

SOME NEWS ABOUT THE INDEPENDENCE NUMBER OF A GRAPH

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Abstract

For a finite undirected graph G on n vertices some continuous optimization problems taken over the n -dimensional cube are presented and it is proved that their optimum values equal the independence number of G .

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1 Introduction and Results

Let G be a finite simple and undirected graph on $V(G) = \{1, 2, \dots, n\}$ with its edge set $E(G)$. A subset I of $V(G)$, such that the subgraph of G induced by I is edgeless, is called an *independent set* of G , and the maximum cardinality of an independent set of G is named the *independence number* $\alpha(G)$ of G . $N(i)$ and d_i denote the set and the number of neighbours of $i \in V(G)$ in G , respectively, and let $\Delta(G) = \max\{d_i \mid i \in V(G)\}$ and $C^n = \{(x_1, x_2, \dots, x_n) \mid 0 \leq x_i \leq 1, i = 1, 2, \dots, n\}$. For events A and B and for a random variable Z of an arbitrary random space, $P(A)$, $P(A|B)$, and $\mathcal{E}(Z)$ denote the probability of A , the conditional probability of A given B , and the expectation of Z , respectively. Since the computation of $\alpha(G)$ is difficult (INDEPENDENT SET is an NP-complete problem; see [6]), much work was done to establish bounds on $\alpha(G)$ (e.g., see [1, 3, 4, 5, 8, 10, 12, 13, 14, 15, 16, 17]), to find efficient algorithms forming a large independent set of G (e.g., see [2, 7, 8, 9, 10, 12]), or to replace the combinatorial optimization problem to determine $\alpha(G)$ by a continuous one

(e.g., see [9, 11]). The last approach leads to bounds on $\alpha(G)$ as well as to efficient algorithms (e.g., see [8, 9]). In the present paper some new continuous optimization problems taken over C^n are presented and it is proved that their optimum values equal $\alpha(G)$. Theorem 1 gives a remarkable result of T.S. Motzkin and E.G. Straus [11] and Theorem 2 is proved in [9].

Theorem 1.

$$\alpha(G) = \max_{(0,0,\dots,0) \neq (x_1,x_2,\dots,x_n) \in C^n} \frac{\left(\sum_{i \in V(G)} x_i \right)^2}{\sum_{i \in V(G)} x_i^2 + 2 \sum_{ij \in E(G)} x_i x_j}.$$

$$\textbf{Theorem 2. } \alpha(G) = \max_{(x_1,x_2,\dots,x_n) \in C^n} \sum_{i \in V(G)} (x_i \prod_{j \in N(i)} (1 - x_j)).$$

A classical lower bound on $\alpha(G)$ due to Y. Caro and V.K. Wei [3, 17] is given by the following theorem.

$$\textbf{Theorem 3. } \alpha(G) \geq \sum_{i \in V(G)} \frac{1}{1+d_i}.$$

The next Theorems 4, 5, 6, and 7 are the main results of the present paper.

$$\textbf{Theorem 4. } \alpha(G) = \max_{(x_1,x_2,\dots,x_n) \in C^n} e_G(x_1, x_2, \dots, x_n), \text{ where}$$

$$e_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} \left(\frac{x_i}{1 + \sum_{j \in N(i)} x_j} + \frac{(1-x_i) \prod_{j \in N(i)} (1-x_j)}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right).$$

$$\textbf{Theorem 5. } \alpha(G) = \max_{(x_1,x_2,\dots,x_n) \in C^n} f_G(x_1, x_2, \dots, x_n), \text{ where}$$

$$f_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} \left(x_i + \frac{(1-x_i) \prod_{j \in N(i)} (1-x_j)}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right) - \sum_{ij \in E(G)} x_i x_j.$$

The following Theorem 6 looks more "complicate", but it is "stronger" than Theorem 4 and Theorem 5 (see Remark 1).

$$\textbf{Theorem 6. } \alpha(G) = \max_{(x_1,x_2,\dots,x_n) \in C^n} g_G(x_1, x_2, \dots, x_n), \text{ where}$$

$$g_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} \left(\left(x_i + \frac{1-x_i}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)} \right) \prod_{j \in N(i)} (1-x_j) \right)$$

$$+ \sum_{i \in V'} \frac{x_i(1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j} \text{ and } V' = \left\{ i \in V(G) \mid \sum_{j \in N(i)} x_j > 0 \right\}.$$

A "weaker" (see Remark 1), but a more "transparent" and (see Remark 2) an "algorithmically realizable" version of Theorem 5 is the following one.

Theorem 7. $\alpha(G) = \max_{(x_1, x_2, \dots, x_n) \in C^n} h_G(x_1, x_2, \dots, x_n)$, where

$$h_G(x_1, x_2, \dots, x_n) = \sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j.$$

2 Proofs

Throughout the proofs we will use the well-known fact that for a random subset M of a given finite set N ,

$$\mathcal{E}(|M|) = \sum_{y \in N} P(y \in M) = \sum_{k=0}^{|N|} kP(|M| = k).$$

Let I be a maximum independent set of G and let $x_i^* = 1$ if $i \in I$ and $x_i^* = 0$ if $i \notin I$. Since $(1 - x_i^*) \prod_{j \in N(i)} (1 - x_j^*) = 0$ for $i \in V(G)$ and $\sum_{ij \in E(G)} x_i^* x_j^* = 0$, we obtain

Lemma 1. $\alpha(G) = e_G(x_1^*, x_2^*, \dots, x_n^*) = f_G(x_1^*, x_2^*, \dots, x_n^*) = g_G(x_1^*, x_2^*, \dots, x_n^*) = h_G(x_1^*, x_2^*, \dots, x_n^*)$.

With Lemma 1, it is clear that Theorem 7 follows from Theorem 5.

Now, let (x_1, x_2, \dots, x_n) be an arbitrary member of C^n . We form a set $X \subseteq V(G)$ by random and independent choice of $i \in V(G)$, where $P(i \in X) = x_i$. Let H_1 , H_2 , and H_3 be the subgraph of G induced by the vertices of X , by the vertices $i \in X$ with $N(i) \cap X \neq \emptyset$, and by the vertices $i \notin X$ with $N(i) \cap X = \emptyset$, respectively. Furthermore, let Y be a smallest subset of $V(H_2)$ covering all edges of H_2 , i.e., the graph induced by $V(H_2) - Y$ is edgeless, and let I_1 and I_3 be a maximum independent set of H_1 and H_3 , respectively. It can be seen easily that $|Y| = |V(H_2)| - \alpha(H_2)$, $|Y| \leq |E(H_2)|$ and that $(X - Y) \cup I_3$ and $I_1 \cup I_3$ are independent sets of G . Because of these remarks and the property of the expectation to be an average value, we have Lemma 2 as follows.

Lemma 2. $\alpha(G) \geq \mathcal{E}(|X - Y|) + \mathcal{E}(\alpha(H_3))$, $\alpha(G) \geq \mathcal{E}(\alpha(H_1)) + \mathcal{E}(\alpha(H_3))$,
 $\mathcal{E}(|X - Y|) = \mathcal{E}(|X|) - \mathcal{E}(|Y|) \geq \mathcal{E}(|X|) - \mathcal{E}(|E(H_2)|)$, and
 $\mathcal{E}(|X - Y|) = \mathcal{E}(|X|) - \mathcal{E}(|V(H_2)|) + \mathcal{E}(\alpha(H_2))$.

Lower bounds on $\mathcal{E}(\alpha(H_1))$, $\mathcal{E}(\alpha(H_2))$, and $\mathcal{E}(\alpha(H_3))$ are given in Lemma 3.

Lemma 3. $\mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j}$,
 $\mathcal{E}(\alpha(H_2)) \geq \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}$, where $V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}$,
and
 $\mathcal{E}(\alpha(H_3)) \geq \sum_{i \in V(G)} \frac{(1 - x_i) \prod_{j \in N(i)} (1 - x_j)}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1 - x_l)}$.

Proof. For $i \in V(G)$ define the random variable Z_i^1 with $Z_i^1 = \frac{1}{1+k}$ if $i \in X$ and $|N(i) \cap X| = k \geq 0$, and $Z_i^1 = 0$ if $i \notin X$. Using Theorem 3,

$$\begin{aligned} \mathcal{E}(\alpha(H_1)) &\geq \mathcal{E}\left(\sum_{i \in V(G)} Z_i^1\right) = \sum_{i \in V(G)} \mathcal{E}(Z_i^1) \\ &= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k) \\ &= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k) \\ &= \sum_{i \in V(G)} x_i \sum_{k=0}^{d_i} \frac{1}{1+k} P(|N(i) \cap X| = k). \end{aligned}$$

For $i \in V(G)$ we have $\sum_{k=0}^{d_i} P(|N(i) \cap X| = k) = 1$. With Jensen's inequality

$$\sum_{l=1}^m \tau_l \phi(y_l) \geq \phi\left(\sum_{l=1}^m \tau_l y_l\right) \text{ for any convex function } \phi \text{ and any } \tau_l \geq 0 \text{ for } l = 1, 2, \dots, m \text{ with } \sum_{l=1}^m \tau_l = 1,$$

$$\mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} x_i \frac{1}{1 + \sum_{k=0}^{d_i} k P(|N(i) \cap X| = k)} = \sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j}.$$

Now, let $V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}$. For $i \in V(G)$ let Z_i^2 be the random variable with $Z_i^2 = \frac{1}{1+k}$ if $i \in X$ and $|N(i) \cap X| = k \geq 1$, and $Z_i^2 = 0$ otherwise. Then,

$$\begin{aligned}
\mathcal{E}(\alpha(H_2)) &\geq \mathcal{E}\left(\sum_{i \in V(G)} Z_i^2\right) = \sum_{i \in V(G)} \mathcal{E}(Z_i^2) \\
&= \sum_{i \in V(G)} \sum_{k=1}^{d_i} \frac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k) \\
&= \sum_{i \in V(G)} \sum_{k=1}^{d_i} \frac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k) \\
&= \sum_{i \in V(G)} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} P(|N(i) \cap X| = k).
\end{aligned}$$

$$\begin{aligned}
P(|N(i) \cap X| = 0) + \sum_{k=1}^{d_i} P(|N(i) \cap X| = k) &= 1 \text{ for } i \in V(G) \text{ and with} \\
\mu_i = P(|N(i) \cap X| = 0) &= \prod_{j \in N(i)} (1 - x_j) \text{ and } \sigma_{ik} = P(|N(i) \cap X| = k),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(\alpha(H_2)) &\geq \sum_{i \in V(G)} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} = \sum_{i \in V(G), \mu_i < 1} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} \\
&= \sum_{i \in V'} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} = \sum_{i \in V'} x_i (1 - \mu_i) \sum_{k=1}^{d_i} \frac{\sigma_{ik}}{(1+k)(1-\mu_i)}.
\end{aligned}$$

For $\lambda_{ik} = \frac{\sigma_{ik}}{1-\mu_i}$ we have $\lambda_{ik} \geq 0$, $\sum_{k=1}^{d_i} \lambda_{ik} = 1$ if $i \in V'$, and again using Jensen's inequality,

$$\begin{aligned}
\mathcal{E}(\alpha(H_2)) &\geq \sum_{i \in V'} x_i (1 - \mu_i) \frac{1}{1 + \sum_{k=1}^{d_i} k \lambda_{ik}} \\
&= \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{k=1}^{d_i} k P(|N(i) \cap X| = k)} = \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}.
\end{aligned}$$

Finally, let us consider the random variable Z_i^3 with $Z_i^3 = \frac{1}{1+k}$ if $i \in V(H_3)$ and $|N(i) \cap V(H_3)| = k \geq 0$, and $Z_i^3 = 0$ if $i \notin V(H_3)$. Then

$$\begin{aligned}
\mathcal{E}(\alpha(H_3)) &\geq \mathcal{E}\left(\sum_{i \in V(G)} Z_i^3\right) = \sum_{i \in V(G)} \mathcal{E}(Z_i^3) \\
&= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in V(H_3) \text{ and } |N(i) \cap V(H_3)| = k) \\
&= \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in V(H_3)) P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3)) \\
&= \sum_{i \in V(G)} ((1 - x_i) \prod_{j \in N(i)} (1 - x_j) \sum_{k=0}^{d_i} \frac{1}{1+k} P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3)))
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i \in V(G)} ((1-x_i) \prod_{j \in N(i)} (1-x_j) \frac{1}{1 + \sum_{k=0}^{d_i} k P(|N(i) \cap V(H_3)|=k \mid i \in V(H_3))}) \\
&= \sum_{i \in V(G)} ((1-x_i) \prod_{j \in N(i)} (1-x_j) \frac{1}{1 + \sum_{j \in N(i)} \prod_{l \in N(j) \setminus (N(i) \cup \{i\})} (1-x_l)}),
\end{aligned}$$

and Lemma 3 is proved. ■

$$\begin{aligned}
&\text{Theorem 4, 5, and 6 follow with } \mathcal{E}(|X|) = \sum_{i \in V(G)} x_i, \mathcal{E}(|E(H_2)|) \\
&= \sum_{ij \in E(G)} x_i x_j, \mathcal{E}(|V(H_2)|) = \sum_{i \in V(G)} x_i (1 - \prod_{j \in N(i)} (1-x_j)), \text{ Lemma 1, 2, and 3.}
\end{aligned}$$

3 Remarks

For $\phi, \psi \in \{e, f, g, h\}$ define $\phi \leq \psi$ if $\phi_G(x_1, x_2, \dots, x_n) \leq \psi_G(x_1, x_2, \dots, x_n)$ for every graph G on n vertices and for every $(x_1, x_2, \dots, x_n) \in C^n$. We write $\phi <> \psi$ if neither $\phi \leq \psi$ nor $\psi \leq \phi$.

Remark 1. $h \leq f \leq g$, $e \leq g$ and $e <> f$.

Proof. We will use the following Lemma 4, which can be seen easily by induction on r .

Lemma 4. For an integer $r \geq 1$ and $a_1, a_2, \dots, a_r \in [0, 1]$,

$$\sum_{q=1}^r a_q + \prod_{q=1}^r (1 - a_q) \geq 1.$$

The inequality $h \leq f$ is obvious. To see $f \leq g$, first notice that $\sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j = \sum_{i \in V(G)} x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j)$. If $\sum_{j \in N(i)} x_j = 0$ for an $i \in V(G)$ then $x_i = x_i (\prod_{j \in N(i)} (1 - x_j)) = x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j)$. Hence, with the abbreviation $\mu_i = \prod_{j \in N(i)} (1 - x_j)$ and $\rho_i = \sum_{j \in N(i)} x_j$ for $i \in V(G)$ we have to show $\sum_{i \in V'} (x_i (\mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i})) \geq \sum_{i \in V'} (x_i (1 - \frac{1}{2} \rho_i))$, where again $V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}$.

Using Lemma 4, even $\mu_i + \frac{(1-\mu_i)^2}{1-\mu_i+\rho_i} \geq 1 - \frac{1}{2} \rho_i$ for all $i \in V'$.

To prove $e \leq g$, we have to show

$$\sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j} \leq \sum_{i \in V(G)} x_i \prod_{j \in N(i)} (1 - x_j) + \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}.$$

Since $\frac{x_i}{1 + \sum_{j \in N(i)} x_j} = x_i \prod_{j \in N(i)} (1 - x_j)$ if $\sum_{j \in N(i)} x_j = 0$, it is sufficient to establish

$$\frac{1}{1 + \rho_i} \leq \mu_i + \frac{(1 - \mu_i)^2}{1 - \mu_i + \rho_i} \text{ if } \sum_{j \in N(i)} x_j > 0, \text{ what is verified easily.}$$

For a cycle C_n on n vertices $e_{C_n}(\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}) < f_{C_n}(\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$, $e_{C_n}(\frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3}) > f_{C_n}(\frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3})$ and Remark 1 is proved. ■

With $h \leq f$ and $e < f$ it is clear that $e \leq h$ does not hold. It remains open, whether $h < e$ or $h \leq e$.

Theorems 1, 2, 4, 5, 6, and 7 are of that type that the independence number $\alpha(G)$ of a graph G on n vertices equals the optimum value of a continuous optimization problem $O(G)$ to maximize a certain function ϕ_G over C^n . Hence, $\phi_G(x_1, x_2, \dots, x_n)$ is a lower bound on $\alpha(G)$ for every $(x_1, x_2, \dots, x_n) \in C^n$. Let $(x'_1, x'_2, \dots, x'_n) \in C^n$ be the solution of an arbitrary approximation algorithm for $O(G)$. How to find an independent set I of G in polynomial time such that $|I| \geq \phi_G(x'_1, x'_2, \dots, x'_n)$? In [8] and [9] efficient algorithms forming I with $|I| \geq \phi_G(x'_1, x'_2, \dots, x'_n)$ are given if $O(G)$ is the optimization problem of Theorem 1 or of Theorem 2. Remark 2 shows that this is also possible if we consider the case $\phi_G = h_G$. In case $\phi_G = e_G$, $\phi_G = f_G$ or $\phi_G = g_G$ the problem remains open, whether such an algorithm exists.

Remark 2. There is an $\mathcal{O}(\Delta(G)n)$ -algorithm with

INPUT: $(x_1, x_2, \dots, x_n) \in C^n$,

OUTPUT: an independent set $I \subseteq V(G)$ with $|I| \geq \sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j$.

Proof. First we give the Algorithm:

1. For $i = 1$ to n do if $\sum_{j \in N(i)} x_j < 1$ then $x_i := 1$ else $x_i := 0$.
2. For $i = 1$ to n do if $(x_i = 1 \text{ and } \prod_{j \in N(i)} (1 - x_j) = 0)$ then $x_i := 0$.
3. $I := \{i \in V(G) \mid x_i = 1\}$.

STOP

It is obvious that the algorithm is an $\mathcal{O}(\Delta(G)n)$ -algorithm. For the input vector $(x_1, x_2, \dots, x_n) \in C^n$ set

$$\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j = a.$$

After step 1, the current (x_1, x_2, \dots, x_n) is a 0-1-vector and

$$\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j \geq a$$

because

$$\frac{\partial}{\partial x_i} \left(\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j \right) = 1 - \sum_{j \in N(i)} x_j, \text{ i.e., } \sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j$$

is multilinear.

In step 2, $\prod_{j \in N(i)} (1 - x_j) = 0$ if and only if there is at least one $j \in N(i)$ such that $x_j = 1$. With $x_i = 0$ instead of $x_i = 1$ the sum $\sum_{k \in V(G)} x_k$ decreases by 1 and the sum $\sum_{kj \in E(G)} x_k x_j$ decreases by at least 1, hence

$\sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j$ does not decrease.

After step 2,

$$x_k x_j = 0 \text{ for all } kj \in E(G), |I| = \sum_{k \in V(G)} x_k = \sum_{k \in V(G)} x_k - \sum_{kj \in E(G)} x_k x_j \geq a$$

and Remark 2 is proved. ■

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