Discussiones Mathematicae
Graph Theory 20 (2000) 81–91

ABOUT UNIQUELY COLORABLE MIXED HYPERTREES

Angela Niculitsa*

Department of Mathematical Cybernetics
Moldova State University
Mateevici 60, Chişinău, MD-2009, Moldova

and

Vitaly Voloshin†

Institute of Mathematics and Informatics
Moldovan Academy of Sciences
Academiei, 5, Chişinău, MD-2028, Moldova

Abstract

A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where X is the vertex set and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A k-coloring of $\mathcal{H}$ is a mapping $c : X \to [k]$ such that each $\mathcal{C}$-edge has two vertices with the same color and each $\mathcal{D}$-edge has two vertices with distinct colors. $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called a mixed hypertree if there exists a tree $T = (X, \mathcal{E})$ such that every $\mathcal{D}$-edge and every $\mathcal{C}$-edge induces a subtree of T. A mixed hypergraph $\mathcal{H}$ is called uniquely colorable if it has precisely one coloring apart from permutations of colors. We give the characterization of uniquely colorable mixed hypertrees.

Keywords: colorings of graphs and hypergraphs, mixed hypergraphs, unique colorability, trees, hypertrees, elimination ordering.

1991 Mathematics Subject Classification: 05C15.

---

*Partially supported by DAAD, TU-Dresden.
†Partially supported by DFG, TU-Dresden.

1 Preliminaries

We use the standard concepts of graphs and hypergraphs from [1, 2] and updated terminology on mixed hypergraphs from [4, 5, 6, 7].

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where X is the *vertex set*, $|X| = n$, and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges*, respectively.

A proper k-coloring of a mixed hypergraph is a mapping $c : X \to [k]$ from the vertex set X into a set of k colors so that each $\mathcal{C}$-edge has two vertices with the same color and each $\mathcal{D}$-edge has two vertices with different colors. The *chromatic polynomial* $P(\mathcal{H}, k)$ gives the number of different proper k-colorings of $\mathcal{H}$.

A *strict* k-coloring is a proper coloring using all k colors. By $c(x)$ we denote the color of vertex $x \in X$ in the coloring c. The maximum number of colors in a strict coloring of $\mathcal{H}$ is the *upper chromatic number* $\bar{\chi}(\mathcal{H})$; the minimum number is the *lower chromatic number* $\chi(\mathcal{H})$.

If for a mixed hypergraph $\mathcal{H}$ there exists at least one coloring, then it is called colorable. Otherwise $\mathcal{H}$ is called uncolorable. Throughout the paper we consider colorable mixed hypergraphs.

If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a mixed hypergraph, then the subhypergraph *induced* by $X' \subseteq X$ is the mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ defined by setting $\mathcal{C}' = \{C \in \mathcal{C} : C \subseteq X'\}$, $\mathcal{D}' = \{D \in \mathcal{D} : D \subseteq X'\}$ and denoted by $\mathcal{H}' = \mathcal{H}/X'$.

The mixed hypergraph $\mathcal{H} = (X, \emptyset, \mathcal{D})$ ($\mathcal{H} = (X, \mathcal{C}, \emptyset)$) is called "$\mathcal{D}$-hypergraph" ("$\mathcal{C}$-hypergraph") and denoted by $\mathcal{H}_{\mathcal{D}}$ ($\mathcal{H}_{\mathcal{C}}$). If $\mathcal{H}_{\mathcal{D}}$ contains only $\mathcal{D}$-edges of size 2 then from the coloring point of view it coincides with classical graph ([2]). We call it $\mathcal{D}$-graph.

For each k, let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each $\mathcal{C}$- and $\mathcal{D}$- edge. In fact r_k equals the number of different strict k-colorings of $\mathcal{H}$ if we disregard permutations of colors. The vector $R(\mathcal{H}) = (r_1, \ldots, r_n) = (0, \ldots, 0, r_{\chi(\mathcal{H})}, \ldots, r_{\bar{\chi}(\mathcal{H})}, 0, \ldots, 0)$ is the *chromatic spectrum* of $\mathcal{H}$.

For the simplicity we assume that two strict k-colorings are considered the same if they can be obtained from each other by permutation of colors. In this case the number of different strict k-colorings coincides with $r_k(\mathcal{H})$. A mixed hypergraph $\mathcal{H}$ is called a *uniquely colorable* (*uc* for short) [5] if it has just one strict coloring.

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called *uc-orderable* [5] if there exists the ordering of the vertex set X, say $X = \{x_1, x_2, \ldots, x_n\}$, with the following property: each subhypergraph $\mathcal{H}_i = \mathcal{H}/X_i$ induced by the vertex set $X_i = \{x_i, x_{i+1}, \ldots, x_n\}$ is uniquely colorable. The corresponding sequence $x_1, \ldots, x_n$ will be called a *uc-ordering* of $\mathcal{H}$.

A sequence $x_0, x_1, \ldots, x_{t+1}$ of vertices is called a *$\mathcal{D}$-path* if $(x_i, x_{i+1}) \in \mathcal{D}$, $0 \leq i \leq t$. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called reduced if $|C| \geq 3$ for each $C \in \mathcal{C}$, and $|D| \geq 2$ for each $D \in \mathcal{D}$, and moreover, no one $\mathcal{C}$-edge ($\mathcal{D}$-edge) is included in another $\mathcal{C}$-edge ($\mathcal{D}$-edge).

As it follows from the splitting-contraction algorithm [6, 7] colorings properties of arbitrary mixed hypergraph may be obtained from some reduced mixed hypergraph. Therefore, throughout the paper we consider reduced mixed hypergraphs.

Let $\mathcal{C}(x)(\mathcal{D}(x))$ denote the set of $\mathcal{C}$-edges ($\mathcal{D}$-edges) containing vertex $x \in X$. Call the set

$$N(x) = \{y : \ y \in X, \ y \neq x, \ \mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset, \ \ \textit{or} \ \ \mathcal{D}(x) \cap \mathcal{D}(y) \neq \emptyset\}$$

the *neighbourhood* of the vertex x in a mixed hypergraph $\mathcal{H}$. In other words, the neighbourhood of a vertex x consists of those vertices which are contained in common $\mathcal{C}$-edges or $\mathcal{D}$-edges with x except x itself.

A vertex x is called *simplicial* [8] in a mixed hypergraph if its neighbourhood induces a uniquely colorable mixed subhypergraph. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called *pseudo-chordal* [8] if there exists an ordering σ of the vertex set X, $\sigma = (x_1, x_2, \ldots, x_n)$, such that the vertex x_j is simplicial in the subhypergraph induced by the set $\{x_j, x_{j+1}, \ldots, x_n\}$ for each $j = 1, 2, \ldots, n-1$.

Definition [8]**.** A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called a *mixed hypertree* if there exists a tree $T = (X, \mathcal{E})$ such that every $\mathcal{C}$-edge induces a subtree of T and every $\mathcal{D}$-edge induces a subtree of T.

Such a tree T is called further a *host tree.* The edge set of a host tree T is denoted by $\mathcal{E} = \{e_1, e_2, \ldots, e_{n-1}\}, \ e_i = (x, y), \ x, y \in X, \ i = 1, 2, \ldots, n-1$.

2 Uniquely Colorable Mixed Hypertrees

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be an arbitrary mixed hypergraph.

Definition. A sequence of vertices of $\mathcal{H}$, $x = x_0, x_1, \ldots, x_k = y, \ k \geq 1$, is called (x, y)-invertor iff:

(1) $x_i \neq x_{i+1},\ i = 0, 1, \ldots, k-1$;
(2) $(x_i, x_{i+1}) \in \mathcal{D},\ i = 0, 1, \ldots, k-1$;
(3) if $x_j \neq x_{j+2}$ then $(x_j, x_{j+1}, x_{j+2}) \in \mathcal{C},\ j = 0, 1, \ldots, k-2$.

In $\mathcal{H}$ for two vertices $x, y \in X$ there may exist many (x, y)-invertors. The *shortest* (x, y)-invertor contains minimal number of vertices. Two (x, y)-invertors are different if they have at least one distinct vertex. A (x, y)-invertor with $x = y$ is called *cyclic invertor.*

Definition. In a mixed hypertree, a cyclic invertor is called simple if all $\mathcal{C}$-edges are different and every $\mathcal{D}$-edge appears consecutively precisely two times.

Let $\mu = (z_0, z_1, \ldots, z_k = z_0),\ k \geq 6$, be some simple cyclic invertor in a mixed hypertree. Without loss of generality assume that $z_0 \neq z_1 \neq z_2 \neq z_0$. From the definition of simple cyclic invertor it follows that $z_0 \neq z_2 \neq \ldots \neq z_{k-2}$ and $z_1 = z_3 = \ldots = z_{k-1} = y$, where y is the center of some star in the host tree T.

Theorem 1. *If* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *is a mixed hypertree then*
(1) $\chi(\mathcal{H}) \leq 2$;
(2) *if, in addition,* $|\mathcal{D}| \leq n-2$ *then* $r_2(\mathcal{H}) \geq 2$.

Proof. (1) It follows from the possibility to start at any vertex and to color $\mathcal{H}$ alternatively by the colors 1 and 2 along the host tree T.

(2) Let $T = (X, \mathcal{E})$ be a host tree of the mixed hypertree $\mathcal{H}$. Since $|\mathcal{D}| \leq n-2$ in T there exists an edge $e = (x, y) \notin \mathcal{D}$. Starting with the vertices x, y we can construct two different colorings with two colors in the following way. First, put $c(x) = c(y) = 1$ and color all the other vertices alternatively along the tree T with the colors $2,\ 1, 2, \ldots$. Second, apply the same procedure starting with $c(x) = 1$ and $c(y) = 2$. ■

Theorem 2. *A mixed hypertree* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *is uniquely colorable if and only if for every two vertices* $x, y \in X$ *there exists an* (x, y)*-invertor.*

Proof. $\Rightarrow$ Let c be the unique strict coloring of the mixed hypertree $\mathcal{H}$. We show that for any two vertices $x, y \in X$ there exists an (x, y)-invertor.

Suppose $\mathcal{H}$ has two vertices $u, v \in X$ such that there is no (u, v)-invertor in $\mathcal{H}$. Consider the unique (u, v)-path in the host tree T of $\mathcal{H}$. The assumption implies that either in $\mathcal{H}$ there is no $\mathcal{D}$-path connecting u and v or in the sequence $u = x_1, x_2, \ldots, x_p = v$ there exists a triple of pairwise different vertices x_j, x_{j+1}, x_{j+2} not belonging to $\mathcal{C}$.

If there is no $\mathcal{D}$-path connecting u and v then by Theorem 1(2) $\mathcal{H}$ has two different colorings with two colors. This contradicts to the unique colorability of mixed hypertree $\mathcal{H}$.

Assume that in the sequence $u = x_1, x_2, \ldots, x_p = v$ there exists a triple of pairwise different vertices x_j, x_{j+1}, x_{j+2} such that $(x_j, x_{j+1}, x_{j+2}) \notin \mathcal{C}$. Evidently, x_{j+1} is not pendant in T. Let T_1 and T_2 be two connected components obtained after deletion of vertex x_{j+1} from the host tree T.

There are two cases. (1) $c(x_j) = c(x_{j+2})$. From Theorem 1(1) it follows that the number of colors in the unique coloring c of $\mathcal{H}$ is 2. Recolor the vertex x_{j+2} and all vertices on even distance from x_{j+2} in the component T_2 with the new color. The obtained coloring is a proper coloring of $\mathcal{H}$ different from c, a contradiction.

(2) $c(x_j) \neq c(x_{j+2})$. Since $(x_j, x_{j+1}), (x_{j+1}, x_{j+2}) \in \mathcal{D}$ we have that $c(x_j) \neq c(x_{j+1}) \neq c(x_{j+2})$. Consequently, $\mathcal{H}$ is colored with at least three colors. But according to Theorem 1 every mixed hypertree can be colored with two colors, a contradiction.

$\Leftarrow$ Assume that any two vertices $x, y \in X$ are joined by an (x, y)-invertor. Suppose $\mathcal{H}$ has at least two strict colorings c_1 and c_2. Then there exist two vertices, say x', y', such that $c_1(x') = c_1(y')$ but $c_2(x') \neq c_2(y')$. Consider (x', y')-invertor $x' = x_0, x_1, \ldots, x_k = y'$. From the definition of invertor follows that if k is even then in all possible colorings the vertices x' and y' have the same color. If k is odd then in all possible colorings the vertices x' and y' have distinct colors. Consider the unique (x', y')-path connecting the vertices x', y' on the host tree T. One can see that the parity of k coincides with the parity of length of the path. Moreover, it is true for any other (x', y')-invertor. Therefore, in all colorings either $c(x') = c(y')$ or $c(x') \neq c(y')$, a contradiction. ■

Corollary 1. *If $\mathcal{H}$ is a uniquely colorable mixed hypertree then $\mathcal{D} = \mathcal{E}$.*

Definition. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph. The $\mathcal{C}$-edge $C \in \mathcal{C}$ is called *redundant* if $R(\mathcal{H}) = R(\mathcal{H}_1)$, where $\mathcal{H}_1 = (X, \mathcal{C} \setminus \{C\}, \mathcal{D})$.

In a uniquely colorable mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ any $\mathcal{C}$-edge of size ≥ 4 is redundant because there is no invertor containing such $\mathcal{C}$-edge.

Theorem 3. *In a uniquely colorable mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ a $\mathcal{C}$-edge C of size 3 is redundant if and only if there exists a simple cyclic invertor containing C.*

Proof. Let $C = (x_1, x_2, x_3)$ be the redundant $\mathcal{C}$-edge. By definition $\mathcal{H}' = (X, \mathcal{C}', \mathcal{D})$ where $\mathcal{C}' = \mathcal{C} \setminus \{C\}$ is a uniquely colorable mixed hypertree. Then for the vertices x_1 and x_3 in $\mathcal{H}'$ there exisits an (x_1, x_3)-invertor: $x_1 = z_0, z_1, \ldots, z_k = x_3$. Construct the (x_1, x_1)-invertor in the following way: $x_1 = z_0, z_1, \ldots, z_k = x_3, x_2, x_1$. This invertor is a simple cyclic invertor of $\mathcal{H}$ containing C.

Conversely, suppose that $\mathcal{C}$-edge, $C = (x_1, x_2, x_3)$ is contained in some simple cyclic invertor $x_1 = z_0, z_1, \ldots, z_k = x_3, x_2, x_1$. Then the vertices x_1 and x_3 are joined by two different (x_1, x_3)-invertors: $(x_1, x_2, x_3) = C$ and $(x_1 = z_0, z_1, \ldots, z_k = x_3) = (x_1, x_3)'$-invertor. In each (x, y)-invertor containing C replace this $\mathcal{C}$-edge by $(x_1, x_3)'$-invertor. Thus, $\mathcal{H}' = (X, \mathcal{C} \setminus \{C\}, \mathcal{D})$ is uniquely colorable, i.e., the $\mathcal{C}$-edge C is redundant. ■

Let us have a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. Consider $X = X_1 \cup X_2 \cup \ldots \cup X_i$ any i-coloring of $\mathcal{H}$, $\chi(\mathcal{H}) \leq i \leq \bar{\chi}(\mathcal{H})$. Choose any X_j and construct *touching graph* $L_j = (X_j, V_j)$ in the following way: if some $C \in \mathcal{C}$ satisfies $C \cap X_j = \{x, y\}$ and $|C \cap X_k| \leq 1$, $k \neq j$, for some $x, y \in X_j$, then $(x, y) \in V_j$ (cf. pair graphs [3]).

Theorem 4. *If a mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is uniquely colorable then in its 2-coloring the touching graphs L_1 and L_2 are connected.*

Proof. By Theorem 1(2), Corollary 1 we obtain $|\mathcal{D}| = n - 1$, $\bar{\chi} = 2$ for each uniquely colorable mixed hypertree. If at least one touching graph is disconnected, then we can construct a new coloring of $\mathcal{H}$ with 3 colors by assigning new color to the vertices of one component. This assures the proper coloring also of any $\mathcal{C}$-edge of size ≥ 4. ■

Corollary 2. *The minimal number of $\mathcal{C}$-edges in any uniquely colorable mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is $n - 2$.*

Proof. Let $\mathcal{H}$ be a uniquely colorable mixed hypertree. Consider its unique 2-coloring, say $X = X_1 \cup X_2$, and construct the touching graphs $L_1 = (X_1, V_1)$, $L_2 = (X_2, V_2)$. The minimal number of edges in L_i to be connected is $|X_i| - 1$, and in this case each of L_i is a tree, $i = 1, 2$. Since every edge in L_i corresponds to some $\mathcal{C}$-edge of $\mathcal{H}$, we obtain that the minimal number of $\mathcal{C}$-edges is:

$$|X_1| - 1 + |X_2| - 1 = |X| - 2.$$

■

Corollary 3. *In a uniquely colorable mixed hypertree* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *the number of redundant* $\mathcal{C}$*-edges is* $|\mathcal{C}| - n + 2$.

Proof. Indeed, consider touching graphs L_i, and construct a spanning trees T_i, $i = 1, 2$. Each elementary cycle in L_i generates some simple cyclic invertor in $\mathcal{H}$. Therefore, each $\mathcal{C}$-edge of $\mathcal{H}$ which has a size ≥ 4 or corresponds to some edge of L_i which is a chord with respect to T_i, is redundant. Hence, the assertion follows. ■

Remark. Redundant $\mathcal{C}$-edge may become not redundant after deleting from $\mathcal{C}$ some another redundant $\mathcal{C}$-edges.

Definition. A mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called complete if every edge of the host tree T forms a $\mathcal{D}$-edge of $\mathcal{H}$, and every path on three vertices of T forms a $\mathcal{C}$-edge in $\mathcal{H}$.

Therefore, having the host tree T for the complete mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ we obtain that $\mathcal{D} = \mathcal{E}$.

Denote by M the number of $\mathcal{C}$-edges of a complete mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. Then

$$M = \sum_{\substack{x \in T \\ d(x) \geq 2}} \binom{d(x)}{2},$$

where $d(x)$ is the degree of vertex x in the host tree T.

Examples show that for any $k > 1$ one can construct a mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $|\mathcal{D}| = n - 1$, $n - 2 \leq |\mathcal{C}| \leq M$ and $\bar{\chi}(\mathcal{H}) = k$. Therefore these bounds on $|\mathcal{D}|$ and $|\mathcal{C}|$ are not sufficient for the mixed hypertrees to be uniquely colorable.

Proposition 1. *A uniquely colorable mixed hypertree with* $|\mathcal{C}| = n - 2$ *is a pseudo-chordal mixed hypergraph.*

Proof. Since $\mathcal{H}$ is uniquely colorable mixed hypertree and $|\mathcal{C}| = n - 2$ then it contains no redundant $\mathcal{C}$-edges and, moreover, all $\mathcal{C}$-edges have the size 3. It follows that there exists a pendant vertex, say x, of the host tree $T = (X, \mathcal{E})$ which belongs to precisely one $\mathcal{C}$-edge, say (x, y, z). The neighbourhood of x induces a complete $\mathcal{D}$-graph on 2 vertices, which itself is uniquely colorable. Consequently, the vertex x is simplicial in $\mathcal{H}$. Deleting the vertex x and $\mathcal{C}$-edge and $\mathcal{D}$-edge containing it, obtain $\mathcal{H}'$ which is uniquely colorable mixed hypertree with minimal number of $\mathcal{C}$-edges. Indeed, if $\mathcal{H}'$ would be not uniquely colorable, then two distinct colorings of $\mathcal{H}'$ formed different colorings of $\mathcal{H}$ because $c(x) = c(z)$, a contradiction. ∎

Remark. Redundant $\mathcal{C}$-edges enlarge the neighbourhood of some vertices without affecting any coloring. Therefore, to recognise the pseudo-chordality we need to delete the redundant $\mathcal{C}$-edges.

From the Theorem 4, Corollaries 2–4 and Proposition 1 we conclude that a uc-orderable mixed hypertree $\mathcal{H}$ can be recognised by consecutive elemination of pendant vertices of $\mathcal{D}$-graph $\mathcal{H}_{\mathcal{D}}$ in special ordering by applying the following

Algorithm (*uc-ordering*).

Input: A mixed hypertree $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, $\sigma - n$-dimensional empty vector.

Idea: Simultanious decomposition of $\mathcal{H}_{\mathcal{D}}$, spanning trees T_1 and T_2 of touching graphs L_1, L_2, respectively, by pendant vertices.

Iterations:

1. If there is a vertex $x \in X$ belonging to none $\mathcal{C}$-edge of size 3 or $\mathcal{D}$-edge of size 2 then return **NON UC**. Otherwise remove from $\mathcal{C}$ all elements of size ≥ 4.
2. Color $\mathcal{D}$-graph $\mathcal{H}_{\mathcal{D}}$ with two colors.
3. Construct touching graphs L_1 and L_2.
4. If $L_i,\ i = 1, 2$, is not connected then return **NON UC**.
5. For L_i construct spanning tree $T_i,\ i = 1, 2$.
6. $i := 1$.
7. While in T_i there exists a vertex x pendant in both T_i and $\mathcal{H}_{\mathcal{D}}$ then delete it from T_i and $\mathcal{H}_{\mathcal{D}}$ and include x in σ.
8. If at least one of T_1 and T_2 is not empty then go to 9. Otherwise return **UC, σ-uc-ordering**.
9. If $i = 1$ then assign $i := i + 1$, otherwise $i := i - 1$. Go to 7.

Remark. All chords of graph L_i with respect to spanning tree $T_i,\ i = 1, 2$, correspond to redundant $\mathcal{C}$-edges in $\mathcal{H}$. The trees T_1 and T_2 provide existence of unique (x, y)-invertor for any $x, y \in X$. The last assures at any step of the algorithm the existence of a vertex, say x, pendant in both $\mathcal{H}_{\mathcal{D}}$ and one of T_1 or T_2. Notice that not every elimination of pendant vertices generates a uc- ordering in $\mathcal{H}_{\mathcal{D}}$.

Example. Given the mixed hypertree $\mathcal{H}$ with $X = \{0,1,2,3,4,5,6,7,8, 9,10, 11,12, 13, 14, 15\}$, $\mathcal{C} = \{(0,1,2);\ (0,1,3);\ (0,2,3);\ (0,3,4);\ (0,4,5);\ (0,5,6);\ (0,4,6);\ (0,2,7);\ (0,7,8);\ (7,8,9);\ (9,8,10);\ (9,10,11);\ (9,11,12);\ (9,10,12);\ (9,12,13);\ (9,13,14);\ (9,14,15);\ (9, 13, 15)\}$, and $\mathcal{D} = \{(0,1);\ (0,2);\ (0,3);\ (0,4);\ (0,5);\ (0,6);\ (0,7);\ (7,8);\ (8,9);\ (9,10);\ (9,11);\ (9,12);\ (9,13);\ (9,14); (9,15)\}$, see the figures 1 and 2 (the $\mathcal{C}$-edges are depicted by triangles).

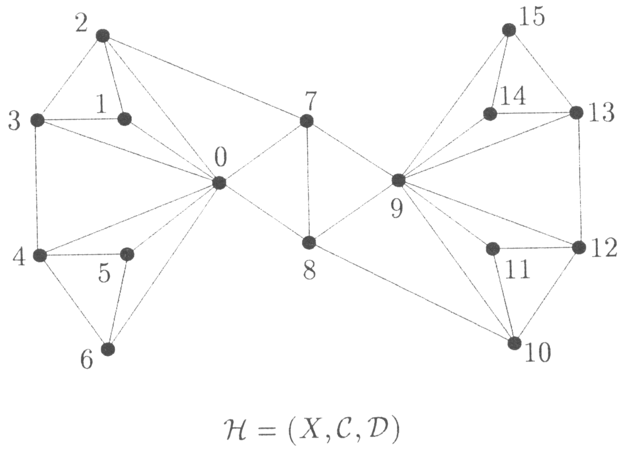

Figure 1

Apply the algorithm. Each vertex of $\mathcal{H}$ belongs to at least one $\mathcal{D}$-edge of size 2 and at least one $\mathcal{C}$-edge of size 3. Color $\mathcal{H}_{\mathcal{D}}$ with 2 colors. Denote by $X_1 = \{0,\ 8,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15\}$ and $X_2 = \{1,\ 2,\ 3,\ 4,\ 5,\ 6, 7,\ 9\}$ two color classes of $\mathcal{H}_{\mathcal{D}}$. Construct the following touching graphs $L_1 = (X_1, V_1)$ and $L_2 = (X_2, V_2)$, where $V_1 = \{(0,8); (8,10); (10,11); (10,12); (11,12); (12,13); (13,14); (13,15); (14,15)\}$ and $V_2 = \{(1,2);\ (1,3);\ (2,3);\ (2,7);\ (3,4);\ (4,5); (4,6);\ (5,6);\ (7,9)\}$. Choose the respective trees T_1 and T_2 (Figure 3).

Consecutively applying the algorithm we obtain one of uc-orderings of $\mathcal{H}$: $\sigma = \{15,14,13,11,12,10,1,5,9,6,4,3,2,8,0,7\}$. At the 7-th step of the algorithm, after including of vertex 10 in σ, we alternate the trees because T_1 has no pendant vertex which is also pendant in $\mathcal{H}_{\mathcal{D}}$. The next alternations of trees are made after adding to σ of vertices 2 and 0. From the above algorithm we have

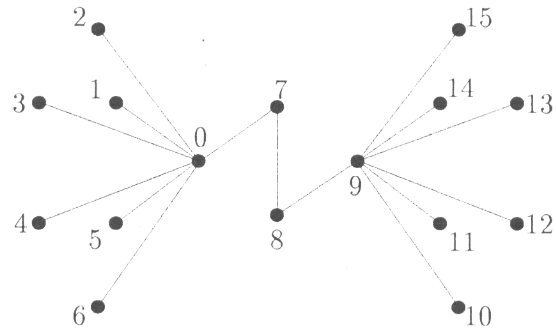

Figure 2. $\mathcal{H}_{\mathcal{D}} = (X, \emptyset, \mathcal{D})$

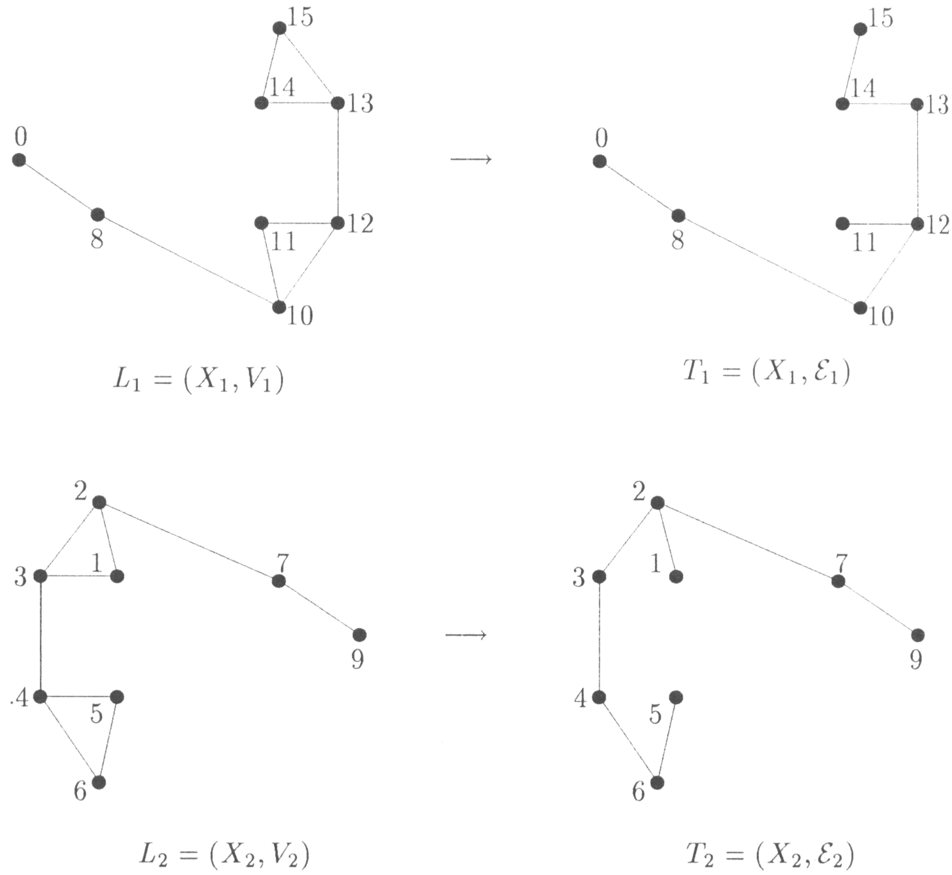

Figure 3

Theorem 5. *A mixed hypertree is uniquely colorable if and only if it is uc-orderable.*

Therefore, combining the Theorems 2, 5 and relation between chromatic polynomial and chromatic spectrum [6, 7], we obtain the following

Theorem 6. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypertree. Then the following five statements are equivalent:*

(1) $R(\mathcal{H}) = (0, 1, 0, \ldots, 0)$;

(2) $P(\mathcal{H}, k) = k(k-1)$;

(3) $\mathcal{H}$ *is uniquely colorable;*

(4) *Every two vertices* $x, y \in X$ *are joined by an* (x, y)*-invertor;*

(5) $\mathcal{H}$ *is uc-orderable.*

References

[1] C. Berge, Hypergraphs: combinatorics of finite sets (North Holland, 1989).

[2] C. Berge, Graphs and Hypergraphs (North Holland, 1973).

[3] K. Diao, P. Zhao and H. Zhou, *About the upper chromatic number of a co-hypergraph*, submitted.

[4] Zs. Tuza and V. Voloshin, *Uncolorable mixed hypergraphs*, Discrete Appl. Math. **99** (2000) 209–227.

[5] Zs. Tuza, V. Voloshin and H. Zhou, *Uniquely colorable mixed hypergraphs*, submitted.

[6] V. Voloshin, *The mixed hypergraphs*, Computer Science J. Moldova, **1** (1993) 45–52.

[7] V. Voloshin, *On the upper chromatic number of a hypergraph*, Australasian J. Combin. **11** (1995) 25–45.

[8] V. Voloshin and H. Zhou, *Pseudo-chordal mixed hypergraphs*, Discrete Math. **202** (1999) 239–248.

Received 16 April 1999
Revised 24 March 2000