

## ABOUT UNIQUELY COLORABLE MIXED HYPERTREES

ANGELA NICULITSA\*

*Department of Mathematical Cybernetics*  
*Moldova State University*  
*Mateevici 60, Chişinău, MD-2009, Moldova*

AND

VITALY VOLOSHIN†

*Institute of Mathematics and Informatics*  
*Moldovan Academy of Sciences*  
*Academiei, 5, Chişinău, MD-2028, Moldova*

### Abstract

A mixed hypergraph is a triple  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  where  $X$  is the vertex set and each of  $\mathcal{C}$ ,  $\mathcal{D}$  is a family of subsets of  $X$ , the  $\mathcal{C}$ -edges and  $\mathcal{D}$ -edges, respectively. A  $k$ -coloring of  $\mathcal{H}$  is a mapping  $c : X \rightarrow [k]$  such that each  $\mathcal{C}$ -edge has two vertices with the same color and each  $\mathcal{D}$ -edge has two vertices with distinct colors.  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is called a mixed hypertree if there exists a tree  $T = (X, \mathcal{E})$  such that every  $\mathcal{D}$ -edge and every  $\mathcal{C}$ -edge induces a subtree of  $T$ . A mixed hypergraph  $\mathcal{H}$  is called uniquely colorable if it has precisely one coloring apart from permutations of colors. We give the characterization of uniquely colorable mixed hypertrees.

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## 1 Preliminaries

We use the standard concepts of graphs and hypergraphs from [1, 2] and updated terminology on mixed hypergraphs from [4, 5, 6, 7].

A *mixed hypergraph* is a triple  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  where  $X$  is the *vertex set*,  $|X| = n$ , and each of  $\mathcal{C}$ ,  $\mathcal{D}$  is a family of subsets of  $X$ , the  *$\mathcal{C}$ -edges* and  *$\mathcal{D}$ -edges*, respectively.

A proper  $k$ -coloring of a mixed hypergraph is a mapping  $c : X \rightarrow [k]$  from the vertex set  $X$  into a set of  $k$  colors so that each  $\mathcal{C}$ -edge has two vertices with the same color and each  $\mathcal{D}$ -edge has two vertices with different colors. The *chromatic polynomial*  $P(\mathcal{H}, k)$  gives the number of different proper  $k$ -colorings of  $\mathcal{H}$ .

A *strict  $k$ -coloring* is a proper coloring using all  $k$  colors. By  $c(x)$  we denote the color of vertex  $x \in X$  in the coloring  $c$ . The maximum number of colors in a strict coloring of  $\mathcal{H}$  is the *upper chromatic number*  $\bar{\chi}(\mathcal{H})$ ; the minimum number is the *lower chromatic number*  $\chi(\mathcal{H})$ .

If for a mixed hypergraph  $\mathcal{H}$  there exists at least one coloring, then it is called colorable. Otherwise  $\mathcal{H}$  is called uncolorable. Throughout the paper we consider colorable mixed hypergraphs.

If  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is a mixed hypergraph, then the subhypergraph *induced* by  $X' \subseteq X$  is the mixed hypergraph  $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$  defined by setting  $\mathcal{C}' = \{C \in \mathcal{C} : C \subseteq X'\}$ ,  $\mathcal{D}' = \{D \in \mathcal{D} : D \subseteq X'\}$  and denoted by  $\mathcal{H}' = \mathcal{H}/X'$ .

The mixed hypergraph  $\mathcal{H} = (X, \emptyset, \mathcal{D})$  ( $\mathcal{H} = (X, \mathcal{C}, \emptyset)$ ) is called " *$\mathcal{D}$ -hypergraph*" (" *$\mathcal{C}$ -hypergraph*") and denoted by  $\mathcal{H}_{\mathcal{D}}$  ( $\mathcal{H}_{\mathcal{C}}$ ). If  $\mathcal{H}_{\mathcal{D}}$  contains only  $\mathcal{D}$ -edges of size 2 then from the coloring point of view it coincides with classical graph ([2]). We call it  *$\mathcal{D}$ -graph*.

For each  $k$ , let  $r_k$  be the number of partitions of the vertex set into  $k$  nonempty parts (color classes) such that the coloring constraint is satisfied on each  $\mathcal{C}$ - and  $\mathcal{D}$ - edge. In fact  $r_k$  equals the number of different strict  $k$ -colorings of  $\mathcal{H}$  if we disregard permutations of colors. The vector  $R(\mathcal{H}) = (r_1, \dots, r_n) = (0, \dots, 0, r_{\chi(\mathcal{H})}, \dots, r_{\bar{\chi}(\mathcal{H})}, 0, \dots, 0)$  is the *chromatic spectrum* of  $\mathcal{H}$ .

For the simplicity we assume that two strict  $k$ -colorings are considered the same if they can be obtained from each other by permutation of colors. In this case the number of different strict  $k$ -colorings coincides with  $r_k(\mathcal{H})$ . A mixed hypergraph  $\mathcal{H}$  is called a *uniquely colorable* (*uc* for short) [5] if it has just one strict coloring.

A mixed hypergraph  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is called *uc-orderable* [5] if there exists the ordering of the vertex set  $X$ , say  $X = \{x_1, x_2, \dots, x_n\}$ , with the following property: each subhypergraph  $\mathcal{H}_i = \mathcal{H}/X_i$  induced by the vertex set  $X_i = \{x_i, x_{i+1}, \dots, x_n\}$  is uniquely colorable. The corresponding sequence  $x_1, \dots, x_n$  will be called a *uc-ordering* of  $\mathcal{H}$ .

A sequence  $x_0, x_1, \dots, x_{t+1}$  of vertices is called a  $\mathcal{D}$ -path if  $(x_i, x_{i+1}) \in \mathcal{D}$ ,  $0 \leq i \leq t$ . A mixed hypergraph  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is called reduced if  $|C| \geq 3$  for each  $C \in \mathcal{C}$ , and  $|D| \geq 2$  for each  $D \in \mathcal{D}$ , and moreover, no one  $\mathcal{C}$ -edge ( $\mathcal{D}$ -edge) is included in another  $\mathcal{C}$ -edge ( $\mathcal{D}$ -edge).

As it follows from the splitting-contraction algorithm [6, 7] colorings properties of arbitrary mixed hypergraph may be obtained from some reduced mixed hypergraph. Therefore, throughout the paper we consider reduced mixed hypergraphs.

Let  $\mathcal{C}(x)(\mathcal{D}(x))$  denote the set of  $\mathcal{C}$ -edges ( $\mathcal{D}$ -edges) containing vertex  $x \in X$ . Call the set

$$N(x) = \{y : y \in X, y \neq x, \mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset, \text{ or } \mathcal{D}(x) \cap \mathcal{D}(y) \neq \emptyset\}$$

the *neighbourhood* of the vertex  $x$  in a mixed hypergraph  $\mathcal{H}$ . In other words, the neighbourhood of a vertex  $x$  consists of those vertices which are contained in common  $\mathcal{C}$ -edges or  $\mathcal{D}$ -edges with  $x$  except  $x$  itself.

A vertex  $x$  is called *simplicial* [8] in a mixed hypergraph if its neighbourhood induces a uniquely colorable mixed subhypergraph. A mixed hypergraph  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is called *pseudo-chordal* [8] if there exists an ordering  $\sigma$  of the vertex set  $X$ ,  $\sigma = (x_1, x_2, \dots, x_n)$ , such that the vertex  $x_j$  is simplicial in the subhypergraph induced by the set  $\{x_j, x_{j+1}, \dots, x_n\}$  for each  $j = 1, 2, \dots, n-1$ .

**Definition** [8]. A mixed hypergraph  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is called a *mixed hypertree* if there exists a tree  $T = (X, \mathcal{E})$  such that every  $\mathcal{C}$ -edge induces a subtree of  $T$  and every  $\mathcal{D}$ -edge induces a subtree of  $T$ .

Such a tree  $T$  is called further a *host tree*. The edge set of a host tree  $T$  is denoted by  $\mathcal{E} = \{e_1, e_2, \dots, e_{n-1}\}$ ,  $e_i = (x, y)$ ,  $x, y \in X$ ,  $i = 1, 2, \dots, n-1$ .

## 2 Uniquely Colorable Mixed Hypertrees

Let  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  be an arbitrary mixed hypergraph.

**Definition.** A sequence of vertices of  $\mathcal{H}$ ,  $x = x_0, x_1, \dots, x_k = y$ ,  $k \geq 1$ , is called  $(x, y)$ -invertor iff:

- (1)  $x_i \neq x_{i+1}$ ,  $i = 0, 1, \dots, k-1$ ;
- (2)  $(x_i, x_{i+1}) \in \mathcal{D}$ ,  $i = 0, 1, \dots, k-1$ ;
- (3) if  $x_j \neq x_{j+2}$  then  $(x_j, x_{j+1}, x_{j+2}) \in \mathcal{C}$ ,  $j = 0, 1, \dots, k-2$ .

In  $\mathcal{H}$  for two vertices  $x, y \in X$  there may exist many  $(x, y)$ -invertors. The *shortest*  $(x, y)$ -inverter contains minimal number of vertices. Two  $(x, y)$ -invertors are different if they have at least one distinct vertex. A  $(x, y)$ -inverter with  $x = y$  is called *cyclic inverter*.

**Definition.** In a mixed hypertree, a cyclic inverter is called simple if all  $\mathcal{C}$ -edges are different and every  $\mathcal{D}$ -edge appears consecutively precisely two times.

Let  $\mu = (z_0, z_1, \dots, z_k = z_0)$ ,  $k \geq 6$ , be some simple cyclic inverter in a mixed hypertree. Without loss of generality assume that  $z_0 \neq z_1 \neq z_2 \neq z_0$ . From the definition of simple cyclic inverter it follows that  $z_0 \neq z_2 \neq \dots \neq z_{k-2}$  and  $z_1 = z_3 = \dots = z_{k-1} = y$ , where  $y$  is the center of some star in the host tree  $T$ .

**Theorem 1.** If  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is a mixed hypertree then

- (1)  $\chi(\mathcal{H}) \leq 2$ ;
- (2) if, in addition,  $|\mathcal{D}| \leq n - 2$  then  $r_2(\mathcal{H}) \geq 2$ .

**Proof.** (1) It follows from the possibility to start at any vertex and to color  $\mathcal{H}$  alternatively by the colors 1 and 2 along the host tree  $T$ .

(2) Let  $T = (X, \mathcal{E})$  be a host tree of the mixed hypertree  $\mathcal{H}$ . Since  $|\mathcal{D}| \leq n - 2$  in  $T$  there exists an edge  $e = (x, y) \notin \mathcal{D}$ . Starting with the vertices  $x, y$  we can construct two different colorings with two colors in the following way. First, put  $c(x) = c(y) = 1$  and color all the other vertices alternatively along the tree  $T$  with the colors 2, 1, 2,  $\dots$ . Second, apply the same procedure starting with  $c(x) = 1$  and  $c(y) = 2$ . ■

**Theorem 2.** A mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is uniquely colorable if and only if for every two vertices  $x, y \in X$  there exists an  $(x, y)$ -inverter.

**Proof.**  $\Rightarrow$  Let  $c$  be the unique strict coloring of the mixed hypertree  $\mathcal{H}$ . We show that for any two vertices  $x, y \in X$  there exists an  $(x, y)$ -inverter.

Suppose  $\mathcal{H}$  has two vertices  $u, v \in X$  such that there is no  $(u, v)$ -inverter in  $\mathcal{H}$ . Consider the unique  $(u, v)$ -path in the host tree  $T$  of  $\mathcal{H}$ . The assumption implies that either in  $\mathcal{H}$  there is no  $\mathcal{D}$ -path connecting  $u$  and  $v$  or in

the sequence  $u = x_1, x_2, \dots, x_p = v$  there exists a triple of pairwise different vertices  $x_j, x_{j+1}, x_{j+2}$  not belonging to  $\mathcal{C}$ .

If there is no  $\mathcal{D}$ -path connecting  $u$  and  $v$  then by Theorem 1(2)  $\mathcal{H}$  has two different colorings with two colors. This contradicts to the unique colorability of mixed hypertree  $\mathcal{H}$ .

Assume that in the sequence  $u = x_1, x_2, \dots, x_p = v$  there exists a triple of pairwise different vertices  $x_j, x_{j+1}, x_{j+2}$  such that  $(x_j, x_{j+1}, x_{j+2}) \notin \mathcal{C}$ . Evidently,  $x_{j+1}$  is not pendant in  $T$ . Let  $T_1$  and  $T_2$  be two connected components obtained after deletion of vertex  $x_{j+1}$  from the host tree  $T$ .

There are two cases. (1)  $c(x_j) = c(x_{j+2})$ . From Theorem 1(1) it follows that the number of colors in the unique coloring  $c$  of  $\mathcal{H}$  is 2. Recolor the vertex  $x_{j+2}$  and all vertices on even distance from  $x_{j+2}$  in the component  $T_2$  with the new color. The obtained coloring is a proper coloring of  $\mathcal{H}$  different from  $c$ , a contradiction.

(2)  $c(x_j) \neq c(x_{j+2})$ . Since  $(x_j, x_{j+1}), (x_{j+1}, x_{j+2}) \in \mathcal{D}$  we have that  $c(x_j) \neq c(x_{j+1}) \neq c(x_{j+2})$ . Consequently,  $\mathcal{H}$  is colored with at least three colors. But according to Theorem 1 every mixed hypertree can be colored with two colors, a contradiction.

$\Leftarrow$  Assume that any two vertices  $x, y \in X$  are joined by an  $(x, y)$ -invector. Suppose  $\mathcal{H}$  has at least two strict colorings  $c_1$  and  $c_2$ . Then there exist two vertices, say  $x', y'$ , such that  $c_1(x') = c_1(y')$  but  $c_2(x') \neq c_2(y')$ . Consider  $(x', y')$ -invector  $x' = x_0, x_1, \dots, x_k = y'$ . From the definition of invector follows that if  $k$  is even then in all possible colorings the vertices  $x'$  and  $y'$  have the same color. If  $k$  is odd then in all possible colorings the vertices  $x'$  and  $y'$  have distinct colors. Consider the unique  $(x', y')$ -path connecting the vertices  $x', y'$  on the host tree  $T$ . One can see that the parity of  $k$  coincides with the parity of length of the path. Moreover, it is true for any other  $(x', y')$ -invector. Therefore, in all colorings either  $c(x') = c(y')$  or  $c(x') \neq c(y')$ , a contradiction. ■

**Corollary 1.** *If  $\mathcal{H}$  is a uniquely colorable mixed hypertree then  $\mathcal{D} = \mathcal{E}$ .*

**Definition.** Let  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  be a mixed hypergraph. The  $\mathcal{C}$ -edge  $C \in \mathcal{C}$  is called *redundant* if  $R(\mathcal{H}) = R(\mathcal{H}_1)$ , where  $\mathcal{H}_1 = (X, \mathcal{C} \setminus \{C\}, \mathcal{D})$ .

In a uniquely colorable mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  any  $\mathcal{C}$ -edge of size  $\geq 4$  is redundant because there is no invector containing such  $\mathcal{C}$ -edge.

**Theorem 3.** *In a uniquely colorable mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  a  $\mathcal{C}$ -edge  $C$  of size 3 is redundant if and only if there exists a simple cyclic invector containing  $C$ .*

**Proof.** Let  $C = (x_1, x_2, x_3)$  be the redundant  $\mathcal{C}$ -edge. By definition  $\mathcal{H}' = (X, \mathcal{C}', \mathcal{D})$  where  $\mathcal{C}' = \mathcal{C} \setminus \{C\}$  is a uniquely colorable mixed hypertree. Then for the vertices  $x_1$  and  $x_3$  in  $\mathcal{H}'$  there exists an  $(x_1, x_3)$ -inventor:  $x_1 = z_0, z_1, \dots, z_k = x_3$ . Construct the  $(x_1, x_3)$ -inventor in the following way:  $x_1 = z_0, z_1, \dots, z_k = x_3, x_2, x_1$ . This inventor is a simple cyclic inventor of  $\mathcal{H}$  containing  $C$ .

Conversely, suppose that  $\mathcal{C}$ -edge,  $C = (x_1, x_2, x_3)$  is contained in some simple cyclic inventor  $x_1 = z_0, z_1, \dots, z_k = x_3, x_2, x_1$ . Then the vertices  $x_1$  and  $x_3$  are joined by two different  $(x_1, x_3)$ -inventors:  $(x_1, x_2, x_3) = C$  and  $(x_1 = z_0, z_1, \dots, z_k = x_3) = (x_1, x_3)'$ -inventor. In each  $(x, y)$ -inventor containing  $C$  replace this  $\mathcal{C}$ -edge by  $(x_1, x_3)'$ -inventor. Thus,  $\mathcal{H}' = (X, \mathcal{C} \setminus \{C\}, \mathcal{D})$  is uniquely colorable, i.e., the  $\mathcal{C}$ -edge  $C$  is redundant. ■

Let us have a mixed hypergraph  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ . Consider  $X = X_1 \cup X_2 \cup \dots \cup X_i$  any  $i$ -coloring of  $\mathcal{H}$ ,  $\chi(\mathcal{H}) \leq i \leq \bar{\chi}(\mathcal{H})$ . Choose any  $X_j$  and construct *touching graph*  $L_j = (X_j, V_j)$  in the following way: if some  $C \in \mathcal{C}$  satisfies  $C \cap X_j = \{x, y\}$  and  $|C \cap X_k| \leq 1$ ,  $k \neq j$ , for some  $x, y \in X_j$ , then  $(x, y) \in V_j$  (cf. pair graphs [3]).

**Theorem 4.** *If a mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is uniquely colorable then in its 2-coloring the touching graphs  $L_1$  and  $L_2$  are connected.*

**Proof.** By Theorem 1(2), Corollary 1 we obtain  $|\mathcal{D}| = n - 1$ ,  $\bar{\chi} = 2$  for each uniquely colorable mixed hypertree. If at least one touching graph is disconnected, then we can construct a new coloring of  $\mathcal{H}$  with 3 colors by assigning new color to the vertices of one component. This assures the proper coloring also of any  $\mathcal{C}$ -edge of size  $\geq 4$ . ■

**Corollary 2.** *The minimal number of  $\mathcal{C}$ -edges in any uniquely colorable mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is  $n - 2$ .*

**Proof.** Let  $\mathcal{H}$  be a uniquely colorable mixed hypertree. Consider its unique 2-coloring, say  $X = X_1 \cup X_2$ , and construct the touching graphs  $L_1 = (X_1, V_1)$ ,  $L_2 = (X_2, V_2)$ . The minimal number of edges in  $L_i$  to be connected is  $|X_i| - 1$ , and in this case each of  $L_i$  is a tree,  $i = 1, 2$ . Since every edge in  $L_i$  corresponds to some  $\mathcal{C}$ -edge of  $\mathcal{H}$ , we obtain that the minimal number of  $\mathcal{C}$ -edges is:

$$|X_1| - 1 + |X_2| - 1 = |X| - 2. \quad \blacksquare$$

**Corollary 3.** *In a uniquely colorable mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  the number of redundant  $\mathcal{C}$ -edges is  $|\mathcal{C}| - n + 2$ .*

**Proof.** Indeed, consider touching graphs  $L_i$ , and construct a spanning trees  $T_i, i = 1, 2$ . Each elementary cycle in  $L_i$  generates some simple cyclic invector in  $\mathcal{H}$ . Therefore, each  $\mathcal{C}$ -edge of  $\mathcal{H}$  which has a size  $\geq 4$  or corresponds to some edge of  $L_i$  which is a chord with respect to  $T_i$ , is redundant. Hence, the assertion follows. ■

**Remark.** Redundant  $\mathcal{C}$ -edge may become not redundant after deleting from  $\mathcal{C}$  some another redundant  $\mathcal{C}$ -edges.

**Definition.** A mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  is called complete if every edge of the host tree  $T$  forms a  $\mathcal{D}$ -edge of  $\mathcal{H}$ , and every path on three vertices of  $T$  forms a  $\mathcal{C}$ -edge in  $\mathcal{H}$ .

Therefore, having the host tree  $T$  for the complete mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  we obtain that  $\mathcal{D} = \mathcal{E}$ .

Denote by  $M$  the number of  $\mathcal{C}$ -edges of a complete mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ . Then

$$M = \sum_{\substack{x \in T \\ d(x) \geq 2}} \binom{d(x)}{2},$$

where  $d(x)$  is the degree of vertex  $x$  in the host tree  $T$ .

Examples show that for any  $k > 1$  one can construct a mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  with  $|\mathcal{D}| = n - 1$ ,  $n - 2 \leq |\mathcal{C}| \leq M$  and  $\bar{\chi}(\mathcal{H}) = k$ . Therefore these bounds on  $|\mathcal{D}|$  and  $|\mathcal{C}|$  are not sufficient for the mixed hypertrees to be uniquely colorable.

**Proposition 1.** *A uniquely colorable mixed hypertree with  $|\mathcal{C}| = n - 2$  is a pseudo-chordal mixed hypergraph.*

**Proof.** Since  $\mathcal{H}$  is uniquely colorable mixed hypertree and  $|\mathcal{C}| = n - 2$  then it contains no redundant  $\mathcal{C}$ -edges and, moreover, all  $\mathcal{C}$ -edges have the size 3. It follows that there exists a pendant vertex, say  $x$ , of the host tree  $T = (X, \mathcal{E})$  which belongs to precisely one  $\mathcal{C}$ -edge, say  $(x, y, z)$ . The neighbourhood of  $x$  induces a complete  $\mathcal{D}$ -graph on 2 vertices, which itself is uniquely colorable. Consequently, the vertex  $x$  is simplicial in  $\mathcal{H}$ . Deleting the vertex  $x$  and  $\mathcal{C}$ -edge and  $\mathcal{D}$ -edge containing it, obtain  $\mathcal{H}'$  which

is uniquely colorable mixed hypertree with minimal number of  $\mathcal{C}$ -edges. Indeed, if  $\mathcal{H}'$  would be not uniquely colorable, then two distinct colorings of  $\mathcal{H}'$  formed different colorings of  $\mathcal{H}$  because  $c(x) = c(z)$ , a contradiction. ■

**Remark.** Redundant  $\mathcal{C}$ -edges enlarge the neighbourhood of some vertices without affecting any coloring. Therefore, to recognise the pseudo-chordality we need to delete the redundant  $\mathcal{C}$ -edges.

From the Theorem 4, Corollaries 2–4 and Proposition 1 we conclude that a uc-orderable mixed hypertree  $\mathcal{H}$  can be recognised by consecutive elimination of pendant vertices of  $\mathcal{D}$ -graph  $\mathcal{H}_{\mathcal{D}}$  in special ordering by applying the following

**Algorithm** (*uc-ordering*).

**Input:** A mixed hypertree  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ ,  $\sigma$  –  $n$ -dimensional empty vector.

**Idea:** Simultaneous decomposition of  $\mathcal{H}_{\mathcal{D}}$ , spanning trees  $T_1$  and  $T_2$  of touching graphs  $L_1, L_2$ , respectively, by pendant vertices.

**Iterations:**

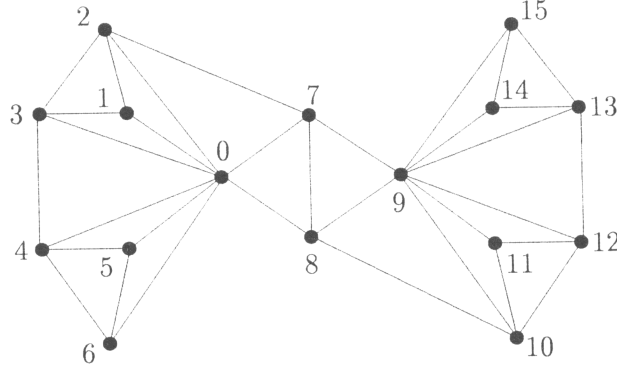
1. If there is a vertex  $x \in X$  belonging to none  $\mathcal{C}$ -edge of size 3 or  $\mathcal{D}$ -edge of size 2 then return **NON UC**. Otherwise remove from  $\mathcal{C}$  all elements of size  $\geq 4$ .
2. Color  $\mathcal{D}$ -graph  $\mathcal{H}_{\mathcal{D}}$  with two colors.
3. Construct touching graphs  $L_1$  and  $L_2$ .
4. If  $L_i$ ,  $i = 1, 2$ , is not connected then return **NON UC**.
5. For  $L_i$  construct spanning tree  $T_i$ ,  $i = 1, 2$ .
6.  $i := 1$ .
7. While in  $T_i$  there exists a vertex  $x$  pendant in both  $T_i$  and  $\mathcal{H}_{\mathcal{D}}$  then delete it from  $T_i$  and  $\mathcal{H}_{\mathcal{D}}$  and include  $x$  in  $\sigma$ .
8. If at least one of  $T_1$  and  $T_2$  is not empty then go to 9. Otherwise return **UC,  $\sigma$ -uc-ordering**.
9. If  $i = 1$  then assign  $i := i + 1$ , otherwise  $i := i - 1$ . Go to 7.

**Remark.** All chords of graph  $L_i$  with respect to spanning tree  $T_i$ ,  $i = 1, 2$ , correspond to redundant  $\mathcal{C}$ -edges in  $\mathcal{H}$ . The trees  $T_1$  and  $T_2$  provide existence of unique  $(x, y)$ -invertor for any  $x, y \in X$ . The last assures at any step of the algorithm the existence of a vertex, say  $x$ , pendant in both  $\mathcal{H}_{\mathcal{D}}$  and one



of  $T_1$  or  $T_2$ . Notice that not every elimination of pendant vertices generates a uc- ordering in  $\mathcal{H}_{\mathcal{D}}$ .

**Example.** Given the mixed hypertree  $\mathcal{H}$  with  $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ ,  $\mathcal{C} = \{(0, 1, 2); (0, 1, 3); (0, 2, 3); (0, 3, 4); (0, 4, 5); (0, 5, 6); (0, 4, 6); (0, 2, 7); (0, 7, 8); (7, 8, 9); (9, 8, 10); (9, 10, 11); (9, 11, 12); (9, 10, 12); (9, 12, 13); (9, 13, 14); (9, 14, 15); (9, 13, 15)\}$ , and  $\mathcal{D} = \{(0, 1); (0, 2); (0, 3); (0, 4); (0, 5); (0, 6); (0, 7); (7, 8); (8, 9); (9, 10); (9, 11); (9, 12); (9, 13); (9, 14); (9, 15)\}$ , see the figures 1 and 2 (the  $\mathcal{C}$ -edges are depicted by triangles).



$$\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$$

Figure 1

Apply the algorithm. Each vertex of  $\mathcal{H}$  belongs to at least one  $\mathcal{D}$ -edge of size 2 and at least one  $\mathcal{C}$ -edge of size 3. Color  $\mathcal{H}_{\mathcal{D}}$  with 2 colors. Denote by  $X_1 = \{0, 8, 10, 11, 12, 13, 14, 15\}$  and  $X_2 = \{1, 2, 3, 4, 5, 6, 7, 9\}$  two color classes of  $\mathcal{H}_{\mathcal{D}}$ . Construct the following touching graphs  $L_1 = (X_1, V_1)$  and  $L_2 = (X_2, V_2)$ , where  $V_1 = \{(0, 8); (8, 10); (10, 11); (10, 12); (11, 12); (12, 13); (13, 14); (13, 15); (14, 15)\}$  and  $V_2 = \{(1, 2); (1, 3); (2, 3); (2, 7); (3, 4); (4, 5); (4, 6); (5, 6); (7, 9)\}$ . Choose the respective trees  $T_1$  and  $T_2$  (Figure 3).

Consecutively applying the algorithm we obtain one of uc-orderings of  $\mathcal{H}$ :  $\sigma = \{15, 14, 13, 11, 12, 10, 1, 5, 9, 6, 4, 3, 2, 8, 0, 7\}$ . At the 7-th step of the algorithm, after including of vertex 10 in  $\sigma$ , we alternate the trees because  $T_1$  has no pendant vertex which is also pendant in  $\mathcal{H}_{\mathcal{D}}$ . The next alternations of trees are made after adding to  $\sigma$  of vertices 2 and 0. From the above algorithm we have

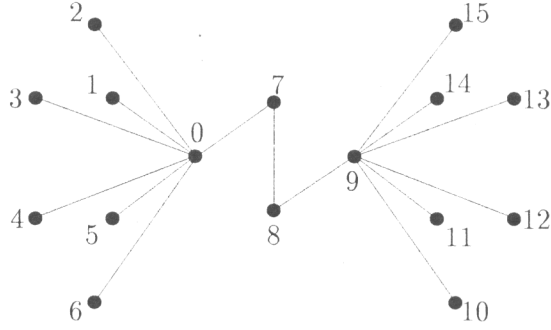
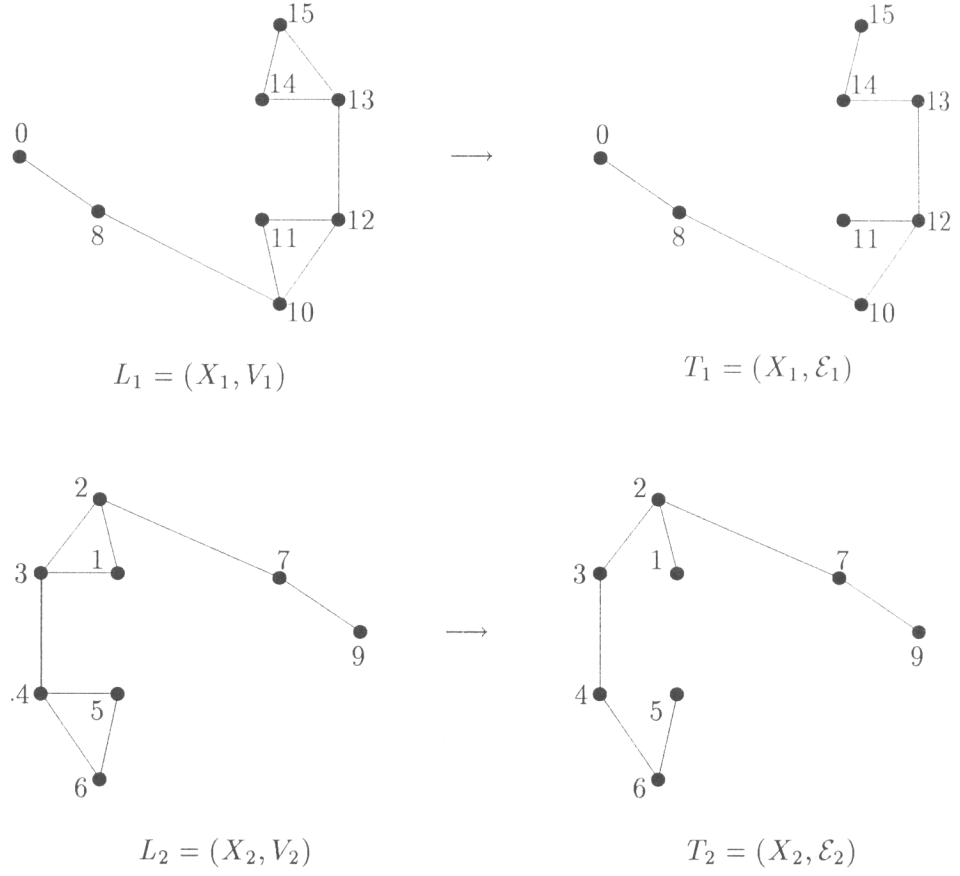
Figure 2.  $\mathcal{H}_{\mathcal{D}} = (X, \emptyset, \mathcal{D})$ 

Figure 3

**Theorem 5.** *A mixed hypertree is uniquely colorable if and only if it is uc-orderable.*

Therefore, combining the Theorems 2, 5 and relation between chromatic polynomial and chromatic spectrum [6, 7], we obtain the following

**Theorem 6.** *Let  $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$  be a mixed hypertree. Then the following five statements are equivalent:*

- (1)  $R(\mathcal{H}) = (0, 1, 0, \dots, 0)$ ;
- (2)  $P(\mathcal{H}, k) = k(k-1)$ ;
- (3)  $\mathcal{H}$  is uniquely colorable;
- (4) Every two vertices  $x, y \in X$  are joined by an  $(x, y)$ -invertor;
- (5)  $\mathcal{H}$  is uc-orderable.

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