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# MEAN VALUE FOR THE MATCHING AND DOMINATING POLYNOMIAL

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**Abstract**

The mean value of the matching polynomial is computed in the family of all labeled graphs with $n$ vertices. We introduce the dominating polynomial of a graph whose coefficients enumerate the dominating sets for a graph and study some properties of the polynomial. The mean value of this polynomial is determined in a certain special family of bipartite digraphs.

**Keywords:** matching, matching polynomial, dominating set.

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## 1 Introduction

The goal of this paper is to compute the average polynomials for the well-known matching polynomial and the dominating polynomial in certain classes of graphs. The matching polynomial first appeared in a paper by Heilman and Lieb [7] as a thermodynamic partition function. For a very interesting introduction to its combinatorial study as well as many of its properties we refer the reader to [3] and [5]. The notion of domination in graphs was introduced last century. This theory can be consulted in the books by Ore [12] and Berge [2]. The paper [9] shows recent developments of the theory and a large account of references on the topic.

Many graph parameters are intrinsically hard to compute (see for example [4]). However, this does not mean that one can not obtain formulas

for them in certain classes of graphs. The class of random graphs, for example, is of particular interest. For instance, computing probabilistic moments for the number of independent sets of vertices in a graph were proved to be useful to compute the number of antichains in partially ordered sets (see [1]).

Typically one has the following situation: Let $P$ be a property of a set of vertices (edges) of a graph $G$ and denote by $p_k(G)$ the number of sets with $k$ vertices (edges) satisfying the property $P$. Then, introduce a generating function, say, the polynomial $p(G,t) = \sum_k p_k(G)\, t^k$ (or an exponential generating function) and ask for a "closed formula" for this generating function of a random graph. Is it possible to find out such a formula for a given graph parameter? This is a difficult question and the answer would not be necessarily affirmative. In the paper we consider this kind of problem for matchings and dominating sets.

In the first part of the paper we calculate the so called average matching polynomial in the class of all labelled graphs with $n$ vertices. The subsequent sections are devoted to introduce here the dominating polynomial of a graph: definition and basic properties. Finally, we determine the average dominating polynomial in a certain class of bipartite digraphs.

For the terminology of graph theory used here, see [11].

## 2 Average Matching Polynomial

Consider a simple graph $G = (V, E)$. Let $M \subseteq E$ be a matching of the graph $G$. If $M$ is a matching, then any $M' \subset M$ is a matching, too. For $|V| = n$, we have that $|M| \leq n/2$ and if the equality holds, then the matching is called *perfect*. Let $\alpha_k(G)$ denote the number of matchings of cardinality $k$ ($k \in \mathbb{N}$) of a graph $G$ and by convention, $\alpha_0(G) = 1$. The matching polynomial is defined by

$$\alpha(G,t) = \sum_{k=0}^{[n/2]} (-1)^k \alpha_k(G)\, t^{n-2k}.$$

There are basic properties of the matching polynomial studied in [5]. We recall some of these properties that will be used later in this paper.

**Theorem 21.** $\alpha(G,t) = \alpha(G-e,t) - \alpha(G-i-j,t)$,
*where* $i, j \in V$ *and* $e = \{i,j\} \in E$.

Applying this theorem to the complete graph, we have the following result.

**Theorem 22.** *For the complete graph $K_n$,*

$$\alpha(K_n, t) = He_n(t) = 2^{-n/2} H_n\left(t/\sqrt{2}\right) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k! 2^k (n-2k)!} t^{n-2k},$$

*where*

$$\begin{aligned} H_n(t) &= (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \quad \text{and} \\ He_n(t) &= (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2} \end{aligned}$$

*are the Hermite and the special Hermite polynomial, respectively.*

Further information on Hermite polynomials can be found in [10].
Let $\mathbb{G}_n$ be the set of all labeled graphs with $n$ vertices. We define

$$\alpha_n(t) = 2^{-\binom{n}{2}} \sum_{G \in \mathbb{G}_n} \alpha(G, t)$$

to be the *mean value of the matching polynomial in the set* $\mathbb{G}_n$ or the *average matching polynomial in* $\mathbb{G}_n$. If $\mathbb{G}_n^k$ denotes the set of all labeled graphs with $n$ vertices and $k$ edges, then we can define the average matching polynomial in this set by

$$\alpha_n^k(t) = \binom{\binom{n}{2}}{k}^{-1} \sum_{G \in \mathbb{G}_n^k} \alpha(G, t). \tag{2.1}$$

It is not difficult to establish that

$$\alpha_n(t) = 2^{-\binom{n}{2}} \sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} \alpha_n^k(t). \tag{2.2}$$

**Lemma 23.**

$$\alpha_n^k(t) = \sum_{j=0}^{[n/2]} (-1)^j \binom{k}{j} \binom{\binom{n}{2}}{j}^{-1} \alpha_j(K_n)\, t^{n-2j}.$$

***Proof.*** Applying the definition of the matching polynomial to (2.1), we have that

$$\begin{aligned}\alpha_n^k(t) &= \binom{\binom{n}{2}}{k}^{-1} \sum_{G\in\mathbb{G}_n^k} \sum_{j=0}^{[n/2]} (-1)^j \alpha_j(G)\, t^{n-2j} \\ &= \sum_{j=0}^{[n/2]} (-1)^j \binom{\binom{n}{2}}{k}^{-1} \left[\sum_{G\in\mathbb{G}_n^k} \alpha_j(G)\right] t^{n-2j}.\end{aligned}$$

We compute the sum in brackets. Let $M(G,j)$ be the set of matchings of cardinality $j$ of $G$, then

$$\sum_{G\in\mathbb{G}_n^k} \alpha_j(G) = \sum_{G\in\mathbb{G}_n^k} \sum_{M\in M(G,j)} 1. \tag{2.3}$$

But any matching of a graph $G\in\mathbb{G}_n^k$ is a matching of the complete graph $K_n$, so the sum in the right of (2.3) is equal to

$$\sum_{M\in M(K_n,j)} \sum_{\substack{G\in\mathbb{G}_n^k \\ M\in M(G,j)}} 1. \tag{2.4}$$

The second sum of (2.4) is the number of graphs belonging to $\mathbb{G}_n^k$ which contain a fixed matching with exactly $j$ edges. Fixing this matching, from the other $\binom{n}{2}-j$ edges of $K_n$, we can choose the $k-j$ missing edges of $G\in\mathbb{G}_n^k$ in

$$\binom{\binom{n}{2}-j}{k-j}$$

ways. Therefore, the expression (2.4) is equal to $\binom{\binom{n}{2}-j}{k-j}\alpha_j(K_n)$ and

$$\begin{aligned}\alpha_n^k(t) &= \sum_{j=0}^{[n/2]} (-1)^j \binom{\binom{n}{2}}{k}^{-1} \binom{\binom{n}{2}-j}{k-j} \alpha_j(K_n)\, t^{n-2j} \\ &= \sum_{j=0}^{[n/2]} (-1)^j \binom{k}{j} \binom{\binom{n}{2}}{j}^{-1} \alpha_j(K_n)\, t^{n-2j}\end{aligned}$$

as desired. ∎

**Theorem 24.**

$$\alpha_n(t) = 2^{-n} H_n(t) = \sum_{j=0}^{[n/2]} \left(-\frac{1}{2}\right)^j \alpha_j(K_n)\, t^{n-2j}.$$

***Proof.*** Applying Lemma 2.3 to the relation (2.2), we obtain that

$$\begin{aligned}\alpha_n(t) &= 2^{-\binom{n}{2}} \sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} \sum_{j=0}^{[n/2]} (-1)^j \binom{k}{j} \binom{\binom{n}{2}}{j}^{-1} \alpha_j(K_n)\, t^{n-2j} \\ &= 2^{-\binom{n}{2}} \sum_{j=0}^{[n/2]} (-1)^j \binom{\binom{n}{2}}{j}^{-1} \alpha_j(K_n)\, t^{n-2j} \left[\sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} \binom{k}{j}\right].\end{aligned}$$

Since

$$\sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} \binom{k}{j} = 2^{\binom{n}{2}-j} \binom{\binom{n}{2}}{j},$$

then

$$\alpha_n(t) = \sum_{j=0}^{[n/2]} \left(-\frac{1}{2}\right)^j \alpha_j(K_n)\, t^{n-2j}$$

as was to be shown. ■

# 3 The Dominating Polynomial, Definition and Properties

Let $G = (V, E)$ be a simple graph and $D \subseteq V$. A set of vertices $D$ is said to be a *dominating set* if for every $y \in V - D$, there exists $x \in D$ such that $\{x, y\} \in E$. For any vertex $x \in V$, let $N(x)$ denote the neighbourhood of $x$, the set of all vertices adjacent to $x$. We write $N[x] = N(x) \cup \{x\}$, the closed neibourhood of $x$. With this notation, $D \subseteq V$ is a dominating set if for every $y \in V - D$, we have that $N[y] \cap D \neq \emptyset$. The family of all dominating sets of a graph $G$ is denoted by $\mathcal{D}_G$. Observe that $V \in \mathcal{D}_G$ and if $D \in \mathcal{D}_G$ and $D \subset D'$, then $D' \in \mathcal{D}_G$.

For any graph $G$, the number of dominating sets of cardinality $k$ is denoted by $\gamma_k(G)$. We define by

$$\gamma(G, t) = \sum_{k=1}^{n} \gamma_k(G)\, t^{n-k}$$

the *dominating polynomial* of the graph $G$, where $n = |V|$. We take (by definition) $\gamma_0(G) = 0$.

For example, the dominating polynomials of the complete graph $K_n$ and the totally disconnected graph $\overline{K}_n$ are

$$\gamma(K_n, t) = \sum_{k=1}^{n} \binom{n}{k} t^{n-k} = (1+t)^n - t^n \text{ and } \gamma\left(\overline{K}_n, t\right) = 1,$$

since every subset of vertices of $K_n$ is a dominating set and there is only one dominating set of $\overline{K}_n$. If $\Phi$ denotes the empty graph, then $\gamma(\Phi, t) = 0$ since $\emptyset \in \mathcal{D}_\Phi$.

Let $\bigcup_{i=1}^{n} G_i$ be a graph composed of disjoint subgraphs $G_1$, $G_2$, ..., $G_n$.

**Theorem 31.** $\gamma(G_1 \cup G_2, t) = \gamma(G_1, t)\,\gamma(G_2, t)$.

***Proof.*** There are no edges between $V(G_1)$ and $V(G_2)$, therefore $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$ are dominating sets of $G_1$ and $G_2$ respectively if and only if $D_1 \cup D_2$ is a dominating set of $G_1 \cup G_2$. It holds that $|D_1 \cup D_2| = |D_1| + |D_2|$. Then

$$\gamma_k(G_1 \cup G_2) = \sum_{i+j=k} \gamma_i(G_1)\,\gamma_j(G_2),$$

which proves the theorem. ∎

As a consequence, we have the following corollary.

**Corollary 32.** $\gamma\left(\bigcup_{i=1}^{n} G_i, t\right) = \prod_{i=1}^{n} \gamma(G_i, t)$.

Let $G_1 + G_2$ be the *sum* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ defined as $G_1 + G_2 = (V_1 \cup V_2, E)$, where

$$E = E_1 \cup E_2 \cup \{\{x, y\} : x \in V_1, y \in V_2\}.$$

**Theorem 33.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any graphs such that $|V_1| = n_1$ and $|V_2| = n_2$. Then*

$$\begin{aligned}\gamma(G_1 + G_2, t) = \gamma(K_{n_1+n_2}, t) &- t^{n_2}\left[\gamma(K_{n_1}, t) - \gamma(G_1, t)\right] \\ &- t^{n_1}\left[\gamma(K_{n_2}, t) - \gamma(G_2, t)\right].\end{aligned}$$

***Proof.*** Let $D$ be a dominating set of $G_1 + G_2$ such that $|D| = k$. We define the following sets:

$$\begin{aligned} S_{1,2} &= \{D \subseteq V_1 \cup V_2 : D \cap V_1 \neq \emptyset \text{ and } D \cap V_2 \neq \emptyset\}, \\ S_{\overline{1},2} &= \{D \subseteq V_1 : D \text{ is not a dominating set in } G_1\} \text{ and} \\ S_{1,\overline{2}} &= \{D \subseteq V_2 : D \text{ is not a dominating set in } G_2\}. \end{aligned}$$

With this notation, we have that

$$|S_{1,2}| = \binom{n_1 + n_2}{k} - \left|S_{\overline{1},2}\right| - \left|S_{1,\overline{2}}\right|.$$

Therefore

$$\gamma_k(G_1 + G_2) = \binom{n_1 + n_2}{k} - \left[\binom{n_1}{k} - \gamma_k(G_1)\right] - \left[\binom{n_2}{k} - \gamma_k(G_2)\right].$$

Multiplying by $t^{n-k}$ and summing for all $k = 1, ..., n$, the desired result is established. ■

We have the following consequence of this theorem.

**Corollary 34.** *The dominating polynomial of the complete $n$-partite graph $K_{m_1,m_2,...,m_n} = \overline{K}_{m_1} + \overline{K}_{m_2} + ... + \overline{K}_{m_n}$ :*

$$\gamma(K_{m_1,m_2,...,m_n}, t) = (1+t)^m - t^m - \sum_{i=1}^{n} t^{m-m_i}\left[(1+t)^{m_i} - t^{m_i} - 1\right],$$

*where $m = \sum_{i=1}^{n} m_i$.*

Consider now a digraph $\Gamma = (U, A)$, where $U$ and $A$ denote the set of vertices and arcs, respectively. The sets of the ex-neighbourhood and in-neighbourhood of a vertex $x$ are denoted by $N^+(x)$ and $N^-(x)$, respectively and write $N^+[x]$ and $N^-[x]$ for the respective closed neighbourhoods. We say that $D \subseteq U$ is a *dominating set* of $\Gamma$ if for every vertex $v \in U - D$, there exists $u \in D$ such that $(u, v) \in A$, that is, $N^-[v] \cap D \neq \emptyset$. From this definition, observe that if $D$ is a dominating set of $\Gamma$, then there exists (at least one) $u \in D$ such that $N^+(u) \neq \emptyset$. The dominating polynomial of a digraph $\Gamma$ is defined similarly as for graphs. The properties proved before are valid in this case, too.

Let us call a bipartite digraph $\Gamma = (U_1, U_2, A)$ one-way if its arcs are all directed from part $U_1$ to part $U_2$. In that follows we will use a particular notion of dominating sets defined for one-way bipartite digraphs. Let us consider subsets $D \subseteq U_1$ and say that $D$ is an OW-*dominating set* if for every vertex $v \in U_2$, there exists $u \in D$ such that $(u, v) \in A$, that is, $N^-[v] \cap D \neq \emptyset$. The family of OW-dominating sets of the one-way bipartite digraph $\Gamma$ is denoted by $\mathcal{D}_\Gamma^{\rightarrow}$. With these definitions, $\mathcal{D}_\Gamma^{\rightarrow} \subseteq 2^{U_1}$. Observe that if $U_1 = \emptyset$, then $\gamma(\Gamma, t) = 0$ (there is no dominating set) and if $U_2 = \emptyset$, then $\gamma(\Gamma, t) = (1+t)^n$ ($n = |U_1|$) by convention.

Let $G = (V, E)$ be a simple graph. We construct a one-way bipartite digraph $\widetilde{G} = (U_1, U_2, A)$ from the graph $G$ such that $U_1$ and $U_2$ are disjoint copies of the set $V$ and

$$A = \{(i, i) : i \in V\} \cup \{(i, j), (j, i) : \{i, j\} \in E\}.$$

**Lemma 35.** $\gamma(G, t) = \gamma\left(\widetilde{G}, t\right)$.

***Proof.*** It is enough to show that $\mathcal{D}_G = \mathcal{D}_{\widetilde{G}}^{\rightarrow}$. The relation $\mathcal{D}_G \subseteq \mathcal{D}_{\widetilde{G}}^{\rightarrow}$ is evident. Conversely, suppose that there exists $D \in \mathcal{D}_{\widetilde{G}}^{\rightarrow}$ such that $D \notin \mathcal{D}_G$. Then for every $y \notin D$ we have that $N_{\widetilde{G}}^-[y] \cap D \neq \emptyset$ and there exists $y \notin D$ such that $N[y] \cap D = \emptyset$. But there exists $x \in D$ such that $(x, y) \in A$ and by the construction of $\widetilde{G}$, then $\{x, y\} \in E$, which is a contradiction. ∎

**Theorem 36.** *For any one-way bipartite digraph* $\Gamma = (U_1, U_2, A)$ *and* $i \in U_1$,

$$\gamma(\Gamma, t) = t\gamma(\Gamma - i, t) + \gamma\left(\Gamma - N^+[i], t\right).$$

***Proof.*** The number of OW-dominating sets of cardinality $k$ in $\Gamma$ splits into two parts:
(i) The number of OW-dominating sets of cardinality $k$ not containing vertex $i$, i.e., the number of OW-dominating sets of cardinality $k$ in $\Gamma - i = (U_1 - i, U_2, A - A')$, where $A' = \{(i, i)\} \cup \{(i, j) : j \in U_2\}$.
(ii) The number of OW-dominating sets containing vertex $i$. Let $i \in D$, where $D$ is a OW-dominating set and $|D| = k$. Delete vertex $i$ and all the vertices dominated by it. Then the number of OW-dominating sets of cardinality $k$ is equal to the number of those sets, but of cardinality $k-1$ in

$$\Gamma - N^+[i] = \left(U_1 - i, U_2 - N^+(i), A - A''\right),$$

where

$$A'' = A' \cup \{(x, y) : x \in N^- (y), y \in N^+ (i)\}.$$

These sets can be chosen in $\gamma_{k-1} (\Gamma - N^+ [i])$ ways. Therefore

$$\gamma_k (\Gamma) = \gamma_k (\Gamma - i) + \gamma_{k-1} (\Gamma - N^+ [i]).$$

Multiplying this equality by $t^{n-k}$ and summing for all $k = 1, ..., n$, we obtain the result. ■

Observe that Lemma 3.5 and Theorem 3.6 imply that the recurrence

$$\gamma (G, t) = t\gamma \left(\widetilde{G} - i, t\right) + \gamma \left(\widetilde{G} - N^+ [i], t\right)$$

holds for any graph $G$.

## 4 Average Dominating Polynomial

Let us consider the dominating polynomial $\gamma (\Gamma, t)$ of an one-way bipartite digraph $\Gamma$ as a random variable, whose average value in the family $\mathbb{D}_{n,m}$ of all labeled bipartite graphs with partite sets of size $n$ and $m$ respectively, is defined by

$$\gamma_{n,m} (t) = \frac{1}{2^{nm}} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma (\Gamma, t). \tag{4.1}$$

This polynomial is called the *average dominating polynomial of the family* $\mathbb{D}_{n,m}$.

**Theorem 41.**

$$\gamma_{n,m} (t) = \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{1}{2^{n-k}}\right)^m t^k, \; n, m \geq 1.$$

***Proof.*** Let $\Gamma = (U_1, U_2, A) \in \mathbb{D}_{n,m}$. Applying Theorem 3.6 to (4.1), we obtain that

$$\gamma_{n,m} (t) = \frac{t}{2^{nm}} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma (\Gamma - i, t) + \frac{1}{2^{nm}} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma (\Gamma - N^+ [i], t). \tag{4.2}$$

Observe that

$$\sum_{\Gamma\in\mathbb{D}_{n,m}} \gamma\left(\Gamma - i, t\right) = 2^m \sum_{\Gamma\in\mathbb{D}_{n-1,m}} \gamma\left(\Gamma, t\right),$$

since vertex $i$ can be joined to each one-way bipartite digraph of $\mathbb{D}_{n-1,m}$ in $2^m$ ways. On the other hand, if $|N^+(i)| = k$, then the second sum in the right of (4.2) runs through all labeled one-way bipartite digraphs of the family $\mathbb{D}_{n-1,m-k}$. The labels of the $k$ vertices of $N^+(i) \subseteq U_2$ can be chosen in $\binom{m}{k}$ ways, there are no edges between them and these $k$ vertices can be joined to the $n-1$ vertices of the set $U_1$ in $2^{(n-1)k}$ ways. Then

$$\begin{aligned}\sum_{\Gamma\in\mathbb{D}_{n,m}} \gamma\left(\Gamma - N\left[i\right], t\right) &= \sum_{k=0}^{m} \binom{m}{k} 2^{(n-1)k} \sum_{\Gamma\in\mathbb{D}_{n-1,m-k}} \gamma\left(\Gamma, t\right) \\ &= \sum_{j=0}^{m} \binom{m}{j} 2^{(n-1)(m-j)} \sum_{\Gamma\in\mathbb{D}_{n-1,j}} \gamma\left(\Gamma, t\right).\end{aligned}$$

Therefore

$$\begin{aligned}\gamma_{n,m}(t) &= \frac{t}{2^{(n-1)m}} \sum_{\Gamma\in\mathbb{D}_{n-1,m}} \gamma\left(\Gamma, t\right) \\ &\quad + \frac{1}{2^{nm}} \sum_{j=0}^{m} \binom{m}{j} 2^{(n-1)(m-j)} \sum_{\Gamma\in\mathbb{D}_{n-1,j}} \gamma\left(\Gamma, t\right)\end{aligned}$$

and so

$$\gamma_{n,m}(t) = t\gamma_{n-1,m}(t) + \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} \gamma_{n-1,j}(t). \tag{4.3}$$

Let us consider the following exponential generating function:

$$\gamma(x, y, t) = \sum_{n\geq 0} \sum_{m\geq 0} \gamma_{n,m}(t) \frac{x^n}{n!} \frac{y^m}{m!},$$

where $\gamma_{-1,m}(t) = 0$ and $\gamma_{n,0}(t) = (1+t)^n$. Multiplying (4.3) by $x^n y^m / n!m!$ and summing for all $n, m \geq 0$, we have that

$$\begin{aligned}\sum_{n\geq 0}\sum_{m\geq 0}\gamma_{n,m}(t)\frac{x^n}{n!}\frac{y^m}{m!} &= t\sum_{n\geq 0}\sum_{m\geq 0}\gamma_{n-1,m}(t)\frac{x^n}{n!}\frac{y^m}{m!}\\ &\quad+\sum_{n\geq 0}\sum_{m\geq 0}\frac{1}{2^m}\frac{x^n}{n!}\frac{y^m}{m!}\sum_{j=0}^{m}\binom{m}{j}\gamma_{n-1,j}(t).\end{aligned}$$

From this formula,

$$\frac{\partial\gamma(x,y,t)}{\partial x}=t\gamma(x,y,t)+e^{\frac{y}{2}}\gamma\left(x,\frac{y}{2},t\right), \tag{4.4}$$

since

$$\begin{aligned}&\sum_{n\geq 0}\sum_{m\geq 0}\frac{x^n}{n!}\frac{\left(\frac{y}{2}\right)^m}{m!}\sum_{j=0}^{m}\binom{m}{j}\gamma_{n-1,j}(t)\\ =&\sum_{n\geq 0}\sum_{j\geq 0}\frac{x^n}{n!}\gamma_{n-1,j}(t)\sum_{m\geq j}\binom{m}{j}\frac{\left(\frac{y}{2}\right)^m}{m!}\\ =&\sum_{n\geq 0}\sum_{j\geq 0}\frac{x^n}{n!}\gamma_{n-1,j}(t)\frac{\left(\frac{y}{2}\right)^j}{j!}\sum_{m\geq j}\frac{\left(\frac{y}{2}\right)^{m-j}}{(m-j)!}\\ =&\ e^{\frac{y}{2}}\gamma\left(x,\frac{y}{2},t\right).\end{aligned}$$

Let us find the solution of the partial differential equation (4.4) in the following form

$$\gamma(x,y,t)=e^y f(x,y,t). \tag{4.5}$$

Consequently,

$$\frac{\partial f(x,y,t)}{\partial x}=tf(x,y,t)+f\left(x,\frac{y}{2},t\right).$$

If

$$f(x,y,t)=\sum_{n\geq 0}\sum_{m\geq 0}f_{n,m}(t)\frac{x^n}{n!}\frac{y^m}{m!},$$

then

$$f_{n,m}(t)=\left(t+\frac{1}{2^m}\right)f_{n-1,m}(t)=\left(t+\frac{1}{2^m}\right)^{n-1}f_{1,m}(t).$$

Computing $f_{1,m}(t)$, we have from (4.5) that

$$\gamma_{1,m}(t) = \sum_{j=0}^{m} \binom{m}{j} f_{1,j}(t).$$

Since $\gamma_{1,m}(t) = 1/2^m$ for $m \geq 1$ and $\gamma_{1,0}(t) = 1 + t$, then

$$f_{1,m}(t) = (-1)^m \sum_{j=0}^{m} (-1)^j \binom{m}{j} \gamma_{1,j}(t) = (-1)^m \left(t + \frac{1}{2^m}\right).$$

Therefore

$$f_{n,m}(t) = (-1)^m \left(t + \frac{1}{2^m}\right)^n$$

and

$$\begin{aligned}
\gamma_{n,m}(t) &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(t + \frac{1}{2^j}\right)^n \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{k=0}^{n} \binom{n}{k} t^k \left(\frac{1}{2^j}\right)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} t^k \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\frac{1}{2^{n-k}}\right)^j \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{1}{2^{n-k}}\right)^m t^k
\end{aligned}$$

as desired. ∎

Observe that the recurrence for the dominating polynomial is closed in the family $\mathbb{D}_{n,m}$. The problems of finding a recurrence relation for the dominating polynomial which is closed in the family $\mathbb{G}_n$, and the calculation of the average polynomial in this family remain open. The same questions can be posed for polynomials defined for other invariants of graphs, such as minimal dominating sets, $K_n$-dominating sets (see [9]), vertex-coverings and edge-coverings.

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## References

[1] J.L. Arocha, *Anticadenas en conjuntos ordenados*, An. Inst. Mat. Univ. Nac. Autónoma México **27** (1987) 1–21.

[2] C. Berge, Graphs and Hypergraphs (North-Holland, London, 1973).

[3] E.J. Farrell, *An introduction to matching polynomials*, J. Combin. Theory (B) **27** (1979) 75–86.

[4] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness (Freeman, New York, 1979).

[5] C.D. Godsil and I. Gutman, *On the theory of the matching polynomial*, J. Graph Theory **5** (1981) 137–144.

[6] C.D. Godsil, Algebraic Combinatorics (Chapman and Hall, New York, 1993).

[7] O.J. Heilmann and E.H. Lieb, *Monomers and dimers*, Phys. Rev. Lett. **24** (1970) 1412–1414.

[8] O.J. Heilmann and E.H. Lieb, *Theory of monomer-dimer systems,* Comm. Math. Phys. **25** (1972) 190–232.

[9] M.A. Henning, O.R. Oellermann and H.C. Swart, *The diversity of domination*, Discrete Math. **161** (1996) 161–173.

[10] N.N. Lebedev, Special Functions and their Applications (Dover, New York, 1972).

[11] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).

[12] O. Ore, Theory of Graphs (Amer. Math. Soc., Providence, 1962).

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