Discussiones Mathematicae
Graph Theory 20 (2000) 165–172

2-FACTORS IN CLAW-FREE GRAPHS

Guantao Chen
Georgia State University, Atlanta, GA 30303

Jill R. Faudree
University of Alaska Fairbanks, Fairbanks, AK 99775

Ronald J. Gould
Emory University, Atlanta, GA 30322

and

Akira Saito
Nihon University, Tokyo 156, Japan

Abstract

We consider the question of the range of the number of cycles possible in a 2-factor of a 2-connected claw-free graph with sufficiently high minimum degree. (By claw-free we mean the graph has no induced $K_{1,3}$.) In particular, we show that for such a graph G of order $n \geq 51$ with $\delta(G) \geq \frac{n-2}{3}$, G contains a 2-factor with exactly k cycles, for $1 \leq k \leq \frac{n-24}{3}$. We also show that this result is sharp in the sense that if we lower $\delta(G)$, we cannot obtain the full range of values for k.

Keywords: claw-free, forbidden subgraphs, 2-factors, cycles.

2000 Mathematics Subject Classification: 05C38.

1 Introduction

The question of determining when a graph contains a 2-factor (a 2-regular spanning subgraph) has long been an important one in graph theory. Many results deal with hamiltonian graphs, that is, graphs G containing a cycle that spans the vertex set $V(G)$. (See [4]). One special class of graphs that has drawn considerable interest are the claw-free graphs. Such graphs contain no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$.

In particular, the following was shown in [5].

Theorem 1. *If G is a 2-connected $K_{1,3}$-free graph of order n with $\delta(G) \geq \frac{n-2}{3}$, then G is hamiltonian.*

We can see that this result is sharp by considering the following nonhamiltonian graph G on $n = 3m$ vertices. Let $V(G) = A_1 \cup A_2 \cup A_3$ such that $|A_i| = m$ and $\langle A_i \rangle \cong K_m$ and let $x_i, y_i \in A_i$, $x_i \neq y_i$ for $i = 1, 2, 3$ and so that $\langle x_1, x_2, x_3 \rangle \cong \langle y_1, y_2, y_3 \rangle \cong K_3$. Clearly, the minimum degree of G is $m - 1 = \frac{n-3}{3}$.

Recently the question of determining the number of cycles possible in a 2-factor of a given 2-connected graph satisfying certain degree conditions has been considered in [2].

The purpose of this paper is to investigate this question for 2-connected claw-free graphs. In particular, we will extend Theorem 1 by showing that the same minimum degree condition implies that G contains a 2-factor with exactly k-cycles for $1 \leq k \leq \frac{n-24}{3}$.

We will let $\langle S \rangle_G$ denote the subgraph of G induced by S a subset of $V(G)$. For $A, B \subset V(G)$, $e_G(A, B)$ denotes the number of edges in G with one vertex in A and the other in B. For $H \subset G$ we will sometimes write $e_G(A, H)$ as shorthand for $e_G(A, V(H))$. The independence number of a graph will be denoted by $\alpha(G)$. For a cycle C, we will denote by $\overrightarrow{C}$ the cycle under some orientation and $\overleftarrow{C}$ will denote the cycle under the opposite orientation. For a vertex, a, on a cycle with some orientation, $\overrightarrow{C}$, we define a^+ and a^- to be the immediate successor and predecessor respectively of a on C with respect to this orientation. Also, for a collection of vertex disjoint cycles S each with some orientation, we define $N_S^+(a)$ to be the set $\{a^+ | a \in (N(a) \cap V(S))\}$. Let $I = a_0, a_1, ..., a_k$ where the a_i's are consecutive vertices on a cycle. Then $l(I) = k$, the length of the segment of the cycle. For terms not defined here, see [3].

2 Main Result

In this section we will prove the theorem. However, first we prove the following proposition which gives sufficient conditions for the existence of k disjoint triangles and will lay the foundation for the proof of the theorem.

Proposition 1. *Let G be a claw-free graph of order n, let k be a positive integer, and let c be an integer so that $c \geq 0$. If $n > 3k + 6 - f(k, c)$ where $f(1, 1) = f(2, 0) = 0$ and $f(k, c) = \frac{9c-9}{k+c-2}$ for all other values of k and c and $\delta(G) \geq \max\{k + c, 3\}$ then G contains k disjoint triangles.*

Proof. If $\delta(G) \geq 3$, then $n \geq 4$ and, since G is claw-free, G must contain at least one triangle. Choose m disjoint triangles in G, say $T_1, T_2, ..., T_m$, so that m is as large as possible. Since G is claw-free and $\delta(G) \geq 3$, we know $m \geq 1$. Assume $m < k$. Let

$$A = \bigcup_{i=1}^{m} V(T_i)$$

and $H = G - A$.

If $\Delta(H) \geq 3$, say $\deg_H a \geq 3$ for some $a \in V(H)$, then since G is claw-free, $b_1b_2 \in E(H)$ for some $b_1, b_2 \in N_H(a)$ and $\{a, b_1, b_2\}$ forms a triangle. This contradicts the maximality of m. Therefore, $\Delta(H) \leq 2$.

Claim. For each $x \in A$, $|N_G(x) \cap V(H)| \leq 3$.

Proof. Assume $|N_G(x) \cap V(H)| \geq 4$ for some $x \in A$. Let $x \in V(T_i)$ and $V(T_i) = \{x, y, z\}$. Let a_1, a_2, a_3, a_4 be distinct neighbors of x in H.

If $N_G(a_1) \cap \{a_2, a_3, a_4\} = \emptyset$, then since x and $\{a_1, a_2, a_3\}$ do not form a claw, without loss of generality, $a_2a_3 \in E(G)$. We apply the same argument to x and $\{a_1, a_2, a_4\}$ and $\{a_1, a_3, a_4\}$, and we have $a_2a_4 \in E(G)$ and $a_3a_4 \in E(G)$. But then $\{a_2, a_3, a_4\}$ forms a triangle, which contradicts the maximality of m. Therefore, $N_G(a_1) \cap \{a_2, a_3, a_4\} \neq \emptyset$. Similarly, we have $\deg_{\langle a_1,a_2,a_3,a_4\rangle_G} a_i \geq 1$ for each i, $1 \leq i \leq 4$. Since $\Delta(H) \leq 2$, we know $\langle a_1, a_2, a_3, a_4\rangle_H = \langle a_1, a_2, a_3, a_4\rangle_G$ must contain two independent edges. Thus, without loss of generality, we may assume $a_1a_2, a_3a_4 \in E(G)$.

Consider the subgraph induced by $F = \langle\{a_1, a_2, a_3, a_4, y, z\}\rangle_G$. We want to show that F must contain $K_3 \cup K_2$ as a subgraph because the existence of such a subgraph in F implies that $\langle F \cup \{x\}\rangle$ contains two independent triangles which contradicts the maximality of A.
Since $r(3, 3) = 6$ and F does not contain three independent vertices, we know F must contain a triangle, say T. Since $F - V(T)$ cannot be an independent set, it must contain an edge. Therefore, F contains $K_3 \cup K_2$. ■

Since $\Delta(H) \leq 2$ and $\delta(G) \geq k + c$, we have $e_G(x, A) \geq k + c - 2$, for each $x \in V(H)$. Thus, $e_G(H, A) \geq (k + c - 2)(n - 3m)$. On the other hand, $e_G(u, H) \leq 3$ for each $u \in A$ which implies $e_G(A, H) \leq 3|A| = 9m$. Therefore, $(k+c-2)(n-3m) \leq 9m$. Thus, $(k+c-2)n \leq (3k+3c+3)m$. Then, using the fact that we assumed $m \leq k - 1$, we find $n \leq \frac{3k^2+3ck-(3c+3)}{k+c-2} = 3k + 6 - \frac{9c-9}{k+c-2}$. This contradicts the assumption and completes the proof. ■

Theorem 2. *Let G be a 2-connected, claw-free graph of order $n \geq 51$ with $\delta(G) \geq \frac{1}{3}(n-2)$. Then for each k with $1 \leq k \leq \frac{n-24}{3}$, G has a 2-factor with exactly k components.*

Proof. By the assumption $n \geq 3k+24$ and $\delta(G) \geq \frac{n-2}{3} \geq \frac{3k+22}{3} \geq k+1$. Therefore, by Proposition 1, G has k disjoint cycles $C_1, C_2, ..., C_k$. Choose $C_1, ..., C_k$ such that $\sum_{i=1}^{k} |V(C_i)|$ is as large as possible. Let $D = \bigcup_{i=1}^{k} V(C_i)$ and assume $D \neq V(G)$. Let $H = G - D$.

Claim 1. $|V(H)| \geq 4$.

Proof. Let $h = |V(H)|$ and assume $h \leq 3$.

Since $h \leq 3$, $|D| \geq n-3 \geq 3k+21$. Thus, there exists some cycle, say C_i, such that $|V(C_i)| \geq 4$. Let $x \in V(H)$ and let $|N_G(x) \cap V(C_i)| = t$, say $N_G(x) \cap C_i = \{a_1, ..., a_t\}$. We may assume $a_1, ..., a_t$ appear in consecutive order along some orientation of C_i. Let $I_j = a_j \overrightarrow{C_i} a_{j+1}$ for $1 \leq j \leq t-1$ and let $I_t = a_t \overrightarrow{C_i} a_1$. If $l(I_j) = 1$ for some $1 \leq j \leq t$, then $a_{j+1} = a_j^+$. Let $C_i' = a_{j+1} \overrightarrow{C_i} a_j x a_{j+1}$ and $C_j' = C_j$ for all $j \neq i$. Then $\{C_1', ..., C_k'\}$ is a disjoint collection of cycles of larger total order, a contradiction. Therefore, $l(I_j) \geq 2$ for each j, $1 \leq j \leq t$.

Since G is claw-free, this implies $a_j^- a_j^+ \in E(G)$ for each j, $1 \leq j \leq t$. If $l(I_j) = 2$, then $a_j^{++} = a_{j+1}$. Let $C_i' = x a_{j+1} \overrightarrow{C_i} a_j^- a_j^+ a_j x$ and $C_j' = C_j$ for all $j \neq i$. If $l(I_j) = 3$, then $a_j^{+++} = a_{j+1}$. Let $C_i' = x a_{j+1} a_{j+1}^- a_{j+1}^+ \overrightarrow{C} a_j^- a_j^+ a_j x$ and $C_j' = C_j$ for all $j \neq i$. In either case, the collection $\{C_1', ...C_k'\}$ forms a set of independent cycles of larger order, a contradiction.

Therefore, $l(I_j) \geq 4$ for each j, $1 \leq j \leq t$. This implies $|V(C_i)| = \sum_{j=1}^{t} l(I_j) \geq 4t$ or $|N_G(x) \cap V(C_i)| \leq \frac{1}{4}|V(C_i)|$ for all C_i such that $|V(C_i)| \geq 4$. Note that x has at most one adjacency to every 3-cycle in the collection $C_1, \cdots, C_k$.

We may assume $|V(C_1)| = |V(C_2)| = ... = |V(C_s)| = 3$ and $|V(C_i)| \geq 4$ for $s+1 \leq i \leq k$. Then,

$$\begin{aligned} \frac{n-2}{3} &\leq \deg_H x + e(x, D) \\ &\leq (h-1) + s + \frac{1}{4} \sum_{i=s+1}^{k} |V(C_i)| = (h-1) + \frac{|D| + s}{4} \\ &= \frac{n + 3h + s - 4}{4}, \end{aligned}$$

which implies $n \leq 3s + 9h - 4$. Since $s \leq k$ and $h \leq 3$, we have $n \leq 3k + 23$. This contradicts the assumption. Consequently, we know $|V(H)| \geq 4$. ■

Claim 2. For each $x \in V(H)$ and for each $y \in V(H) - \{x\}$, $\deg_{H-x} y \geq 2$.

Proof. Assume $\deg_{H-x} y \leq 1$ for some $y \in V(H) - \{x\}$. As in Claim 1, we count the number of edges from y to D observing that y can have at most one adjacency to a 3-cycle and y is adjacent to at most one out of every four vertices on cycles of length 4 or more.

We may assume $|V(C_1)| = |V(C_2)| = ... = |V(C_s)| = 3$ and $|V(C_i)| \geq 4$ for $s + 1 \leq i \leq k$. Then $e(y, D) \leq s + \frac{1}{4}\sum_{i=s+1}^{k} |V(C_i)| = s + \frac{1}{4}(|D| - 3s) = \frac{1}{4}|D| + \frac{1}{4}s$. Therefore,

$$\begin{aligned}
\frac{n-2}{3} &\leq \deg_H y + \deg_D y \leq \deg_H y + e(y, D) \\
&\leq 1 + \deg_{H-x} y + \frac{1}{4}|D| + \frac{1}{4}s \\
&\leq 1 + \deg_{H-x} y + \frac{1}{4}(n-4) + \frac{1}{4}k \\
&\leq 1 + \deg_{H-x} y + \frac{1}{4}(n-4) + \frac{n-24}{12}.
\end{aligned}$$

Thus, $\deg_{H-x} y \geq 2$. ■

By Claims 1 and 2, we know that for every $x \in V(H)$, $H - x$ contains a cycle, call it C_x.

Claim 3. For every $x \in V(H)$, the set $N_D^+(x)$ is independent.

Proof. Assume, to the contrary, $a_1^+ a_2^+ \in E(G)$ for some $a_1, a_2 \in N_D(x)$. If a_1 and a_2 lie in the same cycle of D, say C_i, then we increase the total order of D by replacing C_i by $C_i' = a_1^+ \overrightarrow{C_i} a_2 x a_1 \overleftarrow{C_i} a_2^+ a_1^+$. If a_1 and a_2 lie in different cycles of D, we may assume without loss of generality $a_i \in V(C_i), i = 1, 2$. Then let $C_1' = C_x$, $C_2' = x a_1 \overleftarrow{C_1} a_1^+ a_2^+ \overrightarrow{C_2} a_2 x$ and for $j \neq 1, 2$ let $C_j' = C_j$. Then the collection $\{C_1', ...C_k'\}$ forms a set of k disjoint cycles of larger total order, a contradiction. ■

From the results in [7], we know that in a claw-free graph of order n, $\alpha(G) \leq 2n/(\delta(G)+2)$. Thus, by Claims 3 and the bound on $\alpha(G)$, for each $x \in V(H)$ we have that

$$|N_D[x]| = |N_D^+(x) \cup \{x\}| \leq \alpha(G) \leq \frac{2n}{\delta(G) + 2} \leq \frac{2n}{\frac{n-2}{3} + 2} < 6.$$

Therefore, $|N_D(x)| \leq 4$ and we have $\deg_H x \geq \frac{n-14}{3}$.

Let P be a longest path in H and let x be one of its end vertices. Then $N_H(x) \subseteq V(P)$ or a longer path is possible. Therefore, if we choose $y \in N_H(x)$ so that $x\overrightarrow{P}y$ is as long as possible, we form a cycle $C = x\overrightarrow{P}yx$ with $N_H(x) \subseteq V(C)$. This implies $|V(C)| \geq \deg_H x + 1 \geq \frac{n-14}{3} + 1 = \frac{n-11}{3}$. Then by the maximality of D, we know $|V(C_i)| \geq \frac{n-11}{3}$, for all $1 \leq i \leq k$.

Claim 4. The number of independent cycles, k, is 2.

Proof. Assume $k \geq 3$. Then $n = |V(G)| \geq |V(C)| + |V(C_1)| + |V(C_2)| + |V(C_3)| \geq 4(\frac{n-11}{3})$. This forces $n \leq 44$, a contradiction. ■

Since C_1 and C_2 each have at least $\frac{n-11}{3}$ vertices, we know

$$|V(H)| \leq n - |V(C_1)| - |V(C_2)| \leq \frac{n+22}{3}.$$

Claim 5. The subgraph H is hamiltonian connected.

Proof. If H is not hamiltonian-connected, then by a result in [6],

$$\frac{n-14}{3} \leq \delta(H) \leq \frac{1}{2}|V(H)| \leq \frac{n+22}{6}.$$

This forces $n \leq 50$, a contradiction. ■

In particular, H has a hamiltonian cycle, say C_0. By the maximality of D, we know $|V(C_0)| \leq |V(C_i)|$ for $i = 1, 2$. Thus, $|V(C_0)| \leq \frac{1}{3}n$.

Since G is 2-connected, there exist at least two independent edges between C_0 and $C_1 \cup C_2$.

Claim 6. There do not exist two independent edges from C_0 to C_i, for $i = 1, 2$.

Proof. Without loss of generality, let $i = 1$. Assume there are two independent edges, say a_1b_1 and a_2b_2 between C_0 and C_1 (where $a_1, a_2 \in C_0$, $b_1, b_2 \in C_1$). Without loss of generality, we may assume $l(b_1\overrightarrow{C_1}b_2) \geq \frac{1}{2}|V(C_1)|$. Since $\{a_2\overrightarrow{P}a_1b_1\overrightarrow{C_1}b_2a_2, C_2\}$ forms a set of disjoint cycles where P is a hamiltonian a_1, a_2-path in H, we know $l(b_2\overrightarrow{C_1}b_1) \geq |V(C_0)| + 1 \geq \delta(H) + 2 \geq \frac{n-8}{3}$. Then $|V(C_1)| \geq 2l(b_2\overrightarrow{C}b_1) \geq \frac{2n-16}{3}$. Therefore,

$$n = |V(C_0)| + |V(C_1)| + |V(C_2)| \geq 2\left(\frac{n-11}{3}\right) + \frac{2n-16}{3} = \frac{4n-38}{3}.$$

This forces $n \leq 38$ which is a contradiction. ■

Therefore we may assume $a_1b_1, a_2b_2 \in E(G)$ where $a_1, a_2 \in V(C_0)$, $a_1 \neq a_2$, $b_1 \in V(C_1)$, and $b_2 \in V(C_2)$. As a consequence of Claim 7 and 2-connectivity, we know there exists an edge $d_1d_2 \in E(G)$ such that $d_1 \in V(C_1) - b_1$ and $d_2 \in V(C_2)$.

Let $x \in H - \{a_1, a_2\}$. (Since $|V(H)| = |V(C_0)| \geq \frac{n-11}{3}$ we know such an x exists.) Then by Claim 6, $N_{C_1 \cup C_2}(x) \subset \{b_1, b_2\}$. Therefore, $\deg_H x \geq \frac{n-2}{3} - 2 = \frac{n-8}{3}$, and hence $|V(C_0)| \geq \frac{n-5}{3}$.

Claim 7. The graph $H - \{a_1, a_2\}$ has a triangle T and $H - V(T)$ is hamiltonian-connected.

Proof. Let $H' = H - \{a_1, a_2\}$ and assume $\delta(H') \leq \frac{|V(H')|}{2}$. Since $\delta(H') \geq \delta(H) - 2 \geq \frac{n-8}{3} - 2 \geq \frac{n-14}{3}$ and $|V(H')| \leq \frac{n}{3} - 2 = \frac{n-6}{3}$, we get $\frac{n-14}{3} \leq \frac{1}{2}\left(\frac{n-6}{3}\right)$. This forces $n \leq 18$, a contradiction.

Thus $\delta(H') \geq \frac{|V(H')|+1}{2}$ and $|V(H')| \geq \frac{n-5}{3} - 2 \geq 3$, which implies by [1] that H' is pancyclic. Thus H' has a triangle T. Let $H'' = H - V(T)$. Then $|V(H'')| = |V(H)| - 3 \leq \frac{n}{3} - 3 = \frac{n-9}{3}$ and $\delta(H'') \geq \frac{n-14}{3} - 3 \geq \frac{n-23}{3}$. Therefore, since $n \geq 51$, $\delta(H'') > \frac{1}{2}|V(H'')|$. Hence, by [6] H'' is hamiltonian connected. ■

First, suppose $d_2 \neq b_2$. We may assume $l(d_1\overrightarrow{C_1}b_1) \leq \frac{1}{2}(|V(C_1)|)$ and $l(b_2\overrightarrow{C_2}d_2) \leq \frac{1}{2}(|V(C_2)|)$. By the maximality of C_1 and C_2 and the fact that G is claw-free, $b_1^+b_1^-, b_2^+b_2^- \in E(G)$. Let $C' = a_1b_1b_1^-b_1^+\overrightarrow{C_1}d_1d_2\overrightarrow{C_2}b_2^-b_2^+b_2a_2Pa_1$, where P is a hamiltonian a_1a_2-path in $H - T$. Since C' and T are disjoint cycles, $l(d_1^+\overrightarrow{C_1}b_1^{--}) + l(b_2^{++}\overrightarrow{C_2}d_2^-) + 2 \geq |V(H)|$. Thus $\frac{|V(C_1)|+|V(C_2)|}{2} - 4 \geq |V(H)| \geq \frac{n-5}{3}$, which implies that $|V(C_1)| + |V(C_2)| \geq \frac{2n+14}{3}$. Since $|V(H)| = |V(C_0)| \geq \frac{n-5}{3}$, we have $n = |V(H)| + |V(C_1)| + |V(C_2)| \geq \frac{3n+9}{3} = n+3$, a contradiction. Therefore, we know $d_2 = b_2$ which implies that there cannot be three independent edges between the cycles C, C_1, and C_2.

Since G is 2-connected, there exists an edge $b_2'u$ from $C_2 - \{b_2\}$ to $C_0 \cup C_1$

Case 1. We consider the case where $u \in C_0$. If $u \neq a_1$ the three edges a_1b_1, d_1b_2, and $b_2'u$ are independent, a contradiction. Thus, $u = a_1$. But now the two edges a_2b_2 and a_1b_2' between C_0 and C_2 are independent. This contradicts Claim 7.

Case 2. We consider the case where $u \in C_1$. If $u \neq b_1$, then the three edges a_1b_1, ub_2', and a_2b_2 are independent, a contradiction. If $u = b_1$, consider b_1 and $\{a_1, b_1^+, b_2'\}$. We know $b_2'b_1^+ \notin E(G)$ because $u = b_1$.

By Claim 7, $a_1b_2' \notin E(G)$. If $a_1b_1^+ \in E(G)$, then the three edges $a_1b_1^+, b_1b_2'$ and a_2b_2 are independent, a contradiction. Thus, $\langle b_1, b_1^+, a_1, b_2' \rangle_G$ is a claw, a contradiction.

Hence, in all cases we reach a contradiction, and the result is proved. ■

References

[1] J.A. Bondy, *Pancyclic Graphs* I, J. Combin. Theory (B) **11** (1971) 80–84.

[2] S. Brandt, G. Chen, R.J. Faudree, R.J. Gould and L. Lesniak, *On the Number of Cycles in a* 2*-Factor*, J. Graph Theory, **24** (1997) 165–173.

[3] G. Chartrand and L. Lesniak, Graphs & Digraphs (Chapman and Hall, London, 3rd edition, 1996).

[4] R.J. Gould, *Updating the Hamiltonian Problem — A Survey*, J. Graph Theory **15** (1991) 121–157.

[5] M.M. Matthews and D.P. Sumner, *Longest Paths and Cycles in* $K_{1,3}$*-Free Graphs*, J. Graph Theory **9** (1985) 269–277.

[6] O. Ore, *Hamiltonian Connected Graphs*, J. Math. Pures. Appl. **42** (1963) 21–27.

[7] H. Li and C. Virlouvet, *Neighborhood Conditions for Claw-free Hamiltonian Graphs*, Ars Combinatoria **29** (A) (1990) 109–116.

Received 19 February 1999
Revised 24 January 2000