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RECOGNIZING WEIGHTED DIRECTED CARTESIAN GRAPH BUNDLES

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Abstract

In this paper we show that methods for recognizing Cartesian graph bundles can be generalized to weighted digraphs. The main result is an algorithm which lists the sets of degenerate arcs for all representations of digraph as a weighted directed Cartesian graph bundle over simple base digraphs not containing transitive tournament on three vertices. Two main notions are used. The first one is the new relation $\vec{\delta}^*$ defined among the arcs of a digraph as a weighted directed analogue of the well-known relation δ^*. The second one is the concept of half-convex subgraphs. A subgraph H is half-convex in G if any vertex $x \in G \setminus H$ has at most one predecessor and at most one successor.

Keywords: graph bundles, Cartesian graph product, weighted digraphs, half-convexity.

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1 Introduction

Graph bundles [11, 12] generalize the notion of covering graphs and graph products. They can be defined with respect to arbitrary graph products. Various problems on graph bundles were studied recently, including edge coloring, maximum genus, isomorphism classes and chromatic numbers [7, 8, 9, 10, 11]. Here we shall consider the problem of recognition of Cartesian graph bundles.

It is well known that finite connected graphs enjoy unique factorization under the Cartesian multiplication [15]. There are polynomial algorithms for recognition of product graphs with respect to some of the products, including the Cartesian product (see [4] and the references there). Proofs of the unique factorization theorem and a polynomial algorithm for recognition of directed Cartesian product graphs are given in [3, 20]. On the other hand, a graph may have more than one representation as a graph bundle. Natural questions therefore are to find all possible representations of a graph as a graph bundle or to decide whether a graph has at least one representation as a nontrivial graph bundle. A polynomial algorithm for recognition of Cartesian product bundles over triangle free base is known [6]. Recently, an algorithm for recognition of Cartesian graph bundles which works for bundles with $K_4 \setminus e$-free base was put forward [17, 18, 19]. Here we shall generalize the ideas of [6] to obtain an algorithm for recognition of directed graph bundles with weighted vertices. Our motivation was the fact that it is possible to recognize graph bundle with respect to the strong product by recognizing a corresponding weighted Cartesian graph bundle [21].

We will restrict our attention to cases where fibres are connected. This decision is based on the following two facts. First, if the fibres are totally disconnected, then the graph bundle is a covering graph and it is known that deciding whether a graph is a covering graph is NP-hard [1]. Second, if the graph is a Cartesian graph bundle over a disconnected fibre, then there is a representation of this graph as a graph bundle over a connected fibre [10].

The paper is organized as follows. We begin with several definitions and some easily proved facts in Section 2. In Section 3 the new relation $\vec{\delta}^*$ is defined as weighted directed analogue of the relation δ^*. Instead of 2-convexity, half-convexity is presented and the main theorem is proved in Sections 4 an 5. In the last section we present a polynomial algorithm which finds all representations of a digraph as a Cartesian graph bundle provided the base digraphs do not contain the transitive tournament on three vertices TT_3.

2 Weighted Graph Bundles

We call directed edges *arcs* and denote them either by uv or $u \to v$, for $u, v \in V(G)$. In this paper by digraph we mean a directed graph with weighted vertices if not explicitly stated otherwise. If G has undirected edges, we will replace each undirected edge $\{u, v\}$ by two arcs uv and vu. The weight function on G will be denoted by $c_G : V(G) \to \mathbb{R}_+$. A digraph is thus defined by a triple $G = (V, A, c_G)$. By introducing the trivial weight function $c_G(v) = 1(v) = 1$ we get (unweighted) digraphs $G = (V, A) = (V, A, 1)$. Furthermore, if $u \to v \in A$ implies $v \to u \in A$ for all A, $G = (V, A)$ is clearly equivalent to an undirected graph. The results of this paper therefore generalize those of [6] and are also valid for unweighted digraphs.

We will assume that the digraphs are weakly connected. (Or, equivalently, that the underlying undirected graph $\overline{G}$ is connected.) If G is not weakly connected, then we may find graph bundle representations of each connected components. However, in order to combine the representations of more than one connected component we have to decide whether the fibres on different weakly connected components are isomorphic. As there is no known polynomial algorithm for graph isomorphism, recognition of graph bundles (with respect to any product) is at least as hard as the graph isomorphism problem.

Furthermore, $G \cong H$ will denote weighted graph isomorphism, i.e., the existence of a bijection $b : V(G) \to V(H)$ with $c_H(b(v)) = c_G(v)$ for all $v \in V(G)$ and such that there is an arc $v_1 \to v_2$ in G if and only if $b(v_1) \to b(v_2)$ is an arc in H.

Definition 1. The *Cartesian product* $G \Box H$ of digraphs G and H is the digraph with

$$V(G \Box H) = V(G) \times V(H) \text{ and}$$
$$A(G \Box H) = \{(v_1, w_1) \to (v_2, w_2) : (v_1 \to v_2 \in A(G) \text{ and } w_1 = w_2)$$
$$\text{or } (v_1 = v_2 \text{ and } w_1 \to w_2 \in A(H))\}.$$

The weight function is $c_{G \Box H}\,(v, w) = c_G(v) c_H(w)$.

Definition 2. Let B,F and G be weighted digraphs. G is a (*weighted directed Cartesian*) *graph bundle* with fibre F over the base B if there is a mapping $p: G \to B$ which satisfies the following conditions:

(1) It maps adjacent vertices of G to adjacent or identical vertices in B.

(2) Arcs are mapped to arcs (preserving the direction) or collapsed to vertices.
(3) For each vertex $v \in V(B)$, $p^{-1}(v) \cong F$, and for each arc $e \in A(B)$, $p^{-1}(e) \cong \vec{K}_2 \Box F$. By $\vec{K}_2$ we denoted a graph isomorphic to an arc.
(4) If for $x \to y \in A(G)$, $p(x \to y) \in A(B)$, then:

$$\frac{c_G(x)}{c_B(p(x))} = \frac{c_G(y)}{c_B(p(y))}.$$

(5) If for $x \to y \in A(G)$, $p(x \to y) = z \in V(B)$, then for any arc $z \to u \in A(B)$ and the corresponding lifted isomorphism $m : p^{-1}(z) \to p^{-1}(u)$:

$$\frac{c_G(x)}{c_G(m(x))} = \frac{c_G(y)}{c_G(m(y))}.$$

(See Figure 1.)

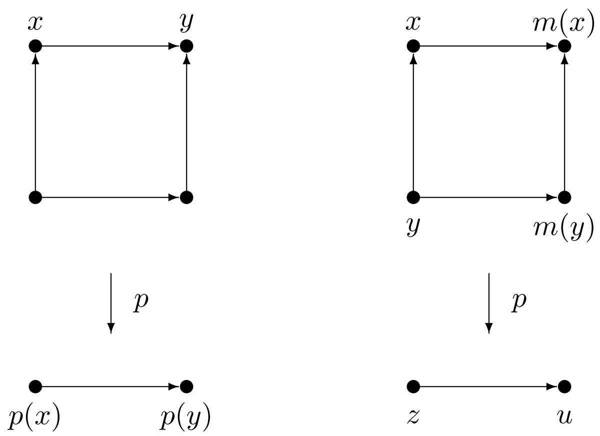


Figure 1. Definition of weighted directed Cartesian graph bundle

We say an arc e is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*. A factorization of a digraph G is a collection of spanning subgraphs H_i of G such that the arc set of G is partitioned into the arc sets of the digraphs H_i. In other words, the set $A(G)$ can be written as a disjoint union of the sets $A(H_i)$. The projection p induces a factorization of G into the digraph consisting of isomorphic copies of the fibre F and the digraph $\tilde{G}$ consisting of all nondegenerate arcs. This factorization is called the *fundamental factorization*. It can be shown that the restriction of p to $\tilde{G}$ is a covering projection of digraphs; see for instance [11] or [12] for details.

We wish to remark that the weighted graph bundles studied in [16] are not defined in the same way as here.

For any degenerate arc $e = x \to y$, $\frac{c_G(y)}{c_G(x)}$ can be understood as the weight of the corresponding arc in F. Similarly, for $y = m(x)$, $\frac{c_G(y)}{c_G(x)}$ can be seen as a weight of a nondegenerate arc of G or, alternatively, the weight of an arc $p(x) \to p(y)$ of B or the weight of the lifted isomorphism m.

3 Relation $\vec{\delta}^\star$

The equivalence relation $\vec{\delta}^\star$ is defined among the arcs of a graph. This relation is a weighted directed analogue of the relation $\delta^\star$ which was first used in the proof of the unique factorization theorem for Cartesian product graphs [15] and later as a starting relation in one of the first algorithms for factoring a graph with respect to the Cartesian product [2]. The algorithm for recognition of Cartesian graph bundles over triangle free bases of [6] is based on $\delta^\star$ as well.

Definition 3. An induced subgraph on four vertices which underlying undirected subgraph induce a cycle, is called chordless square. Let $x_{i,j}, i, j \in \{0, 1\}$ denote the vertices in chordless square, where vertices with both different coordinates are not connected. A chordless square is called *weights and directions consistent chordless square* if for any $i, j, k \in \{0, 1\}$

(1) $x_{i,j} \to x_{i,k} \in A(G) \Rightarrow x_{1-i,j} \to x_{1-i,k} \in A(G)$ and
$x_{i,j} \to x_{k,j} \in A(G) \Rightarrow x_{i,1-j} \to x_{k,1-j} \in A(G)$ (see Figure 2) and

(2) $\dfrac{c_G(x_{i,j})}{c_G(x_{i,k})} = \dfrac{c_G(x_{1-i,j})}{c_G(x_{1-i,k})}$ and $\dfrac{c_G(x_{i,j})}{c_G(x_{k,j})} = \dfrac{c_G(x_{i,1-j})}{c_G(x_{k,1-j})}$,

(or equivalently, there exist constants a_0, a_1, b_0, b_1 such that $c(x_{i,j}) = a_i b_j$).

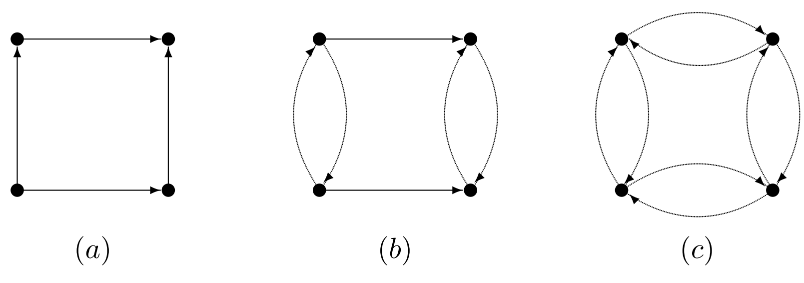


Figure 2. Directions consistent chordless squares

We now define an auxiliary binary relation $\vec{\delta}$.

Definition 4. For any $e, f \in A(G)$ we set $e \; \vec{\delta} \; f$ if at least one of the following conditions is satisfied:

(1) e and f are opposite arcs of a unique weights and directions consistent chordless square.

(2) e and f are incident and there is no weights and directions consistent chordless square spanned on e and f.

(3) $e, f \in \{u \to v, v \to u\}$. (In other words, $e \; \vec{\delta} \; f$ either if $e = f$ or if e and f connect the same pair of vertices, in opposite directions).

By $\vec{\delta}^\star$ we denote the transitive closure of $\vec{\delta}$.

Since $\vec{\delta}$ is symmetric, $\vec{\delta}^\star$ is an equivalence relation.

Note that by definition any pair of incident arcs which belong to distinct $\vec{\delta}^\star$-equivalence classes span a unique weights and directions consistent chordless square. We say that $\vec{\delta}^\star$ has the *square property*. Furthermore, any equivalence relation $R \supseteq \vec{\delta}$ also has the square property.

Let R have the square property and let e be an arc. For any arc f not in the same class as e and incident to e we can define a *translation* of e along f, $T_f(e)$, to be the (unique) opposite arc of the weights and directions consistent chordless square spanned by the arcs e and f.

Equivalence classes of R will be denoted by Greek letters, possibly equipped by indexes. In particular, the class containing the arc e_i will be denoted by φ_i. We are mainly interested in *nontrivial* equivalence relations R, i.e., equivalence relations having at least two equivalence classes.

The following two facts can be easily shown for relation $\vec{\delta}^\star$.

Lemma 1. *Each vertex in a weakly connected digraph G is incident to at least one arc of each $\vec{\delta}^\star$ class.*

Proof. (sketch) Assume the vertex v is in the underlying graph at distance at least k to the closest arc e of the class φ. Take a shortest path in the underlying graph from v to e and use the last arc of this path to find an arc of class φ which is of distance $k-1$ to v (in underlying graph), which is a contradiction. ∎

Lemma 2. *If the arc $u \to v$ is in class φ_1, then for any other $\vec{\delta}^\star$-class $\varphi_2 \neq \varphi_1$, the vertices u and v have the same (in- and out-) φ_2-degree, and $\vec{\delta}^\star$ induces a bijection between the φ_2-arcs incident to u and φ_2-arcs incident to v.*

Proof. Follows directly from definition. ∎

4 Half-Convexity

Let R be an equivalence relation on the arc set $A(G)$ of a weakly connected digraph G and let φ be an equivalence class of R. Denote by G_φ the spanning subgraph of G containing the arcs of φ and let $G_\varphi(v)$ be the weakly connected component of G_φ that contains $v \in V(G)$.

We define a digraph B_φ without parallel arcs and a projection $p_\varphi : G \to B_\varphi$ by the following rules:

(1) Let the vertex set of B_φ be $V(B_\varphi) = \{G_\varphi(v) : v \in V(G)\}$.

(2) For each vertex $v \in V(G)$ let $p_\varphi(v) = G_\varphi(v)$ and
for each arc $e = u \to v \in A(G) \setminus \varphi$ let $p_\varphi(u \to v) = G_\varphi(u) \to G_\varphi(v)$.

(3) There are no other arcs in B_φ except those forced by rule no. 2.

By definition let the weight of a vertex $G_\varphi(v)$ be the minimum weight $c_B(G_\varphi(v)) = \min c_G(v)$ over all $v \in G_\varphi(v)$.

Note that B_φ may have loops and directed 2-cycles, even if G has not. On the other hand, B_φ by definition has no parallel arcs.

Clearly, B_φ has no loops if and only if each weakly connected component of G_φ is an induced subgraph of G. In this case B_φ is a simple digraph, i.e., digraph without loops and parallel arcs.

In the algorithms for recognition of Cartesian graph products and graph bundles it is possible to use convexity properties. For example, it is well known that layers of a Cartesian product graph must be convex subgraphs in the product graph (see, for example [5]) and fibres of a Cartesian graph bundle are 2-convex subgraphs [6]. A subgraph H of a graph G is k-convex, if for every pair of vertices of distance $\leq k$ in G, every shortest path is in H. The usual convexity is the same as ∞-convexity and a subgraph is induced if and only if it is 1-convex.

A natural generalization of this notion to digraphs would be to expect all directed shortest paths of length at most k to be in H. However, this may not hold for fibres of a Cartesian graph bundle, as the following example shows (see Figure 3).

What we need is that nondegenerate arcs define unique isomorphisms between neighboring fibres. Therefore

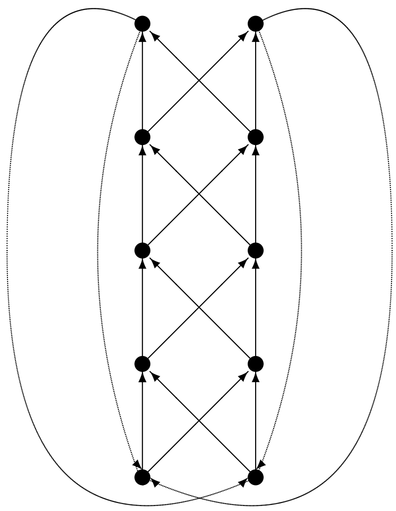
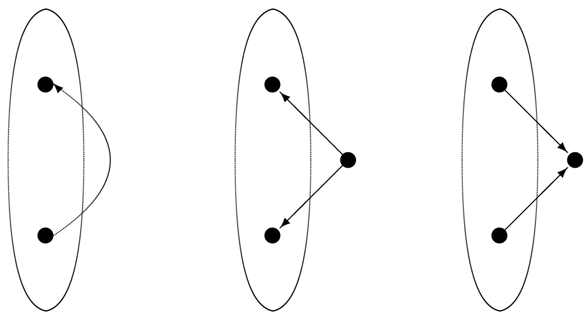


Figure 3. Fibre which is not 2-convex

Definition 5. A weakly connected subgraph H is *half-convex* in G if it is an induced subgraph in G and there is no obstruction of type 1 or type 2, see Figure 4.



type 0 type 1 type 2

Figure 4. Definition of half-convexity. Obstructions

For general H, H is half-convex in G if and only if each of its weakly connected components is half-convex.

Let R be an equivalence relation on $A(G)$ and let φ be an equivalence class of R. We say φ is *half-convex* if G_φ is half-convex. Furthermore, we define R to be *half-convex* if each equivalence class of R is half-convex. R is *weakly half-convex* if at least one equivalence class of R is half-convex.

Any half-convex subgraph H clearly has the following property: For any two vertices $u, v \in V(H)$ there is no vertex $w \in V(G) \setminus V(H)$ such that $w \to v$ and $w \to u$ are both arcs in G, and there is no vertex $w \in V(G) \setminus V(H)$ such that $v \to w$ and $u \to w$ are both arcs in G.

It may be interesting to note that if $\overline{H}$ is 2-convex in $\overline{G}$ then H is half-convex in G. Recall that $\overline{G}$ denotes the underlying graph of G. On the other hand, H being half-convex in G does not imply $\overline{H}$ 2-convex in $\overline{G}$.

5 Fibres are Half-Convex

Note that half-convexity of equivalence class φ implies B_φ is a simple digraph, because φ and hence G_φ are half-convex and therefore each weakly connected component of G_φ is an induced subgraph.

Lemma 3. *Let R be a weakly half-convex equivalence relation on $A(G)$ with the square property and let φ be a half-convex equivalence class of R. Let $e = u \to v$ be an arc from $A(G) \backslash \varphi$. Then e induces a unique isomorphism α between $G_\varphi(u)$ and $G_\varphi(v)$. Furthermore, there is a constant $k = k(e)$ such that for any vertex $w \in G_\varphi(u)$, $c_G(\alpha(w)) = kc_G(w)$.*

Proof. Define the set of arcs M_e connecting $G_\varphi(u)$ to $G_\varphi(v)$ as follows:
(1) $e \in M_e$,
(2) if $e' \in M_e$, $f \in A(G_\varphi(u))$ then $T_f(e') \in M_e$,
(3) if $e' \in M_e$, $f \in A(G_\varphi(v))$ then $T_f(e') \in M_e$,

where $T_f(e)$ is the translation of e along f. Since φ is half-convex, M_e is a matching. Because $G_\varphi(u)$ and $G_\varphi(v)$ are weakly connected, M_e is a perfect matching on $G_\varphi(u) \cup G_\varphi(v)$ and hence defines a 1-1 map $\alpha : V(G_\varphi(u)) \to V(G_\varphi(v))$. By Lemma 2 we can verify that $\alpha : G_\varphi(u) \to G_\varphi(v)$ is a local isomorphism which in turn implies that it is an isomorphism.

Since any two incident arcs $e' = x \to y \in M_e$ and $x \to x'$ or $x' \to x = f \in A(G_\varphi(u))$ span a weights and directions consistent chordless square

$$\frac{c_G(\alpha(x))}{c_G(x)} = \frac{c_G(\alpha(x'))}{c_G(x')}.$$

Because $G_\varphi(u)$ and $G_\varphi(v)$ are weakly connected, for any $w \in G_\varphi(u)$ there is a sequence of weights and directions consistent chordless squares connecting $u \to v$ and $w \to \alpha(w)$ which implies

$$\frac{c_G(v)}{c_G(u)} = \frac{c_G(\alpha(u))}{c_G(u)} = \frac{c_G(\alpha(w))}{c_G(w)}.$$

Hence there is a constant k such that

$$k = k(e) = \frac{c_G(v)}{c_G(u)} = \frac{c_G(\alpha(w))}{c_G(w)}$$

for any $w \in G_\varphi(u)$. ■

Theorem 1. *Let G be any digraph and R any nontrivial weakly half-convex equivalence relation having the square property with φ being a half-convex equivalence class of R. Then $(G, p_\varphi, B_\varphi)$ is a Cartesian graph bundle.*

Proof. First, it is easy to see that B_φ is simple digraph, because φ and hence G_φ are half-convex and therefore each weakly connected component of G_φ is an induced subgraph.

Second, we have to show that all $p^{-1}(x)$ are isomorphic and that $p^{-1}(e)$ induce isomorphisms between fibres.

It is enough to show that for each $e = a \to b \in A(B_\varphi)$, $p^{-1}(e)$ is a perfect matching which induces an isomorphism between two weakly connected components $G_\varphi(u)$ and $G_\varphi(v)$ such that $p(u) = a$ and $p(v) = b$. Since $p^{-1}(e)$ is M_e of the previous lemma, $p^{-1}(e)$ induces an isomorphism between fibres.

Now we show the consistency of weights. By Lemma 3 any arc $e = x \to y$ from $A(G) \setminus \varphi$ induces a unique isomorphism α between $G_\varphi(x)$ and $G_\varphi(y)$ (which maps vertices with minimum weights from $G_\varphi(x)$ to vertices with minimum weights from $G_\varphi(y)$) and there is a constant

$$k = k(e) = \frac{c_{B_\varphi}(G_\varphi(y))}{c_{B_\varphi}(G_\varphi(x))} = \frac{c_{B_\varphi}(p(y))}{c_{B_\varphi}(p(x))}$$

such that for any vertex $w \in G_\varphi(x)$, $c_G(\alpha(w)) = kc_G(w)$. Since $y = \alpha(x)$

$$c_G(y) = kc_G(x) = \frac{c_{B_\varphi}(p(y))}{c_{B_\varphi}(p(x))} c_G(x)$$

and therefore

$$\frac{c_G(x)}{c_{B_\varphi}(p(x))} = \frac{c_G(y)}{c_{B_\varphi}(p(y))}.$$

Finally, for any arc $e = x \to y$, $x \in G_\varphi(y) = u \in V(B_\varphi)$ and for any arc $u \to v$ or $v \to u \in A(B_\varphi)$ and the corresponding isomorphism $\alpha : G_\varphi(y) \to p^{-1}(v)$ there is a constant k such that for any vertex $w \in G_\varphi(x)$, $c_G(\alpha(w)) = kc_G(w)$, by Lemma 3. Therefore

$$\frac{1}{k} = \frac{c_G(x)}{c_G(\alpha(x))} = \frac{c_G(y)}{c_G(\alpha(y))}.$$

Finally, all $G_\varphi(v)$ are isomorphic because G is weakly connected. ■

The theory developed so far can now be used for representing digraph G as a graph bundle. We start with $\vec{\delta}^\star$ and then glue some equivalence classes together as long as the resulting equivalence relation R does not satisfy the conditions of the theorem. Unfortunately, this approach does not recognize all graph bundles. As an example construct a bundle with directed cycle as a fibre and over a directed triangle in Figure 5 as base. From construction it follows that the triangle in base can not be a directed 3-cycle. In the language of tournaments, the forbidden configuration in the base digraph is the transitive tournament on three vertices, following [14] denoted by TT_3.

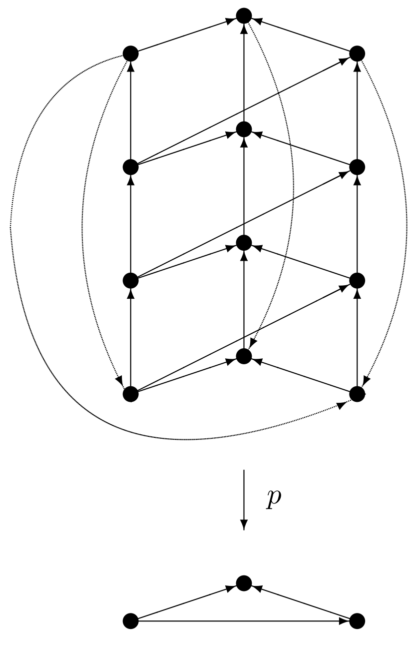


Figure 5. The unsolved configuration over TT_3 in base

Note that a directed $K_{3,3}$ does not provide a counterexample, in contrast to the undirected case. (By a directed $K_{3,3}$ we mean a graph bundle with underlying graph $K_{3,3}$ without 2-loops.)

In general, if (G, p, B) is a graph bundle whose simple base digraph B has no TT_3, then each $\vec{\delta}^\star$ equivalence class either contains only degenerate arcs or only nondegenerate arcs.

Lemma 4. *Let (G, p, B) be a graph bundle whose simple base digraph B has no TT_3. Then each $\vec{\delta}^\star$ equivalence class contains either only degenerate arcs or only nondegenerate arcs. In particular, $\vec{\delta}^\star$ is not trivial.*

Proof. Let R_1 be the union of $\vec{\delta}^\star$ classes containing degenerate arcs and let R_2 be the union of $\vec{\delta}^\star$ classes containing nondegenerate arcs. We claim that R_1 and R_2 have empty intersection. Assume there is a $\vec{\delta}^\star$ equivalence class containing a degenerate arc e' and a nondegenerate arc f'. Then there must be a pair e, f of arcs such that e is degenerate, f is nondegenerate and $e\vec{\delta}f$.

We have two different cases to consider. First, if e and f are incident then by definition of graph bundle there must be a unique weights and directions consistent chordless square spanned by e and f since two adjacent fibres induce a Cartesian product with $\vec{K}_2$. But this is in contradiction with $e\vec{\delta}f$.

The second case occurs when e and f are opposite arcs of a weights and directions consistent chordless square. Then it is easily seen that there must be a TT_3 in the base; see Figure 5. Therefore no $\vec{\delta}^\star$ class can contain both degenerate and nondegenerate arcs if there is no TT_3 in the simple base digraph. ■

Remark. The relation $\vec{\delta}^\star$ may not fail also on some graph bundles with TT_3 in base. A more precise characterization of the graph bundles, not recognized by the present approach is the following: There must be a TT_3 in the base digraph and the composition of the three isomorphisms between fibres over that TT_3 (which is an automorphism on one copy of fibre) must map at least one vertex to one of its neighbors.

Let R be any equivalence relation with the square property defined among arcs of G and let H be a subgraph of G. We define the *half-convex R-closure* $\mathcal{C}(H, R)$ as the subset ρ of the arcs set $A(G)$, such that ρ is the minimal union of equivalence classes of R that satisfies the following two conditions:

(1) $A(H) \subseteq \rho$ and

(2) ρ is half-convex in G.

In order to justify the above definition we must show that the half-convex closure is well defined. It suffices to prove that the intersection of half-convex subgraphs is half-convex.

If φ is a set of arcs, then the half-convex R closure of φ is, by definition, $\mathcal{C}(H,R)$, where H is the subgraph with arc set $A(H) = \varphi$ and the vertices of H are the endpoints of its arcs.

Lemma 5. *If two subgraphs C_1 and C_2 are half-convex, then the intersection $C_1 \cap C_2$ is half-convex.*

Proof. Let $u, v \in C_1 \cap C_2$. If $u \to v$ is in $A(G)$, then the arc $u \to v$ must be in both C_1 and C_2, since C_1 and C_2 are induced subgraphs in G. Assume u and v have a common (forward or backward) neighbor. More precisely, by a forward neighbor we mean a vertex w such that $u \to w$ and $v \to w$, and by a backward neighbor a vertex x such that $x \to u$ and $x \to v$. In either case, the common neighbor has to be both in C_1 and in C_2 because C_1 and C_2 are half-convex. ∎

Corollary 1. *$\mathcal{C}(H,R) = \cap C_i$, the intersection of all C_i for which $H \subseteq C_i$ and the arcs of C_i are a weakly connected component of a digraph induced on a union of R-equivalence classes.*

Corollary 2. *$\mathcal{C}(H,R)$ is a weakly connected component of a digraph induced on a union of R-equivalence classes.*

From the above facts we also infer that the following algorithm computes the *half-convex R-closure* $\mathcal{C}(H,R)$.

Algorithm CLOSURE
Input: H: a connected subgraph of G
Output: C: half-convex R-closure of H
C:= H
repeat
    (*R-closure*)
    while there is an arc $e = uv$ with exactly one endpoint in C
            which is in relation R with some arc of C do
        $C := C \cup \{u, v, e\}$
    (*half-convex closure*)
    Obstructions:= $\emptyset$

for all pairs of vertices x and $y \in C$ do:
if $x \to y \in A(G)$ then Obstructions:= Obstructions $\cup\{x \to y\}$
if $y \to x \in A(G)$ then Obstructions:= Obstructions $\cup\{y \to x\}$
if $\exists z : z \to x \in A(G)$ and $z \to y \in A(G)$ then
Obstructions:= Obstructions $\cup\{z, z \to x, z \to y\}$
if $\exists z : x \to z \in A(G)$ and $y \to z \in A(G)$ then
Obstructions:= Obstructions $\cup\{z, x \to z, y \to z\}$
$C := C \cup$ Obstructions
until Obstructions=$\emptyset$;
C is the half-convex R-closure of H

Note that there are some obvious improvements of this algorithm. For example, if a pair of vertices x, y was checked once and the arcs were added, then there is no need to check the pair x, y again. Here we only want to give a simple algorithm with polynomial running time and a straightforward proof of correctness. A faster algorithm for computing a convex closure relative to an equivalence relation will be given elsewhere [13].

Lemma 6. *Let G be a graph bundle whose simple base digraph contains no TT_3 and let φ be any equivalence class of $\vec{\delta}^\star$ containing only degenerate arcs. If $\rho := \mathcal{C}(\varphi, \vec{\delta}^\star) \neq A(G)$, then G is a graph bundle with fibres being the weakly connected components of G_ρ.*

Proof. Since each weakly connected component of G_ρ is an induced subgraph of G, every arc of $A(G) \setminus \rho$ has its endpoints in distinct weakly connected components of G_ρ. The equivalence relation with two equivalence classes $\{\rho, A(G) \setminus \rho\}$ is weakly half-convex. Therefore, by Lemma 3, all pairs of weakly connected components of G_ρ are pairwise isomorphic. ∎

Lemma 7. *Let G be a graph bundle with fibre F. Assume each equivalence class of $\vec{\delta}^\star$ contains either only degenerate or nondegenerate arcs and let γ be any equivalence class of $\vec{\delta}^\star$. If a weakly connected component of the digraph determined by γ is contained in a fibre, then also the weakly connected component of the half-convex closure $\mathcal{C}(\gamma, R)$ is contained in a fibre. In particular, the digraph determined by the 2-convex closure of γ has at least two weakly connected components.*

Proof. We show that if a weakly connected component of an arbitrary subgraph $H \subset G$ is contained in a fibre, then also the weakly connected component of the half-convex closure is contained in the same fibre. Since the argument is valid for each weakly connected component of H, we may assume, without loss of generality, that H is weakly connected. Let H be a weakly connected subgraph of a fibre F_1. Let H' be obtained from H by a step of the algorithm CLOSURE. At each step either arcs or stars are added. (Stars here are digraphs with a central node and leaves and with all arcs directed either to or from the center.)

(1) If an arc was added, then this arc must also be in F_1 because fibres are induced subgraphs. Furthermore, any other arc of the same $\vec{\delta}^\star$-equivalence class adjacent to a vertex of H' must be in F_1. If not, then this $\vec{\delta}^\star$-equivalence class would contain both degenerate and nondegenerate arcs which we have assumed not to be the case.

(2) If a star (with a new vertex $v \notin F_1$) was added to H, then we have a vertex v in another fibre, say F_2, adjacent to a pair of vertices of H and hence of F_1. But since G is a graph bundle a vertex cannot have more than one neighbor in another fibre. Hence, no vertex not belonging to F_1 can be added and H' must also be a subgraph of F_1.

Thus all obstructions are in F and the arcs of the obstructions degenerate. Since each class of $\vec{\delta}^\star$ contains by assumption either only degenerate or only nondegenerate arcs, the $\vec{\delta}^\star$ closure contains only degenerate arcs. ■

6 The Algorithm

If there is a simple digraph B with no TT_3, such that (G, p, B) is a graph bundle for some p, we can now give a polynomial algorithm which finds representations of G.

Algorithm BUNDLE
Input: G: digraph without loops
Output: C: set of degenerate arcs of some bundle representation
compute $\vec{\delta}^\star$
if the number of $\vec{\delta}^\star$ equivalence classes $> \log_2 n$ **then**
    return("G is not a graph bundle over a TT_3-free simple base.")
else begin
    for all unions of equivalence classes φ of $\vec{\delta}^\star$ **do**
        if $C := \mathcal{C}(\varphi, \vec{\delta}^\star) \neq A(G)$ **then return**(C)
    end-for
end-else
if no representation found **then**
    return("G is not a graph bundle over a TT_3-free simple base.")

Theorem 2. *Let G be a digraph which can be represented as a graph bundle over TT_3-free simple base. Then algorithm BUNDLE lists $\mathcal{C}(\varphi, \vec{\delta}^\star)$, the sets of degenerate arcs of $(G, p_\varphi, G_\varphi)$, for all representations of G as a graph bundle.*

Proof. We first discuss the correctness and time complexity of the block inside the for loop. For any representation over a TT_3-free simple base, the equivalence classes of the relation $\vec{\delta}^\star$ contain either only degenerate or only nondegenerate arcs by Lemma 4. Let φ be an equivalence class of $\vec{\delta}^\star$ with degenerate arcs. Each weakly connected component must be contained in one fibre and by Lemmas 6 and 7 the closure $\mathcal{C}(\varphi, \vec{\delta}^\star)$ is the set of degenerate arcs for a representation of G as a graph bundle.

The number of iterations of the for loop is not more than 2^N, where N is the number of equivalence classes of $\vec{\delta}^\star$. If G is a Cartesian graph bundle, then the relation $\vec{\delta}^\star$ can have at most $\log_2 n$ equivalence classes. (This fact was noted in [2] for Cartesian product graphs. Essentially the same argument is valid for graph bundles, too.)

Hence we can compute the half-convex R closures of all unions of R equivalence classes and therefore list all representations of G as a graph bundle in polynomial time. ■

Remark. In [6], it is claimed that by computing the closures of all $\delta^\star$ equivalence classes, we can find all *minimal* representations of Cartesian graph bundle G over triangle free base. Minimal representation means there is no representation such that its fibres would be proper subsets of minimal representation fibres. Using the observation that the number of equivalence classes of the relation is at most $\log_2 n$, the algorithm of [6] finds *all* representations, too.

We finally briefly discuss a possible generalization of the algorithm given here. By adaptation of results in [17] and [19], one would obtain a method for disabling of $\vec{\delta}$ on induced subgraphs with underlying undirected graph $K_{3,3} \setminus e$, which would give an algorithm for recognizing weighted directed Cartesian graph bundles over more general base digraphs.

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