# LONG INDUCED PATHS IN 3-CONNECTED PLANAR GRAPHS 

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#### Abstract

It is shown that every 3 -connected planar graph with a large number of vertices has a long induced path.


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Let $G$ be an undirected graph without loops and multiple edges. Denote by $p(G)$ the number of vertices in the longest induced path of $G$. Finding long induced paths in graphs is an interesting but difficult problem. However, it is easy to revise all the references devoted to related problems (see [1-7]).

Denote $p_{n}=\min \{p(G)\}$ where the minimum is taken over all triconnected planar graphs of order $n$. The purpose of this note is to prove the following.

Theorem. $\lim _{n \rightarrow \infty} p_{n}=\infty$
Proof. Denote by $G_{n}$ a fixed triconnected planar graph such that $p\left(G_{n}\right)=$ $p_{n}$. Let $\Delta_{n}$ be the maximum degree of $G_{n}$ and let $v_{n}$ be a fixed vertex of maximum degree in $G_{n}$. It is easy to see that the diameter $d$ of any graph is large if it has an small maximum degree. In fact one can prove that $p_{n} \geq d\left(G_{n}\right)+1 \geq \log _{\Delta_{n}} n$. So if $\left\{\Delta_{n}\right\}$ is bounded, then we are done. Hence, we can suppose that $\left\{\Delta_{n}\right\}$ grows.

A well known theorem of Whitney states that, any triconnected planar graph has an unique embedding in the sphere. In this embedding the topological neighborhood of a vertex $v$ is an open disk bounded by a cycle $C_{v}$
of the graph which in general contains more vertices than the ones in the graphical neighborhood of the vertex.

Denote by $G_{n}^{\prime}$ the graph obtained from $G_{n}$ by deleting $v_{n}$ and every other vertex not in $C_{v_{n}}$. Of course, any induced path in $G_{n}^{\prime}$ is an induced path in $G_{n}$. We denote by $n^{\prime}$ the order of $G_{n}^{\prime}$. We know that $n^{\prime} \geq \Delta_{n}$ and therefore $\left\{n^{\prime}\right\}$ is unbounded.

We can think on the graph $G_{n}^{\prime}$ as drawn in the plane in such a way that the cycle $C_{v_{n}}$ bounds the infinite face. Let $D_{n}$ be the dual graph of $G_{n}^{\prime}$ and let us delete from $D_{n}$ the vertex corresponding to the infinite face to obtain $D_{n}^{\prime}$. Since every vertex of $G_{n}^{\prime}$ lies in the boundary of the infinite face then, $D_{n}^{\prime}$ is a tree.

Let us associate to each vertex of $D_{n}^{\prime}$ a weight equal to the number of vertices of the corresponding face in $G_{n}^{\prime}$ minus two. The weight of a path in $D_{n}^{\prime}$ is by definition the sum of the weights of its vertices. Observe that a path of weight $w$ in $D_{n}^{\prime}$ corresponds to a subgraph $P$ of $G_{n}^{\prime}$ which is a path of faces separated by edges. It is easy to see that $P$ has exactly $w+2$ vertices. Deleting a vertex from each of the two end faces of $P$ we split the boundary of $P$ into two paths. Again, the fact that every vertex of $G_{n}^{\prime}$ lies in the boundary of the infinite face implies that these two paths are induced in $G_{n}^{\prime}$ and one of them has at least $w / 2$ vertices. Therefore, if we denote by $w_{n}$ the maximum weight of a path in $D_{n}^{\prime}$ then, to prove the proposition we must show that $\left\{w_{n}\right\}$ is unbounded.

Denote by $k=k(n)$ the size of the biggest interior face in $G_{n}^{\prime}$ and by $m=m(n)$ the number of vertices in $D_{n}^{\prime}$. If we triangulate all interior faces of $G_{n}^{\prime}$, then the number of all interior triangles with respect to the cycle $C_{v_{n}}$ must be $n^{\prime}-2$, but in the interior of each face there are at most $k-2$ triangles and so $m \geq \frac{n^{\prime}-2}{k-2}$. Let $v$ be a vertex in $D_{n}^{\prime}$ of eccentricity equal to the diameter $d=d(n)$ of $D_{n}^{\prime}$ and denote by $V_{i}$ the set of vertices at distance $i$ from $v$.

It is clear that

$$
\frac{n^{\prime}-2}{k-2} \leq m=\sum_{i=0}^{d}\left|V_{i}\right| \leq \sum_{i=0}^{d} k^{i} \leq \frac{k^{d+1}-2}{k-2}
$$

and therefore $\log _{3} n^{\prime} \leq(d+1) \log _{3} k$. Since any vertex has weight no less than one then $w_{n} \geq d+1$. On the other hand, $w_{n} \geq k-2 \geq \log _{3} k$ for any $k \geq 3$. Hence, $w_{n} \geq \sqrt{\log _{3} n^{\prime}}$ and the proof is completed.

Remark. The method in the proof of the proposition gives a lower bound $O(\log n)$ for maximal outerplanar graphs with $n$ vertices. However, this an
easier result that can be proved in several other ways. In this case the bound is asymptotically sharp. It is reached in the family $\left\{\mathbf{S}_{i}\right\}$ shown in the figure.


Figure 1. Polygon triangulations with $p=O(\log n)$.

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