Discussiones Mathematicae
Graph Theory 20 (2000) 129–138

GEODETIC SETS IN GRAPHS

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Abstract

For two vertices u and v of a graph G, the closed interval $I[u,v]$ consists of u, v, and all vertices lying in some $u-v$ geodesic in G. If S is a set of vertices of G, then $I[S]$ is the union of all sets $I[u,v]$ for $u, v \in S$. If $I[S] = V(G)$, then S is a geodetic set for G. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set. A set S of vertices in a graph G is uniform if the distance between every two distinct vertices of S is the same fixed number. A geodetic set is essential if for every two distinct vertices $u, v \in S$, there exists a third vertex w of G that lies in some $u-v$ geodesic but in no $x-y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$. It is shown that for every integer $k \geq 2$, there exists a connected graph G with $g(G) = k$ which contains a uniform, essential minimum geodetic set. A minimal geodetic set S has no proper subset which is a geodetic set. The maximum cardinality of a minimal geodetic set is the upper geodetic number $g^+(G)$. It is shown that every two integers a and b with $2 \leq a \leq b$ are realizable as the geodetic and upper geodetic numbers, respectively, of some graph and when $a < b$ the minimum order of such a graph is $b + 2$.

Keywords: geodetic set, geodetic number, upper geodetic number.

AMS Subject Classification: 05C12.

[1]Research supported in part by the Western Michigan University Faculty Research and Creative Activities Grant.

1 Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* , *rad* G, and the maximum eccentricity is its *diameter*, *diam* G. A $u-v$ path of length $d(u, v)$ is also referred to as a $u-v$ *geodesic*. Please see the books [2, 5] for graph notation and terminology. We define the *closed interval* $I[u, v]$ as the set consisting of u, v, and all vertices lying in some $u-v$ geodesic of G, and for a nonempty subset S of $V(G)$,

$$I[S] = \bigcup_{u,v \in S} I[u, v].$$

The set S is *convex* if $I[S] = S$. A set S of vertices of G is defined in [1, 3] to be a *geodetic set* in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a *minimum geodetic set*. The cardinality of a minimum geodetic set in G is the *geodetic number* $g(G)$.

The graph G_1 of Figure 1 has geodetic number 2 as $S_1 = \{w_1, y_1\}$ is the unique minimum geodetic set of G_1. On the other hand, each 2-element subset S of the vertex set of G_2 has the property that $I[S]$ is properly contained in $V(G_2)$. Thus $g(G_2) \geq 3$. Since $S_2 = \{u_2, v_2, x_2\}$ is a geodetic set, $g(G_2) = 3$.

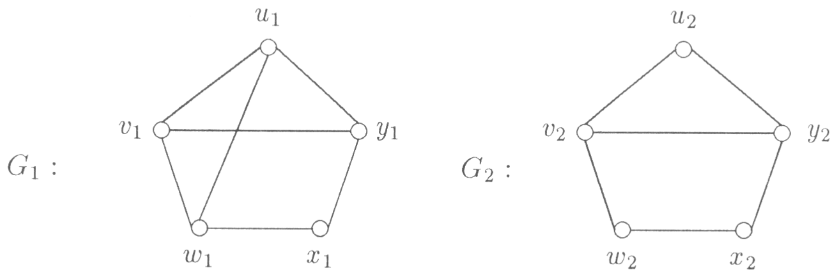

Figure 1. Illustrating the geodetic number

The closed intervals $I[u, v]$ in a connected graph G were studied and characterized by Nebeský [7, 8] and were also investigated extensively in the book by Mulder [6], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The intervals of an oriented graph have been studied in [4].

2 Uniform and Essential Minimum Geodetic Sets

A graph F is called a *minimum geodetic subgraph* if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum geodetic set in G. Those graphs that are minimum geodetic subgraphs were characterized in [1].

Theorem A. *A nontrivial graph F is a minimum geodetic subgraph if and only if every vertex of F has eccentricity* 1 *or no vertex of F has eccentricity* 1.

As a consequence of this theorem, there exists a graph G containing a minimum geodetic set S such that $\langle S\rangle$ is complete or S is independent. In the former case, $d_G(u,v)=1$ for all distinct $u,v\in S$; while in the latter case, $d_G(u,v)\geq 2$ for all distinct $u,v\in S$. This is illustrated in Figure 2.

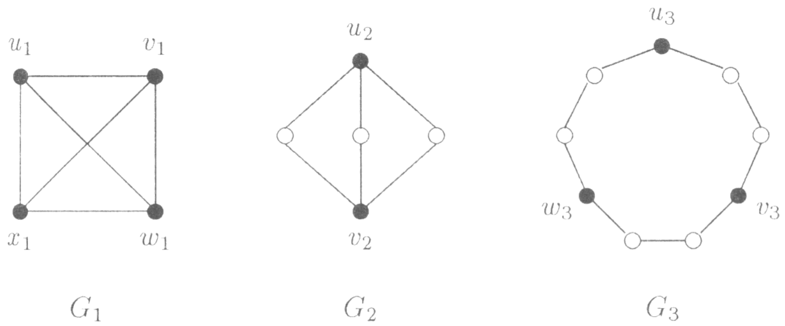

Figure 2. Graphs with uniform minimum geodetic sets

The graphs G_1, G_2, and G_3 in Figure 2 contain minimum geodetic sets $S_1=\{u_1,v_1,w_1,x_1\}$, $S_2=\{u_2,v_2\}$, and $S_3=\{u_3,v_3,w_3\}$, respectively, with an added property. For every two distinct vertices $y,z\in S_i$, $i=1,2,3$, $d_{G_i}(y,z)=i$. This suggests the following definition. A set S of vertices in a connected graph G is *uniform* if the distance between every two vertices of S is the same fixed number. Obviously, if S is uniform, then $\langle S\rangle$ is complete or S is independent. Hence each minimum geodetic set indicated in Figure 2 is uniform.

We define a geodetic set S to be *essential* if for every two vertices u,v in S, there exists a vertex $w\neq u,v$ of G that lies in a $u-v$ geodesic but in no $x-y$ geodesic for $x,y\in S$ and $\{x,y\}\neq\{u,v\}$. For example the set $S=\{x,y,z\}$ is an essential geodetic set of the graph G of Figure 3, while S is not uniform in G.

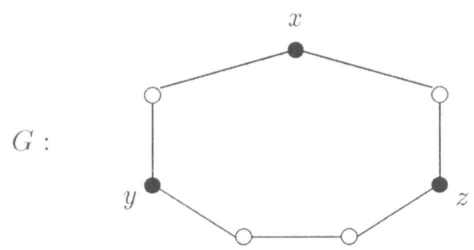

Figure 3. A graph G with an essential geodetic set

We now show that it is possible for a graph to have a minimum geodetic set with a specified number of vertices designated as essential as well as uniform.

Theorem 21. *For each integer $k \geq 2$, there exists a connected graph G with $g(G) = k$ which contains a uniform, essential minimum geodetic set.*

Proof. Let $K_k^{(k-1)}$ denote the multigraph of order k for which every two vertices of $K_k^{(k-1)}$ are joined by $k-1$ edges. Let $G_k = S\left(K_k^{(k-1)}\right)$ be the subdivision graph of $K_k^{(k-1)}$. Clearly *diam* $G_k = 3$ if $k \geq 3$. We show by induction that $g(G_k) = k$ and $V\left(K_k^{(k-1)}\right)$ is a uniform, essential minimum geodetic set for G_k.

To begin the inductive proof, for $k = 2$, the graph $G_2 = S\left(K_2^{(1)}\right)$ is a path of order 3. Therefore, $g(G_2) = 2$ and the two end-vertices of G_2 form a uniform, essential minimum geodetic set for G_2. Now we take $g(G_{k-1}) = k-1$, where $k-1 \geq 2$, and $V\left(K_{k-1}^{(k-2)}\right)$ is a uniform, essential minimum geodetic set for G_{k-1}. We now consider G_k.

Let $S = V\left(K_k^{(k-1)}\right) = \{v_1, v_2, \cdots, v_k\}$. For each pair i, j, $1 \leq i < j \leq k$, label the $k-1$ vertices of degree 2 that are adjacent to both v_i and v_j by $v_{i,j}^1, v_{i,j}^2, \cdots, v_{i,j}^{k-1}$. Since $I[S] = V(G_k)$, it follows that $g(G_k) \leq k$.

Suppose, to the contrary, that $g(G_k) = m < k$ and let $W = \{w_1, w_2, \cdots, w_m\}$ be a minimum geodetic set of G_k. We consider three cases.

Case 1. *W is a proper subset of* $\{v_1, v_2, \cdots, v_k\}$. Then $I[W] = V(G_m)$, where $G_m = S\left(K_m^{m-1}\right)$ with $V\left(K_m^{m-1}\right) = W$. Therefore, $I[W] \neq V(G_k)$, contradicting the fact that W is a geodetic set of G_k.

Case 2. $W = \{v_{i,j}^1, v_{i,j}^2, \cdots, v_{i,j}^{k-1}\}$ *where* $1 \leq i < j \leq k$. Then $I[W] = W \cup \{v_i, v_j\} \subset V(G_k)$, once again contradicting the fact that W is a geodetic set of G_k.

Case 3. *There exist integers i, j, p, q, where $1 \le i < j \le k$ and $1 \le p < q \le k-1$, such that $v_{i,j}^p \in W$ and $v_{i,j}^q \notin W$.* Since $I[W] = V(G_k)$, there exist $x, y \in W$ such that $v_{i,j}^q$ lies on an $x-y$ geodesic in G_k. Since $v_{i,j}^q \notin W$, it follows that $2 \le d(x,y) \le 3$.

Suppose first that $d(x,y) = 2$. We show that

$$I[W] = I\left[W - \{v_{i,j}^p\}\right].$$

In this case, $\{x,y\} = \{v_i, v_j\}$, say $x = v_i$ and $y = v_j$. So $v_{i,j}^q$ lies in the geodesic $x, v_{i,j}^q, y$ in G_k. It follows that $v_{i,j}^p$ lies in the geodesic $x, v_{i,j}^p, y$ in G_k, so $v_{i,j}^p \in I[x,y]$. Let $v \notin W$ be a vertex that lies in some $v_{i,j}^p - w$ geodesic in G_k, where $w \in W$. If $d(v_{i,j}^p, w) = 2$, then $v \in \{x, y\}$. This contradicts the fact that $v \notin W$, so $d(v_{i,j}^p, w) = 3$. Thus v lies in either the geodesic v_i, v, w or in the geodesic v_j, v, w in G_k. Therefore, $I[W] = I\left[W - \{v_{i,j}^p\}\right]$, contradicting the fact that W is a minimum geodetic set of G_k.

Suppose next that $d(x,y) = 3$. We show that a geodetic set W' of a graph G_{k-1} can be formed from W, where $|W'| \le k-2$ and which will contradict the induction hypothesis.

In this case, exactly one of x and y belongs to $\{v_i, v_j\}$, say $x = v_i$ and $y \ne v_j$. Then y is a subdivision vertex, so $\deg y = 2$ in G_k, and $v_{i,j}^q$ lies in the $x-y$ geodesic $x, v_{i,j}^q, v_j, y$ in G_k. This implies that $v_{i,j}^p$ also lies in an $x-y$ geodesic, namely the geodesic $x, v_{i,j}^p, v_j, y$, in G_k. So $v_{i,j}^p \in I[x,y]$. Now let $v \notin W$ be a vertex that lies in some $v_{i,j}^p - w$ geodesic in G_k, where $w \in W$. If $d(v_{i,j}^p, w) = 2$, then $v = v_j$. This implies that v lies in the $x-y$ geodesic $x, v_{i,j}^p, v, y$ in G_k, so $v \in I[x,y]$ and $d(v_{i,j}^p, w) = 3$. Then $w \in \{v_1, v_2, \cdots, v_k\}$, say $w = v_h$. Let

$$W' = W - W \bigcap \{v_{i,j}^\ell, v_{j,h}^\ell : 1 \le \ell \le k-1\}.$$

Since $v_{i,j}^p, y \in W \bigcap \{v_{i,j}^\ell, v_{j,h}^\ell : 1 \le \ell \le k-1\}$, it follows that $|W'| \le k-2$. Let $G_{k-1} = S\left(K_{k-1}^{(k-2)}\right)$, where $V\left(K_{k-1}^{(k-2)}\right) = \{v_1, v_2, \cdots, v_{j-1}, v_{j+1}, \cdots, v_k\}$. We show that $I[W'] = V(G_{k-1})$, contradicting the induction hypothesis.

Let $v \notin W'$ be a vertex of G_{k-1}. Since $I[W] = V(G_k)$, it follows that v lies in some $u-w$ geodesic P in G_k, where $u, w \in W$. Observe that at least one of u, w must be in W', for otherwise, P contains no vertex in G_{k-1}. Assume first that $u, w \in W'$. Then P is also a geodesic in G_{k-1} giving the desired result. Therefore, exactly one of u and w belongs to W', say $w \in W'$. If $d(u,w) = 2$, then $v \in \{v_i, v_h\}$, contradicting $v \notin W'$, therefore $d(u,w) = 3$. Then v lies in either the geodesic v_i, v, w, or in the geodesic v_h, v, w in G_{k-1}. It follows that $I[W'] = V(G_{k-1})$, which contradicts the induction hypothesis.

Therefore $S = V\left(K_k^{(k-1)}\right)$ is a minimum geodetic set of G_k. Then $v_{i,j}^{\ell}$, where $1 \leq i < j \leq k$ and $1 \leq \ell \leq k-1$, lies in exactly one geodesic, namely the geodesic $v_i, v_{i,j}^{\ell}, v_j$, in G_k. Moreover, $d(u,w) = 2$ for all $u, w \in S$. Therefore, S is a uniform, essential minimum geodetic set for G_k. ■

3 Minimal Geodetic Sets

A geodetic set S in a connected graph G is called a *minimal geodetic set* if no proper subset of S is a geodetic set. Of course, every minimum geodetic set is a minimal geodetic set, but the converse is not true. For example, let $G = K_{2,3}$ of Figure 4 with partite sets $V_1 = \{x, y\}$ and $V_2 = \{u, v, w\}$. Then $\{u, v, w\}$ is a minimal geodetic set of $K_{2,3}$ but is not a minimum geodetic set of $K_{2,3}$ since $\{x, y\}$ is its unique minimum geodetic set. We define the *upper geodetic number* $g^+(G)$ as the maximum cardinality of a minimal geodetic set of G. Obviously, $g(G) \leq g^+(G)$. The graph G of Figure 4 has geodetic number 2 and upper geodetic number 3.

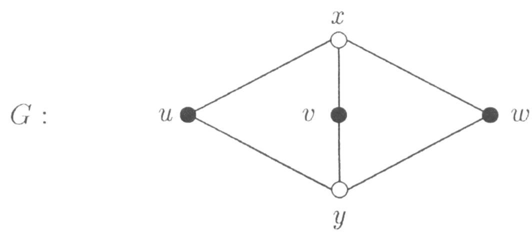

Figure 4. A graph G with a minimal geodetic set

We now show that every two integers a and b with $2 \leq a \leq b$ are realizable as the geodetic number and upper geodetic number, respectively, of some graph. Furthermore, we determine the minimum order of such a graph. Certainly, this minimum order is at least b. Indeed, if $a = b$, then the only geodetic set of K_b is its vertex set; so $g(K_b) = g^+(K_b) = b$ and the minimum order is b. Indeed, if G is a graph of order n with $g^+(G) = n$, then $G = K_n$ and so $g(G) = g^+(G)$. Before taking this observation one step further, we present a lemma.

Lemma 3.1. *Let G be a nontrivial connected graph of order n with $g^+(G) = n-1$ and let S be a minimal geodetic set of maximum cardinality such that $V(G) - S = \{v\}$. Then G does not contain nonadjacent vertices $u, w \in S$ such that u and w are mutually adjacent to both v and some vertex of S.*

Proof. Suppose, to the contrary, that there exist vertices $x, y, z \in S$ such that $xy \notin E(G)$ and x and y are mutually adjacent to both v and z. Then z lies in the geodesic x, z, y, while v lies in the geodesic x, v, y. Hence $S - \{z\}$ is a geodetic set, contradicting the minimality of S. ■

Theorem 3.2. *Let G be a nontrivial connected graph of order n. If $g^+(G) = n-1$, then $g(G) = g^+(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, where $S = \{v_1, v_2, \cdots, v_{n-1}\}$ is a minimal geodetic set of maximum cardinality. First, we claim that every vertex in S is adjacent to v_n. Suppose, to the contrary, that some $v \in S$ is not adjacent to v_n. Among the pairs x, y of distinct vertices of S for which v lies in some $x - y$ geodesic, we choose a pair such that $d(x, y)$ is minimum. If $v \neq x, y$, then v_n lies in some $u - w$ geodesic of length 2, where $u, w \in S$ and $u, w \neq v$. This implies that $S - \{v\}$ is a geodetic set, a contradiction. Therefore, either $x = v$ or $y = v$, say the former. We consider two cases.

Case 1. $yv_n \in E(G)$. Then there are two subcases.

Subcase 1.1. Among the vertices of S adjacent to v_n, there exists some vertex z not adjacent to y.

Here v_n lies in the geodesic y, v_n, z in G. By Lemma 3.1, $xz \notin E(G)$. Since $P : x, y, v_n, z$ is a path in G, it follows that $d(x, z) \leq 3$. Assume first that $d(x, z) = 2$. Then there exists a vertex $w \in S$ adjacent to both x and z. By Lemma 3.1, $wy \notin E(G)$. Then x lies in the geodesic y, x, w in G, implying that $S - \{x\}$ is a geodetic set, producing a contradiction. Therefore, $d(x, z) = 3$. Thus P is a geodesic and $S - \{y\}$ is a geodetic set, which is a contradiction.

Subcase 1.2. Every vertex of S that is adjacent to v_n is also adjacent to y. Since v_n lies in some $u - w$ geodesic for $u, w \in S$, it follows that $\deg v_n \geq 3$. Necessarily, $uw \notin E(G)$, this is impossible by Lemma 3.1.

Case 2. $yv_n \notin E(G)$.
Then v_n lies in some $u - v$ geodesic of length 2. By Lemma 3.1, y is not adjacent to both u and v, say $yu \notin E(G)$. Let $d(y, u) = \ell$ and let $y =$

$w_0, w_1, w_2, \cdots, w_\ell = u$ be a $y-u$ geodesic. Since $yv_n \notin E(G)$, it follows that $w_1 \neq v_n$. If $w_1 \neq v$, then $S-\{w_1\}$ is a geodetic set, which is a contradiction. Thus $w_1 = v$. Then y, v, v_n, u is a geodesic and $S - \{v\}$ is a geodetic set, contrary to hypothesis.

This completes the proof of the claim. Therefore, for every pair x, y of nonadjacent vertices in S, the vertex v_n lies in the geodetic x, v_n, y. Clearly, $diam(G) = 2$.

Next we show that

$$G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$$

where $n_1, n_2, \cdots, n_r, r$ are positive integers with $n_1 + n_2 + \cdots + n_r = n - 1$ and $V(K_1) = \{v_n\}$, which implies that $g(G) = g^+(G) = n - 1$. Suppose, to the contrary, that this is not the case. Then there exist $x, y, z \in S$ such that $d(x, y) = 2$ and $xz, zy \in E(G)$. It follows that z and v_n both lie in some $x - y$ geodesic. So $S - \{z\}$ is a geodetic set, which is a contradiction. ■

We can now complete the proof of the realizability of every two integers a and b with $2 \leq a \leq b$ as the geodetic number and upper geodetic number, respectively, of some graph.

Theorem 3.3. *For every two positive integers a and b, where $2 \leq a < b$, there exists a graph G with $g(G) = a$ and $g^+(G) = b$.*

Proof. Let $F = \overline{K}_{b-a+1} + \overline{K}_2$, where $V(K_{b-a+1}) = \{v_1, v_2, \cdots, v_{b-a+1}\}$ and $V(K_2) = \{x, y\}$. The graph G is formed from F by adding $a-1$ pendant edges yu_i $(1 \leq i \leq a-1)$ to the vertex y of F (see Figure 5). The graph G has the unique minimum geodetic set $S = \{x, u_1, u_2, \cdots, u_{a-1}\}$ and so $g(G) = a$.

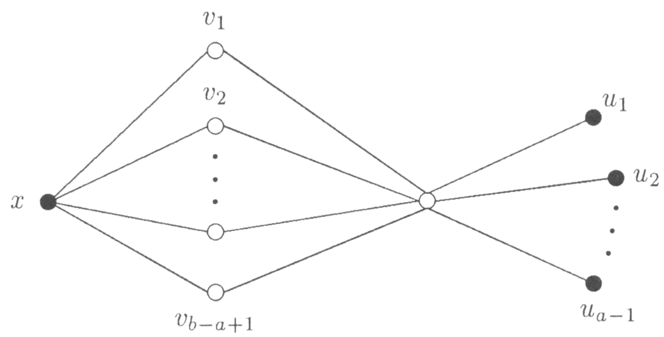

Figure 5. A graph G with $g(G) = a$ and $g^+(G) = b$

Now let

$$S' = \{u_1, u_2, \cdots, u_{a-1}, v_1, v_2, \cdots, v_{b-a+1}\}.$$

Then $I[S'] = V(G)$. We show that S' is a minimal geodetic set of G. Let $v \in S'$. We show that $I[S' - \{v\}] \neq V(G)$. Assume first that $v = u_i$ for some i $(1 \leq i \leq a-1)$. Then $I[S' - \{u_i\}] = V(G) - \{u_i\}$. So $v = v_j$ for some j $(1 \leq j \leq b-a+1)$. Then $I[S' - \{v_j\}] = V(G) - \{v_j\}$. It follows that $I\,[S' - \{v\}] \neq V(G)$ for every $v \in S'$. Since $|S'| = b$, we have that $g^+(G) \geq b$.

Next we show that there is no minimal geodetic set W of G with $|W| > b$, which implies that $g^+(G) = b$. Note that the graph G has order $n = b+2$. Since $g(G) = a < b$, it suffices to show that G does not contain an $(n-1)$-element minimal geodetic set. Suppose, to the contrary, that W is a minimal geodetic set of G where $|W| = n-1$. Let $v \notin W$. Since every geodetic set of G must contain all end-vertices of G, it follows that $v = x$, for otherwise, the geodetic set $S = \{x, u_1, u_2, \cdots, u_{a-1}\}$ is a proper subset of W, which contradicts the fact that W is minimal. Then $y \in W$. It follows that $I[W] = I\,[W - \{y\}] = V(G)$. Once again, this contradicts W being a minimal geodetic set of G. ■

The proof of Theorem 3.3 shows that if $b - a \geq 2$ and k is an integer with $a < k < b$, then there need not be a graph G with $g(G) = a$ and $g^+(G) = b$ containing a minimal geodetic set of cardinality k, that is, a graph G need not contain an 'intermediate' minimal geodetic set.

The following corollary gives the smallest order of a graph satisfying the hypothesis of Theorem 3.3. The proof is a direct consequence of Theorem 3.2 and 3.3.

Corollary 3.4. *For every two positive integers a and b, where $2 \leq a < b$, the smallest order of a graph G with $g(G) = a$ and $g^+(G) = b$ is $b + 2$.*

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Received 27 September 1999
Revised 26 January 2000