m \geq 2$.
(2) $f_1(r,m) = r(2m+1) + 3$, $\quad f'_1(r,m) = rm + 1$
for $r \geq 1$, $m \geq 3$.

Lemma 61. *If x is an integer then $x \in \text{Image}\,(f_0) \cup \text{Image}\,(f_1)$ iff $x \geq 4$ and $x \notin \{5, 7, 11, 15, 23\}$.*

Proof. Take $X = \text{Image}\,(f_0) \cup \text{Image}\,(f_1)$. Clearly $x \in X$ implies $x \geq 4$. If x is an even number, $x \geq 4$, then $x \in \text{Image}\,(f_0)$. Let $x = 2x_1 + 1$ with $x_1 \geq 2$ and $x \notin X$. Then $2x_1+2$ has no odd divisor bigger than 3 and $2x_1-2$ has no odd divisor bigger than 5. So, $x_1+1 = 2^t.i_1$ and $x_1 - 1 = 2^s.i_2$ where $i_1 \in \{1,3\}$ and $i_2 \in \{1,3,5\}$. It follows that either $t \leq 1$ or $s \leq 1$. In the first case $x \in \{5, 11\}$, in the second, $x \in \{5, 9, 13, 7, 15, 23\}$. However $\{9, 13\} \subseteq \text{Image}\,(f_1)$ and therefore $x \in \{5, 7, 11, 15, 23\}$. It can be easily verified that in fact these values do not belong to X. ∎

Let D_j be the (acyclic) digraph whose vertices are the integers bigger than 2 and whose arcs are the pairs of the form $(f'_j(r,m), f_j(r,m))$, $j = 0, 1$ and take $D = D_0 \cup D_1$. It is easy to prove that D_0 and D_1 are arc disjoint. We assign to each arc $\tau = (f'_j(r,m), f_j(r,m))$ the weight $\omega(\tau) = 2m+1$ and a digraph operator $\hat{\tau}$ so that $\hat{\tau}(\alpha) = \vec{C}_{2m+1}(I_m)[\alpha]$ if $j = 0$ and $\hat{\tau}(\alpha) = \vec{C}_{2m+1}(I_{m,m})[\alpha]$ if $j = 1$. If $\pi = (u_0, \tau_1, u_1, \tau_1, u_2, \ldots, u_{n-1}, \tau_n, u_n)$ is a directed path in D we define $\hat{\pi} = \hat{\tau}_n \circ \cdots \circ \hat{\tau}_2 \circ \hat{\tau}_1$ and $\omega(\pi) = \omega(\tau_n) \ldots \omega(\tau_1)$.

Using Corollary 4.3, Proposition 4.5 and Theorems 5.1 and 5.2 we obtain the following

Lemma 62. *If α is a vertex-critical u_0-dichromatic circulant tournament then $\hat{\pi}(\alpha)$ is a vertex-critical u_n-dichromatic circulant tournament such that $o(\hat{\pi}(\alpha)) = o(\alpha)\omega(\pi)$.* ∎

Remark 63. Using Lemma 6.1 it follows immediately that the set of vertices of D with indegree 0 is $\{3, 4, 5, 7, 11, 15, 23\}$.

Lemma 64. *For each integer $n \geq 3$, $n \neq 7$ there is a directed path in D from a vertex in $\{3, 4, 5, 11, 13, 15, 23\}$ to n.*

Proof. Let $B = \{3, 4, 5, 11, 13, 15, 23\}$ and $W = \{w \in V(D)$: there is a directed Bw-path in $D\}$. Since $(3, 6), (4, 8), (5, 9), (5, 10), (6, 12), (8, 14), (8, 16), (9, 17), (9, 18), (10, 20), (11, 19), (11, 20), (11, 21), (11, 22), (12, 24) \in A(D_1)$ then $I_{24} - \{1, 2, 7\} \subseteq W$. We will prove that $K = \mathbb{N} - \{1, 2, 7\} = W$. The proof is by induction. Let $n \geq 25$ such that $s \in W$ whenever $s \leq n - 1$, $s \in K$. Because of Remark 6.3 there exists a k such that $(k, n) \in A(D)$. Now $k < n$ and $k \notin \{1, 2, 7\}$ since the only $\{1, 2, 7\}w$-arcs of D are $(2, 4), (7, 13)$ and $(7, 14)$. Therefore $k \in K$ and so $n \in K$. ∎

Proposition 65. *For every integer $k \in \{3, 4, 5, 11, 13, 15, 23\}$ there exists an infinite family $\mathcal{F}_k$ of vertex-critical k-dichromatic circulant tournaments no two of them having the same order.*

Proof. The families $\mathcal{F}_j$ for $j = 3, 4$ and 5 are the following: $\mathcal{F}_3 = \{\vec{C}_{2m+1}(I_{m,m}) \colon m \geq 3\}$, $\mathcal{F}_5 = \{\vec{C}_3[\vec{C}_{2m+1}(I_{m,m})] \colon m \geq 3\}$ [15]; $\mathcal{F}_4 = \{\vec{C}_{6m+1}(I_{3m,2m}) \colon m \geq 2\}$ [13]. Define now $\mathcal{F}_{11} = \{\vec{C}_{17}(I_{8,5})[\alpha] \colon \alpha \in \mathcal{F}_3\}$; $\mathcal{F}_{13} = \{\vec{C}_{17}(I_{8,7})[\alpha] \colon \alpha \in \mathcal{F}_5\}$; $\mathcal{F}_{15} = \{\vec{C}_{17}(I_{8,6})[\alpha] \colon \alpha \in \mathcal{F}_5\}$. That these last 3 families satisfy the required conditions is a direct consequence of Corollary 4.3, Proposition 4.5 and Theorems 5.1 and 5.2 and the fact that for each $j \in \{11, 13, 15\}$, all the members of $\mathcal{F}_j$ have different orders. Finally define the family $\mathcal{F}_{23} = \{\vec{C}_3[\alpha] \colon \alpha \in \mathcal{F}_{15}\}$ which satisfies the required conditions because of Proposition 4.4 and Theorems 4.2, 5.1 and 5.2. ∎

Theorem 66. *For every integer $k \geq 3$, $k \neq 7$ there exists an infinite family $\mathcal{F}_k$ of pairwise non isomorphic vertex-critical k-dichromatic circulant tournaments.*

Proof. In fact, we will construct for each $k \geq 3$, $k \neq 7$ an infinite family $\mathcal{F}_k$ of vertex-critical k-dichromatic circulant tournaments such that all its members have different orders. By Lemma 6.4 there is in D a directed uk-path π with $u \in \{3, 4, 5, 11, 13, 15, 23\}$. Define $\mathcal{F}_k = \{\hat{\pi}(\alpha) \colon \alpha \in \mathcal{F}_u\}$. By Lemmas 6.2 and 6.5, $\mathcal{F}_k$ has the required properties. ∎

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