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SOME RESULTS CONCERNING THE ENDS OF MINIMAL CUTS OF SIMPLE GRAPHS

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Abstract

Let S be a cut of a simple connected graph G. If S has no proper subset that is a cut, we say S is a minimal cut of G. To a minimal cut S, a connected component of $G - S$ is called a fragment. And a fragment with no proper subset that is a fragment is called an end. In the paper ends are characterized and it is proved that to a connected graph $G = (V, E)$, the number of its ends $\Sigma \leq |V(G)|$.

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In this paper $G = (V, E)$ will always denote finite non-complete connected graph. The notations not mentioned are the same with those in reference [1]. For $A \subset V(G)$ we use $\Gamma(A)$ to denote the adjacent set of A, that is, $\Gamma(A) = \{v|uv \in E(G),\ u \in A\}$, and we put $N(A) = \Gamma(A) - A$. $\langle A\rangle$ is a subgraph induced by $A \subset V(G)$, that is, $\langle A\rangle = G[A]$. A set of vertices $S \subset V(G)$ is called a *cut* of G if there are at least two connected components in $G - S$. A *minimal cut* is a cut without a proper subset that is a cut. If S is a minimal cut, then a connected component of $G - S$ is called a *fragment*. And a fragment with no proper subset that is a fragment is called an *end*. It is obvious that the graphs we discuss all have cuts and furthermore, minimal cuts. Thus a graph has at least two fragments. Since all fragments have ends, a graph has at least two ends.

It is obvious that:

Proposition P. *Let S be a cut of G. Then S is a minimal cut if and only if for any $u \in S$ and a connected component of $G - S$, say $\langle A \rangle$, $N(u) \cap A \neq \emptyset$.*

Definition 1. If S_1 and S_2 are two minimal cuts of G and there are at least two connected components of $G - S_1$ which contain vertices of S_2, then S_1 *interferes* with S_2.

Theorem 2. *Let S_1 and S_2 be two minimal cuts of G and S_1 interferes with S_2, then there exist vertices of S_1 in every fragment of $G - S_2$.*

Proof. Suppose $\langle A \rangle$ is a fragment of $G - S_2$ and $A \cap S_1 = \emptyset$, then because $\langle A \rangle$ is connected, all the vertices in A must belong to a fragment of $G - S_1$. By Proposition P, for each $v \in S_2 - S_1$, $N(v) \cap A \neq \emptyset$, then $\langle A \cup (S_2 - S_1) \rangle$ is connected. Thus the vertices which belong to $S_2 - S_1$ can only be in one fragment of $G - S_1$, and it contradicts the fact that S_1 interferes with S_2.

By Theorem 2, S_1 interferes with S_2 and S_2 interferes with S_1 are equivalent assertions.

Theorem 3. *Let $\langle A \rangle$ be an end of G, then for any $u \in A$, every minimal cut of G that contains u interferes with $N(A)$.*

Proof. Suppose S is a minimal cut of G that contains u and does not interfere with $N(A)$, then by applying Theorem 2 and Definition 1, $S - N(A) \subset A$. And since $N(u) \subset (A \cup N(A))$, from Proposition P, every fragment in $G - S$ contains a vertex in A or $N(A)$. Since S does not interfere with $N(A)$, there is at least one of such fragments that does not contain any vertex in $N(A)$. Thus it contains vertices in A and only in A. But this fragment does not contain u, then it is a proper subset of A, contradicting the fact that $\langle A \rangle$ is an end.

Corollary 4. *Let S be a minimal cut of G. If G does not contain any minimal cuts that interfere with S, then a vertex in S cannot belong to any end of G.*

Theorem 5. *Suppose $\langle A \rangle$ is a fragment of G, then $\langle A \rangle$ is also an end of G if and only if $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$.*

Proof. (a) From Theorem 3, if $\langle A \rangle$ is an end of G, then $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$.

(b) For every non-empty proper subset of A, say A', since $N(A') \subset A \cup N(A)$, and $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$, then $N(A')$ is not a minimal cut, thus $\langle A' \rangle$ is not a fragment and $\langle A \rangle$ is an end.

Theorem 6. *Let $\langle A \rangle$ be a fragment of G, then $\langle A \rangle$ is an end if and only if for any $u \in A$ and $v \in N(A)$, $uv \in E(G)$.*

Proof. (a) Suppose A is an end. There are $u \in A$, $v \in N(A)$ and $uv \notin E$. Consider all the $u - v$ paths in G. There must exist vertices of $(A \cup N(A)) - \{u, v\}$ in every such path. Delete all the vertices in $(A \cup N(A)) - \{u, v\}$ from each path, then we obtain a disconnected graph with no $u - v$ paths, thus the vertices we deleted are a cut of G. But this cut is contained by $A \cup N(A)$ and it is not $N(A)$, contradicting Theorem 5.

(b) Let A' be a proper subset of A. It is obvious that $N(A') \subset (A \cup N(A))$ and $N(A') \cap (A - A') \neq \emptyset$ under the conditions of the theorem, then $N(A')$ is not a minimal cut. Thus $\langle A' \rangle$ is not a fragment, then $\langle A \rangle$ is an end.

Corollary 7. *Let $\langle A \rangle$ be an end of G, then all the minimal cuts that interfere with $N(A)$ contain A.*

Proof. Let S be a minimal cut that interferes with $N(A)$. If there exist $v \in A$ and $v \notin S$, then there must exist a fragment of $G - S$ that contains at least one vertex $u \in N(A)$ and does not contain vertex v, which contradicts Theorem 6.

Theorem 8. *Let $\langle A \rangle$ be an end of G and $\langle B \rangle$ be a fragment of G that does not contain A, then $A \cap B = \emptyset$.*

Proof. Under the conditions of the theorem, if $A \cap B \neq \emptyset$, then since $A - B \neq \emptyset$ and $\langle A \rangle$ is connected, we have $N(B) \cap A \neq \emptyset$.

Thus, if $N(B)$ interferes with $N(A)$, by applying Corollary 7, $A \subset N(B)$. It contradicts $A \cap B \neq \emptyset$. If $N(B)$ does not interfere with $N(A)$, from $N(B) \cap A \neq \emptyset$ we have $N(B) \subset (A \cup N(A))$. If $N(B) \neq N(A)$, it will contradict Theorem 5.

From Theorem 8 we know, for any two distinct ends of G, $\langle A \rangle$ and $\langle B \rangle$, there must be $A \cap B = \emptyset$. Thus by denoting the number of distinct ends of G as Σ, there is

Corollary 9. $\Sigma \leq |V(G)|$.

References

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