Discussiones Mathematicae
Graph Theory 20 (2000) 129–138

GEODETIC SETS IN GRAPHS

Gary Chartrand

Department of Mathematics and Statistics
Western Michigan University, Kalamazoo, MI 49008, USA

Frank Harary

Department of Computer Science
New Mexico State University, Las Cruces, NM 88003, USA

and

Ping Zhang[1]

Department of Mathematics and Statistics
Western Michigan University, Kalamazoo, MI 49008, USA

Abstract

For two vertices u and v of a graph G, the closed interval $I[u,v]$ consists of u, v, and all vertices lying in some $u-v$ geodesic in G. If S is a set of vertices of G, then $I[S]$ is the union of all sets $I[u,v]$ for $u,v \in S$. If $I[S] = V(G)$, then S is a geodetic set for G. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set. A set S of vertices in a graph G is uniform if the distance between every two distinct vertices of S is the same fixed number. A geodetic set is essential if for every two distinct vertices $u,v \in S$, there exists a third vertex w of G that lies in some $u-v$ geodesic but in no $x-y$ geodesic for $x,y \in S$ and $\{x,y\} \neq \{u,v\}$. It is shown that for every integer $k \geq 2$, there exists a connected graph G with $g(G) = k$ which contains a uniform, essential minimum geodetic set. A minimal geodetic set S has no proper subset which is a geodetic set. The maximum cardinality of a minimal geodetic set is the upper geodetic number $g^+(G)$. It is shown that every two integers a and b with $2 \leq a \leq b$ are realizable as the geodetic and upper geodetic numbers, respectively, of some graph and when $a < b$ the minimum order of such a graph is $b+2$.

Keywords: geodetic set, geodetic number, upper geodetic number.

AMS Subject Classification: 05C12.

---

[1]Research supported in part by the Western Michigan University Faculty Research and Creative Activities Grant.

1 Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* , *rad* G, and the maximum eccentricity is its *diameter*, *diam* G. A $u-v$ path of length $d(u, v)$ is also referred to as a $u-v$ *geodesic*. Please see the books [2, 5] for graph notation and terminology. We define the *closed interval* $I[u, v]$ as the set consisting of u, v, and all vertices lying in some $u-v$ geodesic of G, and for a nonempty subset S of $V(G)$,

$$I[S] = \bigcup_{u,v \in S} I[u, v].$$

The set S is *convex* if $I[S] = S$. A set S of vertices of G is defined in [1, 3] to be a *geodetic set* in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a *minimum geodetic set*. The cardinality of a minimum geodetic set in G is the *geodetic number* $g(G)$.

The graph G_1 of Figure 1 has geodetic number 2 as $S_1 = \{w_1, y_1\}$ is the unique minimum geodetic set of G_1. On the other hand, each 2-element subset S of the vertex set of G_2 has the property that $I[S]$ is properly contained in $V(G_2)$. Thus $g(G_2) \geq 3$. Since $S_2 = \{u_2, v_2, x_2\}$ is a geodetic set, $g(G_2) = 3$.

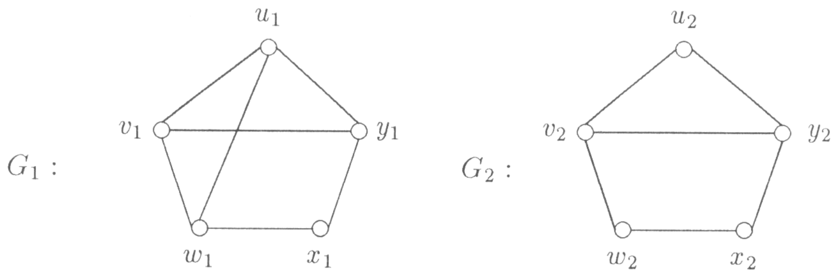

Figure 1. Illustrating the geodetic number

The closed intervals $I[u, v]$ in a connected graph G were studied and characterized by Nebeský [7, 8] and were also investigated extensively in the book by Mulder [6], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The intervals of an oriented graph have been studied in [4].

2 Uniform and Essential Minimum Geodetic Sets

A graph F is called a *minimum geodetic subgraph* if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum geodetic set in G. Those graphs that are minimum geodetic subgraphs were characterized in [1].

Theorem A. *A nontrivial graph F is a minimum geodetic subgraph if and only if every vertex of F has eccentricity* 1 *or no vertex of F has eccentricity* 1.

As a consequence of this theorem, there exists a graph G containing a minimum geodetic set S such that $\langle S \rangle$ is complete or S is independent. In the former case, $d_G(u,v) = 1$ for all distinct $u, v \in S$; while in the latter case, $d_G(u,v) \geq 2$ for all distinct $u, v \in S$. This is illustrated in Figure 2.

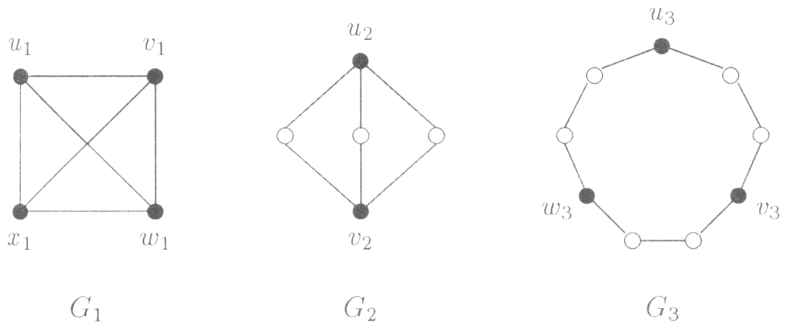

Figure 2. Graphs with uniform minimum geodetic sets

The graphs G_1, G_2, and G_3 in Figure 2 contain minimum geodetic sets $S_1 = \{u_1, v_1, w_1, x_1\}$, $S_2 = \{u_2, v_2\}$, and $S_3 = \{u_3, v_3, w_3\}$, respectively, with an added property. For every two distinct vertices $y, z \in S_i$, $i = 1, 2, 3$, $d_{G_i}(y, z) = i$. This suggests the following definition. A set S of vertices in a connected graph G is *uniform* if the distance between every two vertices of S is the same fixed number. Obviously, if S is uniform, then $\langle S \rangle$ is complete or S is independent. Hence each minimum geodetic set indicated in Figure 2 is uniform.

We define a geodetic set S to be *essential* if for every two vertices u, v in S, there exists a vertex $w \neq u, v$ of G that lies in a $u - v$ geodesic but in no $x - y$ geodesic for $x, y \in S$ and $\{x, y\} \neq \{u, v\}$. For example the set $S = \{x, y, z\}$ is an essential geodetic set of the graph G of Figure 3, while S is not uniform in G.

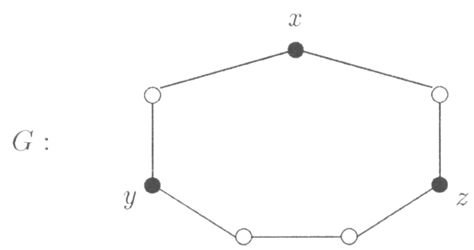

Figure 3. A graph G with an essential geodetic set

We now show that it is possible for a graph to have a minimum geodetic set with a specified number of vertices designated as essential as well as uniform.

Theorem 21. *For each integer $k \geq 2$, there exists a connected graph G with $g(G) = k$ which contains a uniform, essential minimum geodetic set.*

Proof. Let $K_k^{(k-1)}$ denote the multigraph of order k for which every two vertices of $K_k^{(k-1)}$ are joined by $k-1$ edges. Let $G_k = S\left(K_k^{(k-1)}\right)$ be the subdivision graph of $K_k^{(k-1)}$. Clearly *diam* $G_k = 3$ if $k \geq 3$. We show by induction that $g(G_k) = k$ and $V\left(K_k^{(k-1)}\right)$ is a uniform, essential minimum geodetic set for G_k.

To begin the inductive proof, for $k = 2$, the graph $G_2 = S\left(K_2^{(1)}\right)$ is a path of order 3. Therefore, $g(G_2) = 2$ and the two end-vertices of G_2 form a uniform, essential minimum geodetic set for G_2. Now we take $g(G_{k-1}) = k-1$, where $k-1 \geq 2$, and $V\left(K_{k-1}^{(k-2)}\right)$ is a uniform, essential minimum geodetic set for G_{k-1}. We now consider G_k.

Let $S = V\left(K_k^{(k-1)}\right) = \{v_1, v_2, \cdots, v_k\}$. For each pair i, j, $1 \leq i < j \leq k$, label the $k-1$ vertices of degree 2 that are adjacent to both v_i and v_j by $v_{i,j}^1, v_{i,j}^2, \cdots, v_{i,j}^{k-1}$. Since $I[S] = V(G_k)$, it follows that $g(G_k) \leq k$.

Suppose, to the contrary, that $g(G_k) = m < k$ and let $W = \{w_1, w_2, \cdots, w_m\}$ be a minimum geodetic set of G_k. We consider three cases.

Case 1. *W is a proper subset of* $\{v_1, v_2, \cdots, v_k\}$. Then $I[W] = V(G_m)$, where $G_m = S\left(K_m^{m-1}\right)$ with $V\left(K_m^{m-1}\right) = W$. Therefore, $I[W] \neq V(G_k)$, contradicting the fact that W is a geodetic set of G_k.

Case 2. $W = \{v_{i,j}^1, v_{i,j}^2, \cdots, v_{i,j}^{k-1}\}$ *where* $1 \leq i < j \leq k$. Then $I[W] = W \cup \{v_i, v_j\} \subset V(G_k)$, once again contradicting the fact that W is a geodetic set of G_k.

Case 3. *There exist integers i, j, p, q, where $1 \le i < j \le k$ and $1 \le p < q \le k-1$, such that $v_{i,j}^p \in W$ and $v_{i,j}^q \notin W$.* Since $I[W] = V(G_k)$, there exist $x, y \in W$ such that $v_{i,j}^q$ lies on an $x-y$ geodesic in G_k. Since $v_{i,j}^q \notin W$, it follows that $2 \le d(x,y) \le 3$.

Suppose first that $d(x,y) = 2$. We show that

$$I[W] = I\left[W - \{v_{i,j}^p\}\right].$$

In this case, $\{x,y\} = \{v_i, v_j\}$, say $x = v_i$ and $y = v_j$. So $v_{i,j}^q$ lies in the geodesic $x, v_{i,j}^q, y$ in G_k. It follows that $v_{i,j}^p$ lies in the geodesic $x, v_{i,j}^p, y$ in G_k, so $v_{i,j}^p \in I[x,y]$. Let $v \notin W$ be a vertex that lies in some $v_{i,j}^p - w$ geodesic in G_k, where $w \in W$. If $d(v_{i,j}^p, w) = 2$, then $v \in \{x, y\}$. This contradicts the fact that $v \notin W$, so $d(v_{i,j}^p, w) = 3$. Thus v lies in either the geodesic v_i, v, w or in the geodesic v_j, v, w in G_k. Therefore, $I[W] = I\left[W - \{v_{i,j}^p\}\right]$, contradicting the fact that W is a minimum geodetic set of G_k.

Suppose next that $d(x,y) = 3$. We show that a geodetic set W' of a graph G_{k-1} can be formed from W, where $|W'| \le k-2$ and which will contradict the induction hypothesis.

In this case, exactly one of x and y belongs to $\{v_i, v_j\}$, say $x = v_i$ and $y \ne v_j$. Then y is a subdivision vertex, so $\deg y = 2$ in G_k, and $v_{i,j}^q$ lies in the $x-y$ geodesic $x, v_{i,j}^q, v_j, y$ in G_k. This implies that $v_{i,j}^p$ also lies in an $x-y$ geodesic, namely the geodesic $x, v_{i,j}^p, v_j, y$, in G_k. So $v_{i,j}^p \in I[x,y]$. Now let $v \notin W$ be a vertex that lies in some $v_{i,j}^p - w$ geodesic in G_k, where $w \in W$. If $d(v_{i,j}^p, w) = 2$, then $v = v_j$. This implies that v lies in the $x-y$ geodesic $x, v_{i,j}^p, v, y$ in G_k, so $v \in I[x,y]$ and $d(v_{i,j}^p, w) = 3$. Then $w \in \{v_1, v_2, \cdots, v_k\}$, say $w = v_h$. Let

$$W' = W - W \bigcap \{v_{i,j}^\ell, v_{j,h}^\ell : 1 \le \ell \le k-1\}.$$

Since $v_{i,j}^p, y \in W \bigcap \{v_{i,j}^\ell, v_{j,h}^\ell : 1 \le \ell \le k-1\}$, it follows that $|W'| \le k-2$. Let $G_{k-1} = S\left(K_{k-1}^{(k-2)}\right)$, where $V\left(K_{k-1}^{(k-2)}\right) = \{v_1, v_2, \cdots, v_{j-1}, v_{j+1}, \cdots, v_k\}$. We show that $I[W'] = V(G_{k-1})$, contradicting the induction hypothesis.

Let $v \notin W'$ be a vertex of G_{k-1}. Since $I[W] = V(G_k)$, it follows that v lies in some $u-w$ geodesic P in G_k, where $u, w \in W$. Observe that at least one of u, w must be in W', for otherwise, P contains no vertex in G_{k-1}. Assume first that $u, w \in W'$. Then P is also a geodesic in G_{k-1} giving the desired result. Therefore, exactly one of u and w belongs to W', say $w \in W'$. If $d(u,w) = 2$, then $v \in \{v_i, v_h\}$, contradicting $v \notin W'$, therefore $d(u, w) = 3$. Then v lies in either the geodesic v_i, v, w, or in the geodesic v_h, v, w in G_{k-1}. It follows that $I[W'] = V(G_{k-1})$, which contradicts the induction hypothesis.

Therefore $S = V\left(K_k^{(k-1)}\right)$ is a minimum geodetic set of G_k. Then $v_{i,j}^{\ell}$, where $1 \le i < j \le k$ and $1 \le \ell \le k-1$, lies in exactly one geodesic, namely the geodesic $v_i, v_{i,j}^{\ell}, v_j$, in G_k. Moreover, $d(u, w) = 2$ for all $u, w \in S$. Therefore, S is a uniform, essential minimum geodetic set for G_k. ■

3 Minimal Geodetic Sets

A geodetic set S in a connected graph G is called a *minimal geodetic set* if no proper subset of S is a geodetic set. Of course, every minimum geodetic set is a minimal geodetic set, but the converse is not true. For example, let $G = K_{2,3}$ of Figure 4 with partite sets $V_1 = \{x, y\}$ and $V_2 = \{u, v, w\}$. Then $\{u, v, w\}$ is a minimal geodetic set of $K_{2,3}$ but is not a minimum geodetic set of $K_{2,3}$ since $\{x, y\}$ is its unique minimum geodetic set. We define the *upper geodetic number* $g^+(G)$ as the maximum cardinality of a minimal geodetic set of G. Obviously, $g(G) \le g^+(G)$. The graph G of Figure 4 has geodetic number 2 and upper geodetic number 3.

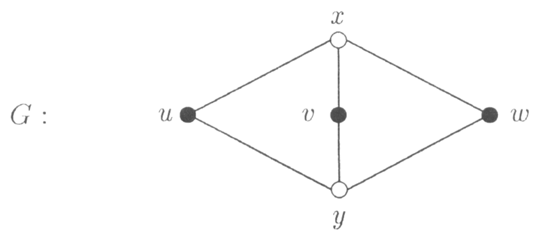

Figure 4. A graph G with a minimal geodetic set

We now show that every two integers a and b with $2 \le a \le b$ are realizable as the geodetic number and upper geodetic number, respectively, of some graph. Furthermore, we determine the minimum order of such a graph. Certainly, this minimum order is at least b. Indeed, if $a = b$, then the only geodetic set of K_b is its vertex set; so $g(K_b) = g^+(K_b) = b$ and the minimum order is b. Indeed, if G is a graph of order n with $g^+(G) = n$, then $G = K_n$ and so $g(G) = g^+(G)$. Before taking this observation one step further, we present a lemma.

Lemma 3.1. *Let G be a nontrivial connected graph of order n with $g^+(G) = n-1$ and let S be a minimal geodetic set of maximum cardinality such that $V(G) - S = \{v\}$. Then G does not contain nonadjacent vertices $u, w \in S$ such that u and w are mutually adjacent to both v and some vertex of S.*

Proof. Suppose, to the contrary, that there exist vertices $x, y, z \in S$ such that $xy \notin E(G)$ and x and y are mutually adjacent to both v and z. Then z lies in the geodesic x, z, y, while v lies in the geodesic x, v, y. Hence $S - \{z\}$ is a geodetic set, contradicting the minimality of S. ■

Theorem 3.2. *Let G be a nontrivial connected graph of order n. If $g^+(G) = n-1$, then $g(G) = g^+(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, where $S = \{v_1, v_2, \cdots, v_{n-1}\}$ is a minimal geodetic set of maximum cardinality. First, we claim that every vertex in S is adjacent to v_n. Suppose, to the contrary, that some $v \in S$ is not adjacent to v_n. Among the pairs x, y of distinct vertices of S for which v lies in some $x-y$ geodesic, we choose a pair such that $d(x, y)$ is minimum. If $v \neq x, y$, then v_n lies in some $u-w$ geodesic of length 2, where $u, w \in S$ and $u, w \neq v$. This implies that $S - \{v\}$ is a geodetic set, a contradiction. Therefore, either $x = v$ or $y = v$, say the former. We consider two cases.

Case 1. $yv_n \in E(G)$. Then there are two subcases.

Subcase 1.1. Among the vertices of S adjacent to v_n, there exists some vertex z not adjacent to y.

Here v_n lies in the geodesic y, v_n, z in G. By Lemma 3.1, $xz \notin E(G)$. Since $P : x, y, v_n, z$ is a path in G, it follows that $d(x, z) \leq 3$. Assume first that $d(x, z) = 2$. Then there exists a vertex $w \in S$ adjacent to both x and z. By Lemma 3.1, $wy \notin E(G)$. Then x lies in the geodesic y, x, w in G, implying that $S - \{x\}$ is a geodetic set, producing a contradiction. Therefore, $d(x, z) = 3$. Thus P is a geodesic and $S - \{y\}$ is a geodetic set, which is a contradiction.

Subcase 1.2. Every vertex of S that is adjacent to v_n is also adjacent to y. Since v_n lies in some $u - w$ geodesic for $u, w \in S$, it follows that $\deg v_n \geq 3$. Necessarily, $uw \notin E(G)$, this is impossible by Lemma 3.1.

Case 2. $yv_n \notin E(G)$.
Then v_n lies in some $u - v$ geodesic of length 2. By Lemma 3.1, y is not adjacent to both u and v, say $yu \notin E(G)$. Let $d(y, u) = \ell$ and let $y =$

$w_0, w_1, w_2, \cdots, w_\ell = u$ be a $y-u$ geodesic. Since $yv_n \notin E(G)$, it follows that $w_1 \neq v_n$. If $w_1 \neq v$, then $S-\{w_1\}$ is a geodetic set, which is a contradiction. Thus $w_1 = v$. Then y, v, v_n, u is a geodesic and $S - \{v\}$ is a geodetic set, contrary to hypothesis.

This completes the proof of the claim. Therefore, for every pair x, y of nonadjacent vertices in S, the vertex v_n lies in the geodetic x, v_n, y. Clearly, $diam(G) = 2$.

Next we show that

$$G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$$

where $n_1, n_2, \cdots, n_r, r$ are positive integers with $n_1 + n_2 + \cdots + n_r = n - 1$ and $V(K_1) = \{v_n\}$, which implies that $g(G) = g^+(G) = n - 1$. Suppose, to the contrary, that this is not the case. Then there exist $x, y, z \in S$ such that $d(x, y) = 2$ and $xz, zy \in E(G)$. It follows that z and v_n both lie in some $x - y$ geodesic. So $S - \{z\}$ is a geodetic set, which is a contradiction. ■

We can now complete the proof of the realizability of every two integers a and b with $2 \leq a \leq b$ as the geodetic number and upper geodetic number, respectively, of some graph.

Theorem 3.3. *For every two positive integers a and b, where $2 \leq a < b$, there exists a graph G with $g(G) = a$ and $g^+(G) = b$.*

Proof. Let $F = \overline{K}_{b-a+1} + \overline{K}_2$, where $V(K_{b-a+1}) = \{v_1, v_2, \cdots, v_{b-a+1}\}$ and $V(K_2) = \{x, y\}$. The graph G is formed from F by adding $a - 1$ pendant edges yu_i $(1 \leq i \leq a - 1)$ to the vertex y of F (see Figure 5). The graph G has the unique minimum geodetic set $S = \{x, u_1, u_2, \cdots, u_{a-1}\}$ and so $g(G) = a$.

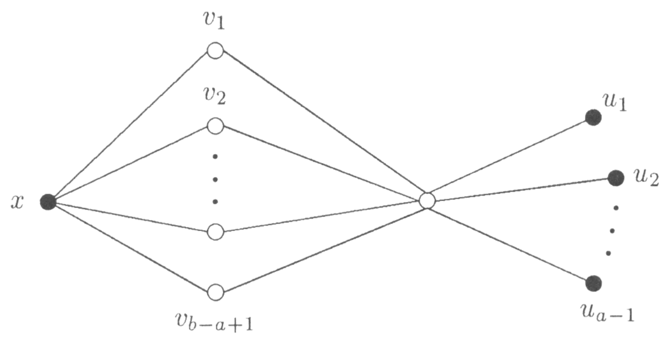

Figure 5. A graph G with $g(G) = a$ and $g^+(G) = b$

Now let

$$S' = \{u_1, u_2, \cdots, u_{a-1}, v_1, v_2, \cdots, v_{b-a+1}\}.$$

Then $I[S'] = V(G)$. We show that S' is a minimal geodetic set of G. Let $v \in S'$. We show that $I[S' - \{v\}] \neq V(G)$. Assume first that $v = u_i$ for some i $(1 \leq i \leq a-1)$. Then $I[S' - \{u_i\}] = V(G) - \{u_i\}$. So $v = v_j$ for some j $(1 \leq j \leq b-a+1)$. Then $I[S' - \{v_j\}] = V(G) - \{v_j\}$. It follows that $I\,[S' - \{v\}] \neq V(G)$ for every $v \in S'$. Since $|S'| = b$, we have that $g^+(G) \geq b$.

Next we show that there is no minimal geodetic set W of G with $|W| > b$, which implies that $g^+(G) = b$. Note that the graph G has order $n = b+2$. Since $g(G) = a < b$, it suffices to show that G does not contain an $(n-1)$-element minimal geodetic set. Suppose, to the contrary, that W is a minimal geodetic set of G where $|W| = n-1$. Let $v \notin W$. Since every geodetic set of G must contain all end-vertices of G, it follows that $v = x$, for otherwise, the geodetic set $S = \{x, u_1, u_2, \cdots, u_{a-1}\}$ is a proper subset of W, which contradicts the fact that W is minimal. Then $y \in W$. It follows that $I[W] = I\,[W - \{y\}] = V(G)$. Once again, this contradicts W being a minimal geodetic set of G. ■

The proof of Theorem 3.3 shows that if $b - a \geq 2$ and k is an integer with $a < k < b$, then there need not be a graph G with $g(G) = a$ and $g^+(G) = b$ containing a minimal geodetic set of cardinality k, that is, a graph G need not contain an 'intermediate' minimal geodetic set.

The following corollary gives the smallest order of a graph satisfying the hypothesis of Theorem 3.3. The proof is a direct consequence of Theorem 3.2 and 3.3.

Corollary 3.4. *For every two positive integers a and b, where $2 \leq a < b$, the smallest order of a graph G with $g(G) = a$ and $g^+(G) = b$ is $b + 2$.*

References

[1] G. Chartrand, F. Harary and P. Zhang, *On the geodetic number of a graph, Networks* (to appear).

[2] G. Chartrand and L. Lesniak, Graphs & Digraphs (third edition, Chapman & Hall, New York, 1996).

[3] G. Chartrand and P. Zhang, *The forcing geodetic number of a graph*, Discuss. Math. Graph Theory **19** (1999) 45–58.

[4] G. Chartrand and P. Zhang, *The geodetic number of an oriented graph*, European J. Combin. **21** (2000) 181–189.

[5] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).

[6] H.M. Mulder, The Interval Function of a Graph (Mathematisch Centrum, Amsterdam, 1980).

[7] L. Nebeský, *A characterization of the interval function of a connected graph*, Czech. Math. J. **44** (119) (1994) 173–178.

[8] L. Nebeský, *Characterizing of the interval function of a connected graph*, Math. Bohem. **123** (1998) 137–144.

Received 27 September 1999
Revised 26 January 2000