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2-FACTORS IN CLAW-FREE GRAPHS

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Abstract

We consider the question of the range of the number of cycles possible in a 2-factor of a 2-connected claw-free graph with sufficiently high minimum degree. (By claw-free we mean the graph has no induced $K_{1,3}$.) In particular, we show that for such a graph G of order $n \geq 51$ with $\delta(G) \geq \frac{n-2}{3}$, G contains a 2-factor with exactly k cycles, for $1 \leq k \leq \frac{n-24}{3}$. We also show that this result is sharp in the sense that if we lower $\delta(G)$, we cannot obtain the full range of values for k.

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1 Introduction

The question of determining when a graph contains a 2-factor (a 2-regular spanning subgraph) has long been an important one in graph theory. Many results deal with hamiltonian graphs, that is, graphs G containing a cycle that spans the vertex set $V(G)$. (See [4]). One special class of graphs that has drawn considerable interest are the claw-free graphs. Such graphs contain no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$.

In particular, the following was shown in [5].

Theorem 1. *If G is a 2-connected $K_{1,3}$-free graph of order n with $\delta(G) \geq \frac{n-2}{3}$, then G is hamiltonian.*

We can see that this result is sharp by considering the following nonhamiltonian graph G on $n = 3m$ vertices. Let $V(G) = A_1 \cup A_2 \cup A_3$ such that $|A_i| = m$ and $\langle A_i \rangle \cong K_m$ and let $x_i, y_i \in A_i$, $x_i \neq y_i$ for $i = 1, 2, 3$ and so that $\langle x_1, x_2, x_3 \rangle \cong \langle y_1, y_2, y_3 \rangle \cong K_3$. Clearly, the minimum degree of G is $m - 1 = \frac{n-3}{3}$.

Recently the question of determining the number of cycles possible in a 2-factor of a given 2-connected graph satisfying certain degree conditions has been considered in [2].

The purpose of this paper is to investigate this question for 2-connected claw-free graphs. In particular, we will extend Theorem 1 by showing that the same minimum degree condition implies that G contains a 2-factor with exactly k-cycles for $1 \leq k \leq \frac{n-24}{3}$.

We will let $\langle S \rangle_G$ denote the subgraph of G induced by S a subset of $V(G)$. For $A, B \subset V(G)$, $e_G(A, B)$ denotes the number of edges in G with one vertex in A and the other in B. For $H \subset G$ we will sometimes write $e_G(A, H)$ as shorthand for $e_G(A, V(H))$. The independence number of a graph will be denoted by $\alpha(G)$. For a cycle C, we will denote by $\overrightarrow{C}$ the cycle under some orientation and $\overleftarrow{C}$ will denote the cycle under the opposite orientation. For a vertex, a, on a cycle with some orientation, $\overrightarrow{C}$, we define a^+ and a^- to be the immediate successor and predecessor respectively of a on C with respect to this orientation. Also, for a collection of vertex disjoint cycles S each with some orientation, we define $N_S^+(a)$ to be the set $\{a^+ | a \in (N(a) \cap V(S))\}$. Let $I = a_0, a_1, ..., a_k$ where the a_i's are consecutive vertices on a cycle. Then $l(I) = k$, the length of the segment of the cycle. For terms not defined here, see [3].

2 Main Result

In this section we will prove the theorem. However, first we prove the following proposition which gives sufficient conditions for the existence of k disjoint triangles and will lay the foundation for the proof of the theorem.

Proposition 1. *Let G be a claw-free graph of order n, let k be a positive integer, and let c be an integer so that $c \geq 0$. If $n > 3k + 6 - f(k, c)$ where $f(1, 1) = f(2, 0) = 0$ and $f(k, c) = \frac{9c-9}{k+c-2}$ for all other values of k and c and $\delta(G) \geq \max\{k + c, 3\}$ then G contains k disjoint triangles.*

Proof. If $\delta(G) \geq 3$, then $n \geq 4$ and, since G is claw-free, G must contain at least one triangle. Choose m disjoint triangles in G, say $T_1, T_2, ..., T_m$, so that m is as large as possible. Since G is claw-free and $\delta(G) \geq 3$, we know $m \geq 1$. Assume $m < k$. Let

$$A = \bigcup_{i=1}^{m} V(T_i)$$

and $H = G - A$.

If $\Delta(H) \geq 3$, say $\deg_H a \geq 3$ for some $a \in V(H)$, then since G is claw-free, $b_1 b_2 \in E(H)$ for some $b_1, b_2 \in N_H(a)$ and $\{a, b_1, b_2\}$ forms a triangle. This contradicts the maximality of m. Therefore, $\Delta(H) \leq 2$.

Claim. For each $x \in A$, $|N_G(x) \cap V(H)| \leq 3$.

Proof. Assume $|N_G(x) \cap V(H)| \geq 4$ for some $x \in A$. Let $x \in V(T_i)$ and $V(T_i) = \{x, y, z\}$. Let a_1, a_2, a_3, a_4 be distinct neighbors of x in H.

If $N_G(a_1) \cap \{a_2, a_3, a_4\} = \emptyset$, then since x and $\{a_1, a_2, a_3\}$ do not form a claw, without loss of generality, $a_2 a_3 \in E(G)$. We apply the same argument to x and $\{a_1, a_2, a_4\}$ and $\{a_1, a_3, a_4\}$, and we have $a_2 a_4 \in E(G)$ and $a_3 a_4 \in E(G)$. But then $\{a_2, a_3, a_4\}$ forms a triangle, which contradicts the maximality of m. Therefore, $N_G(a_1) \cap \{a_2, a_3, a_4\} \neq \emptyset$. Similarly, we have $\deg_{\langle a_1, a_2, a_3, a_4 \rangle_G} a_i \geq 1$ for each i, $1 \leq i \leq 4$. Since $\Delta(H) \leq 2$, we know $\langle a_1, a_2, a_3, a_4 \rangle_H = \langle a_1, a_2, a_3, a_4 \rangle_G$ must contain two independent edges. Thus, without loss of generality, we may assume $a_1 a_2, a_3 a_4 \in E(G)$.

Consider the subgraph induced by $F = \langle \{a_1, a_2, a_3, a_4, y, z\} \rangle_G$. We want to show that F must contain $K_3 \cup K_2$ as a subgraph because the existence of such a subgraph in F implies that $\langle F \cup \{x\} \rangle$ contains two independent triangles which contradicts the maximality of A.
Since $r(3, 3) = 6$ and F does not contain three independent vertices, we know F must contain a triangle, say T. Since $F - V(T)$ cannot be an independent set, it must contain an edge. Therefore, F contains $K_3 \cup K_2$. ∎

Since $\Delta(H) \leq 2$ and $\delta(G) \geq k + c$, we have $e_G(x, A) \geq k + c - 2$, for each $x \in V(H)$. Thus, $e_G(H, A) \geq (k + c - 2)(n - 3m)$. On the other hand, $e_G(u, H) \leq 3$ for each $u \in A$ which implies $e_G(A, H) \leq 3|A| = 9m$. Therefore, $(k + c - 2)(n - 3m) \leq 9m$. Thus, $(k + c - 2)n \leq (3k + 3c + 3)m$. Then, using the fact that we assumed $m \leq k - 1$, we find $n \leq \frac{3k^2 + 3ck - (3c+3)}{k+c-2} = 3k + 6 - \frac{9c-9}{k+c-2}$. This contradicts the assumption and completes the proof. ∎

Theorem 2. *Let G be a 2-connected, claw-free graph of order $n \geq 51$ with $\delta(G) \geq \frac{1}{3}(n-2)$. Then for each k with $1 \leq k \leq \frac{n-24}{3}$, G has a 2-factor with exactly k components.*

Proof. By the assumption $n \geq 3k+24$ and $\delta(G) \geq \frac{n-2}{3} \geq \frac{3k+22}{3} \geq k+1$. Therefore, by Proposition 1, G has k disjoint cycles $C_1, C_2, ..., C_k$. Choose $C_1, ..., C_k$ such that $\sum_{i=1}^{k} |V(C_i)|$ is as large as possible. Let $D = \bigcup_{i=1}^{k} V(C_i)$ and assume $D \neq V(G)$. Let $H = G - D$.

Claim 1. $|V(H)| \geq 4$.

Proof. Let $h = |V(H)|$ and assume $h \leq 3$.

Since $h \leq 3$, $|D| \geq n-3 \geq 3k+21$. Thus, there exists some cycle, say C_i, such that $|V(C_i)| \geq 4$. Let $x \in V(H)$ and let $|N_G(x) \cap V(C_i)| = t$, say $N_G(x) \cap C_i = \{a_1, ..., a_t\}$. We may assume $a_1, ..., a_t$ appear in consecutive order along some orientation of C_i. Let $I_j = a_j \overrightarrow{C_i} a_{j+1}$ for $1 \leq j \leq t-1$ and let $I_t = a_t \overrightarrow{C_i} a_1$. If $l(I_j) = 1$ for some $1 \leq j \leq t$, then $a_{j+1} = a_j^+$. Let $C_i' = a_{j+1} \overrightarrow{C_i} a_j x a_{j+1}$ and $C_j' = C_j$ for all $j \neq i$. Then $\{C_1', ..., C_k'\}$ is a disjoint collection of cycles of larger total order, a contradiction. Therefore, $l(I_j) \geq 2$ for each j, $1 \leq j \leq t$.

Since G is claw-free, this implies $a_j^- a_j^+ \in E(G)$ for each j, $1 \leq j \leq t$. If $l(I_j) = 2$, then $a_j^{++} = a_{j+1}$. Let $C_i' = x a_{j+1} \overrightarrow{C_i} a_j^- a_j^+ a_j x$ and $C_j' = C_j$ for all $j \neq i$. If $l(I_j) = 3$, then $a_j^{+++} = a_{j+1}$. Let $C_i' = x a_{j+1} a_{j+1}^- a_{j+1}^+ \overrightarrow{C} a_j^- a_j^+ a_j x$ and $C_j' = C_j$ for all $j \neq i$. In either case, the collection $\{C_1', ... C_k'\}$ forms a set of independent cycles of larger order, a contradiction.

Therefore, $l(I_j) \geq 4$ for each j, $1 \leq j \leq t$. This implies $|V(C_i)| = \sum_{j=1}^{t} l(I_j) \geq 4t$ or $|N_G(x) \cap V(C_i)| \leq \frac{1}{4}|V(C_i)|$ for all C_i such that $|V(C_i)| \geq 4$. Note that x has at most one adjacency to every 3-cycle in the collection $C_1, \cdots, C_k$.

We may assume $|V(C_1)| = |V(C_2)| = ... = |V(C_s)| = 3$ and $|V(C_i)| \geq 4$ for $s+1 \leq i \leq k$. Then,

$$\begin{aligned} \frac{n-2}{3} &\leq \deg_H x + e(x, D) \\ &\leq (h-1) + s + \frac{1}{4} \sum_{i=s+1}^{k} |V(C_i)| = (h-1) + \frac{|D|+s}{4} \\ &= \frac{n+3h+s-4}{4}, \end{aligned}$$

which implies $n \le 3s + 9h - 4$. Since $s \le k$ and $h \le 3$, we have $n \le 3k + 23$. This contradicts the assumption. Consequently, we know $|V(H)| \ge 4$. ■

Claim 2. For each $x \in V(H)$ and for each $y \in V(H) - \{x\}$, $\deg_{H-x}\ y \ge 2$.

Proof. Assume $\deg_{H-x}\ y \le 1$ for some $y \in V(H) - \{x\}$. As in Claim 1, we count the number of edges from y to D observing that y can have at most one adjacency to a 3-cycle and y is adjacent to at most one out of every four vertices on cycles of length 4 or more.

We may assume $|V(C_1)| = |V(C_2)| = ... = |V(C_s)| = 3$ and $|V(C_i)| \ge 4$ for $s + 1 \le i \le k$. Then $e(y, D) \le s + \frac{1}{4}\sum_{i=s+1}^{k} |V(C_i)| = s + \frac{1}{4}(|D| - 3s) = \frac{1}{4}|D| + \frac{1}{4}s$. Therefore,

$$\begin{aligned} \frac{n-2}{3} &\le \deg_H\ y + \deg_D\ y \le \deg_H\ y + e(y, D) \\ &\le 1 + \deg_{H-x}\ y + \frac{1}{4}|D| + \frac{1}{4}s \\ &\le 1 + \deg_{H-x}\ y + \frac{1}{4}(n-4) + \frac{1}{4}k \\ &\le 1 + \deg_{H-x}\ y + \frac{1}{4}(n-4) + \frac{n-24}{12}. \end{aligned}$$

Thus, $\deg_{H-x}\ y \ge 2$. ■

By Claims 1 and 2, we know that for every $x \in V(H)$, $H - x$ contains a cycle, call it C_x.

Claim 3. For every $x \in V(H)$, the set $N_D^+(x)$ is independent.

Proof. Assume, to the contrary, $a_1^+ a_2^+ \in E(G)$ for some $a_1, a_2 \in N_D(x)$. If a_1 and a_2 lie in the same cycle of D, say C_i, then we increase the total order of D by replacing C_i by $C_i' = a_1^+ \overrightarrow{C_i} a_2 x a_1 \overleftarrow{C_i} a_2^+ a_1^+$. If a_1 and a_2 lie in different cycles of D, we may assume without loss of generality $a_i \in V(C_i), i = 1, 2$. Then let $C_1' = C_x$, $C_2' = x a_1 \overleftarrow{C_1} a_1^+ a_2^+ \overrightarrow{C_2} a_2 x$ and for $j \ne 1, 2$ let $C_j' = C_j$. Then the collection $\{C_1', ...C_k'\}$ forms a set of k disjoint cycles of larger total order, a contradiction. ■

From the results in [7], we know that in a claw-free graph of order n, $\alpha(G) \le 2n/(\delta(G)+2)$. Thus, by Claims 3 and the bound on $\alpha(G)$, for each $x \in V(H)$ we have that

$$|N_D[x]| = |N_D^+(x) \cup \{x\}| \le \alpha(G) \le \frac{2n}{\delta(G) + 2} \le \frac{2n}{\frac{n-2}{3} + 2} < 6.$$

Therefore, $|N_D(x)| \leq 4$ and we have $\deg_H x \geq \frac{n-14}{3}$.

Let P be a longest path in H and let x be one of its end vertices. Then $N_H(x) \subseteq V(P)$ or a longer path is possible. Therefore, if we choose $y \in N_H(x)$ so that $x\overrightarrow{P}y$ is as long as possible, we form a cycle $C = x\overrightarrow{P}yx$ with $N_H(x) \subseteq V(C)$. This implies $|V(C)| \geq \deg_H x + 1 \geq \frac{n-14}{3} + 1 = \frac{n-11}{3}$. Then by the maximality of D, we know $|V(C_i)| \geq \frac{n-11}{3}$, for all $1 \leq i \leq k$.

Claim 4. The number of independent cycles, k, is 2.

Proof. Assume $k \geq 3$. Then $n = |V(G)| \geq |V(C)| + |V(C_1)| + |V(C_2)| + |V(C_3)| \geq 4(\frac{n-11}{3})$. This forces $n \leq 44$, a contradiction. ■

Since C_1 and C_2 each have at least $\frac{n-11}{3}$ vertices, we know

$$|V(H)| \leq n - |V(C_1)| - |V(C_2)| \leq \frac{n+22}{3}.$$

Claim 5. The subgraph H is hamiltonian connected.

Proof. If H is not hamiltonian-connected, then by a result in [6],

$$\frac{n-14}{3} \leq \delta(H) \leq \frac{1}{2}|V(H)| \leq \frac{n+22}{6}.$$

This forces $n \leq 50$, a contradiction. ■

In particular, H has a hamiltonian cycle, say C_0. By the maximality of D, we know $|V(C_0)| \leq |V(C_i)|$ for $i = 1, 2$. Thus, $|V(C_0)| \leq \frac{1}{3}n$.

Since G is 2-connected, there exist at least two independent edges between C_0 and $C_1 \cup C_2$.

Claim 6. There do not exist two independent edges from C_0 to C_i, for $i = 1, 2$.

Proof. Without loss of generality, let $i = 1$. Assume there are two independent edges, say a_1b_1 and a_2b_2 between C_0 and C_1 (where $a_1, a_2 \in C_0$, $b_1, b_2 \in C_1$). Without loss of generality, we may assume $l(b_1\overrightarrow{C_1}b_2) \geq \frac{1}{2}|V(C_1)|$. Since $\{a_2\overrightarrow{P}a_1b_1\overrightarrow{C_1}b_2a_2, C_2\}$ forms a set of disjoint cycles where P is a hamiltonian a_1, a_2-path in H, we know $l(b_2\overrightarrow{C_1}b_1) \geq |V(C_0)| + 1 \geq \delta(H) + 2 \geq \frac{n-8}{3}$. Then $|V(C_1)| \geq 2l(b_2\overrightarrow{C}b_1) \geq \frac{2n-16}{3}$. Therefore,

$$n = |V(C_0)| + |V(C_1)| + |V(C_2)| \geq 2\left(\frac{n-11}{3}\right) + \frac{2n-16}{3} = \frac{4n-38}{3}.$$

This forces $n \leq 38$ which is a contradiction. ■

Therefore we may assume $a_1b_1, a_2b_2 \in E(G)$ where $a_1, a_2 \in V(C_0)$, $a_1 \neq a_2$, $b_1 \in V(C_1)$, and $b_2 \in V(C_2)$. As a consequence of Claim 7 and 2-connectivity, we know there exists an edge $d_1d_2 \in E(G)$ such that $d_1 \in V(C_1) - b_1$ and $d_2 \in V(C_2)$.

Let $x \in H - \{a_1, a_2\}$. (Since $|V(H)| = |V(C_0)| \geq \frac{n-11}{3}$ we know such an x exists.) Then by Claim 6, $N_{C_1 \cup C_2}(x) \subset \{b_1, b_2\}$. Therefore, $\deg_H x \geq \frac{n-2}{3} - 2 = \frac{n-8}{3}$, and hence $|V(C_0)| \geq \frac{n-5}{3}$.

Claim 7. The graph $H - \{a_1, a_2\}$ has a triangle T and $H - V(T)$ is hamiltonian-connected.

Proof. Let $H' = H - \{a_1, a_2\}$ and assume $\delta(H') \leq \frac{|V(H')|}{2}$. Since $\delta(H') \geq \delta(H) - 2 \geq \frac{n-8}{3} - 2 \geq \frac{n-14}{3}$ and $|V(H')| \leq \frac{n}{3} - 2 = \frac{n-6}{3}$, we get $\frac{n-14}{3} \leq \frac{1}{2}\left(\frac{n-6}{3}\right)$. This forces $n \leq 18$, a contradiction.

Thus $\delta(H') \geq \frac{|V(H')|+1}{2}$ and $|V(H')| \geq \frac{n-5}{3} - 2 \geq 3$, which implies by [1] that H' is pancyclic. Thus H' has a triangle T. Let $H'' = H - V(T)$. Then $|V(H'')| = |V(H)| - 3 \leq \frac{n}{3} - 3 = \frac{n-9}{3}$ and $\delta(H'') \geq \frac{n-14}{3} - 3 \geq \frac{n-23}{3}$. Therefore, since $n \geq 51$, $\delta(H'') > \frac{1}{2}|V(H'')|$. Hence, by [6] H'' is hamiltonian connected. ■

First, suppose $d_2 \neq b_2$. We may assume $l(d_1\overrightarrow{C_1}b_1) \leq \frac{1}{2}(|V(C_1)|)$ and $l(b_2\overrightarrow{C_2}d_2) \leq \frac{1}{2}(|V(C_2)|)$. By the maximality of C_1 and C_2 and the fact that G is claw-free, $b_1^+b_1^-, b_2^+b_2^- \in E(G)$. Let $C' = a_1b_1b_1^-b_1^+\overrightarrow{C_1}d_1d_2\overrightarrow{C_2}b_2^-b_2^+b_2a_2Pa_1$, where P is a hamiltonian a_1a_2-path in $H - T$. Since C' and T are disjoint cycles, $l(d_1^+\overrightarrow{C_1}b_1^{--}) + l(b_2^{++}\overrightarrow{C_2}d_2^-) + 2 \geq |V(H)|$. Thus $\frac{|V(C_1)|+|V(C_2)|}{2} - 4 \geq |V(H)| \geq \frac{n-5}{3}$, which implies that $|V(C_1)| + |V(C_2)| \geq \frac{2n+14}{3}$. Since $|V(H)| = |V(C_0)| \geq \frac{n-5}{3}$, we have $n = |V(H)|+|V(C_1)|+|V(C_2)| \geq \frac{3n+9}{3} = n+3$, a contradiction. Therefore, we know $d_2 = b_2$ which implies that there cannot be three independent edges between the cycles C, C_1, and C_2.

Since G is 2-connected, there exists an edge $b_2'u$ from $C_2 - \{b_2\}$ to $C_0 \cup C_1$

Case 1. We consider the case where $u \in C_0$. If $u \neq a_1$ the three edges a_1b_1, d_1b_2, and $b_2'u$ are independent, a contradiction. Thus, $u = a_1$. But now the two edges a_2b_2 and a_1b_2' between C_0 and C_2 are independent. This contradicts Claim 7.

Case 2. We consider the case where $u \in C_1$. If $u \neq b_1$, then the three edges a_1b_1, ub_2', and a_2b_2 are independent, a contradiction. If $u = b_1$, consider b_1 and $\{a_1, b_1^+, b_2'\}$. We know $b_2'b_1^+ \notin E(G)$ because $u = b_1$.

By Claim 7, $a_1b'_2 \notin E(G)$. If $a_1b_1^+ \in E(G)$, then the three edges $a_1b_1^+, b_1b'_2$ and a_2b_2 are independent, a contradiction. Thus, $\langle b_1, b_1^+, a_1, b'_2\rangle_G$ is a claw, a contradiction.

Hence, in all cases we reach a contradiction, and the result is proved. ■

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