Discussiones Mathematicae
Graph Theory 20 (2000) 23–38

THE STRONG ISOMETRIC DIMENSION OF FINITE REFLEXIVE GRAPHS

Shannon L. Fitzpatrick

University of Victoria
Victoria, British Columbia

and

Richard J. Nowakowski*

Dalhousie University
Halifax, Nova Scotia

Abstract

The strong isometric dimension of a reflexive graph is related to its injective hull: both deal with embedding reflexive graphs in the strong product of paths. We give several upper and lower bounds for the strong isometric dimension of general graphs; the exact strong isometric dimension for cycles and hypercubes; and the isometric dimension for trees is found to within a factor of two.

Keywords: isometric, embedding, strong product, injective hull, paths, distance, metric.

1991 Mathematics Subject Classification: 05C12, 05C75.

1 Introduction and Preliminaries

We consider reflexive graphs, that is, graphs with a loop at every vertex (the loops are not drawn in the figures). The distance between two vertices x and y in a graph G is the length of a shortest path joining the two and is denoted by $d_G(x, y)$. The reference to G will be dropped when there is no risk of confusion. A graph G is an *isometric subgraph* of H if there is a map $f : V(G) \to V(H)$ such that for all $x, y \in V(G)$, $d_G(x, y) = d_H(f(x), f(y))$. A graph G is a *retract* of a graph H if there are edge-preserving maps $g : G \to H$ and $f : H \to G$ such that $f \circ g$ is the identity

*Partially supported by a grant from the NSERC.

map on G. Isometric subgraphs are clearly induced subgraphs and a retract is an isometric subgraph since walks are mapped to walks. In Figure 1, G is an induced subgraph of both H and I (indicated by the larger circles) but G is only an isometric subgraph of H, whereas in Figure 2, G is an isometric subgraph of both H and I but is only a retract of I.

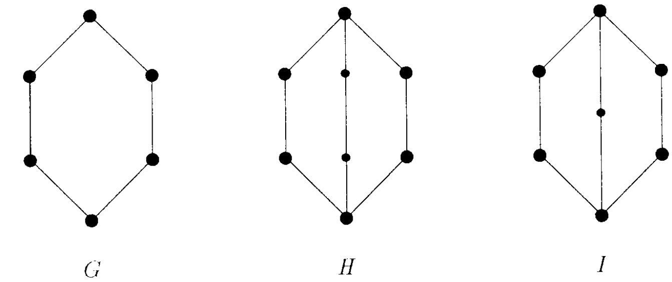

Figure 1. Graph G is an Isometric Subgraph of H but not of I.

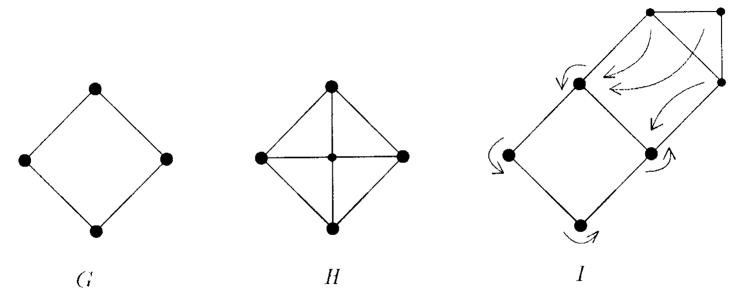

Figure 2. G is an Isometric Subgraph of H and a Retract of I.

The concept of strong isometric dimension has two motivations. It is implicit in the concept of an *injective hull* of a graph [6, 7, 10, 13]. The injective hull of a graph G is the smallest supergraph H of G where H is an *absolute retract*, i.e., H is a retract of a graph whenever it is an induced subgraph of that graph. One way of finding the injective hull for a reflexive graph G is to embed G in a strong product of paths then take the smallest retract of the product that contains that image of G. Isbell [6] gives a construction for the injective hull which proceeds via a metric space defined on functions.
The second motivation comes from the game of 'cop and robber' introduced independently in [12] and [14]. The game rules are: given a connected graph G, the cop chooses a vertex of G, then the robber chooses a vertex. They then move alternately — each can move to an adjacent vertex or pass.

(Passing is equivalent to moving along a loop.) The cop wins if he ever occupies the same vertex as the robber; the robber wins if this situation never occurs. In [12] and [14], the authors characterize those graphs in which the cop has a winning strategy. These are called *cop-win* graphs. In [12], it is shown that the strong product of two cop-win graphs is also cop-win. Clearly, a finite path is a cop-win graph and thus the strong product of any finite set of finite paths is cop-win. For an arbitrary graph G, one can ask for the least number of cops, $c(G)$, required to capture a robber (see [1, 2, 9]). The embedding of G in a product of paths allows the 'holes' of G to appear. These are structures which the robber can use to evade the cops. Generally speaking, the greater the complexity of the 'hole' the more options the robber has and therefore the more cops are needed to capture the robber. However, the degrees of the vertices in a product of k paths are bounded by 3^k and so the size of the strong isometric dimension bounds the possible complexity of the 'holes'. We address this question in [5].

While it is true that the taking of an injective hull is categorically the same as taking the Dedekind-MacNeille completion of an order, it is not true that the strong isometric dimension of a graph is the equivalent concept to that of dimension in ordered sets. In the ordered set situation, the order is embedded in a product of chains, but the order is not isometric. The corresponding idea would be to embed the order in a product of fences (see [7]).

We write, $a \simeq b$ if a is either equal or adjacent to b, $a \sim b$ if a is adjacent to but not equal to b, and $a \perp b$ if a is neither adjacent nor equal to b. The *strong* product of a set of graphs $\{G_i : i = 1, 2, \ldots, k\}$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the $V(G_i)$ and there is an edge between $\bar{a} = (a_1, a_2, \ldots, a_k)$ and $\bar{b} = (b_1, b_2, \ldots, b_k)$ if and only if $a_i \simeq b_i$ for $i = 1, 2, \ldots, k$. See Figure 3 for an example of $K_3 \boxtimes P_3$. The distance between $\bar{a} = (a_1, a_2, \ldots, a_k)$ and $\bar{b} = (b_1, b_2, \ldots, b_k)$ is therefore given by $d(\bar{a}, \bar{b}) = \max\{d(a_i, b_i) : i = 1, 2, \ldots k\}$. The *Cartesian* product of $\{G_i : i = 1, 2, \ldots, k\}$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the $V(G_i)$ and there is an edge between $\bar{a} = (a_1, a_2, \ldots, a_k)$ and $\bar{b} = (b_1, b_2, \ldots, b_k)$ if and only if there is some j such that $a_i = b_i$ for $i \neq j$ and $a_j \sim b_j$. For other terms please see [3].

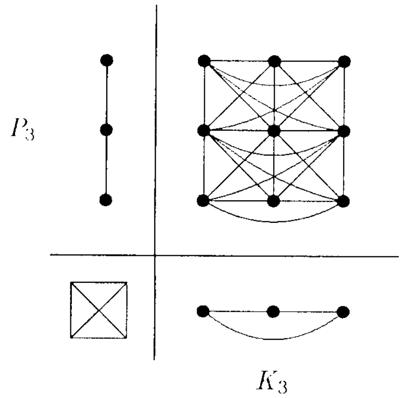

Figure 3. The Strong Product of K_3 and P_3

The *isometric dimension* of a graph G has been defined as the least number of paths needed so as to be able to isometrically embed G in the Cartesian product of the paths, see [15]. This is not always possible unless there is a relaxation of the isometry condition. In this paper, we are interested in finding when a given graph is an isometric subgraph of the strong product of paths. The *strong isometric dimension* of a graph G is the least number k such that there is a set of k paths $\{P_1, P_2, \ldots, P_k\}$ with G an isometric subgraph of $\boxtimes_{i=1}^{k} P_i$. We denote this by $idim(G) = k$. One of our first results is that if G is a finite, reflexive, connected graph then $idim(G)$ exists.

Figure 4 shows that $idim(C_4) \leq 2$ and since C_4 is not an induced subgraph of any path then $idim(C_4) = 2$. Cycles require a lot of space, indeed, in Lemma 28 we show that $idim(C_n) = \lceil n/2 \rceil$. In contrast, the strong product of n edges is the complete graph K_{2^n}, thus $idim(K_m) = \lceil \log_2 m \rceil$.

If G is an isometric subgraph of H then $idim(G) \leq idim(H)$ but this is not necessarily true if G is only an induced subgraph. It is also true that $idim(G \boxtimes H) \leq idim(G) + idim(H)$ but equality need not hold. For example, $idim(K_3 \boxtimes K_5) = idim(K_{15}) = 4$ but $idim(K_3) + idim(K_5) = 2 + 3$.

A *projection* of $H \subseteq \boxtimes_{i=1}^{k} G_i$ onto G_i is a map $\pi_i : H \to G_i$ defined as $\pi_i(a_1, a_2, \ldots, a_k) = a_i$. A *realizer* of G is a set of paths $\{P_i : i = 1, 2, \ldots, k\}$ with $k = idim(G)$ and an edge-preserving map $F : G \to \boxtimes_{i=1}^{k} P_i$ such that $F(G)$ is an isometric subgraph of $\boxtimes_{i=1}^{k} P_i$. We will put $\bar{a} = F(a)$. The vertices of a path in the realizer will be a range of consecutive integers. This will allow us to refer to the next and previous vertex along a path as $\pi_i(\bar{a}) + 1$ and $\pi_i(\bar{a}) - 1$, respectively.

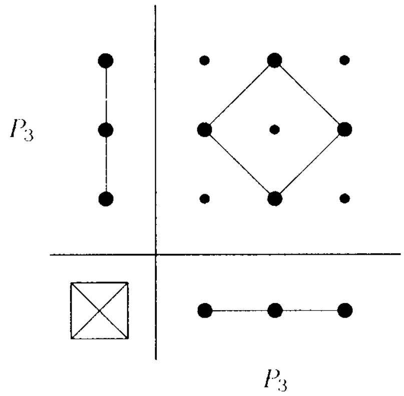

Figure 4. An Isometric Embedding of C_4 in $P_3 \boxtimes P_3$.

We say that vertices $a, b \in V(G)$ are *separated* by (H, f) if H is a graph and $f : G \to H$ is an edge-preserving map where $d(a,b) = d(f(a), f(b))$. Often the separating graph will be a path P_i from a realizer and the projection onto P_i will be the corresponding map. In this case, we have $d(a,b) = d(\pi_i(\bar{a}), \pi_i(\bar{b}))$ and we say that a, b $(\bar{a}, \bar{b})$ are separated in the i^{th} coordinate.

In the next section, we show that for any finite, connected, reflexive graph G, $idim(G)$ exists. We also give upper and lower bounds for $idim(G)$. These bounds allow us to determine the strong isometric dimension of all cycles and hypercubes. In Section 3, we show that if T is a tree with k leaves then $\lceil \log_2(k) \rceil \leq idim(T) \leq 2\lceil \log_2(k) \rceil$. This is done by subdividing and contracting edges of T to give two extremal trees – a star and a tree whose vertices have degree 1 or 3, both trees having k leaves. We pose several problems in the last section.

2 Bounds on the Strong Isometric Dimension of a Graph

Let G be a graph and let $P_v = \{v = v_0, v_1, \ldots, v_k\}$ be an isometric path. Let $P_v^* = \{0, 1, \ldots, k\}$ be a path disjoint from G. The *distance-retraction* map $f_v^* : G \to P_v^*$ is defined by $f_v^*(x) = d(v,x)$ if $d(v,x) \leq k$ else $f_v^*(x) = k$. Note the sense of direction with these maps.

We make use of distance-retraction maps so we first present some of their properties.

Lemma 21. *Let $P_v = \{v = v_0, v_1, \ldots, v_k\}$ be an isometric path of G. Then P_v^* is a retract of G, moreover v is separated by (P_v^*, f_v^*) from $x \in V(G)$ if $d(v, x) \le k$.*

Proof. Define $g : P_v^* \to P_v$ by $g(i) = v_i$ and it is easy to verify that g is edge-preserving. Also, if $x \sim y$ in G then $|d(v, x) - d(v, y)| \le 1$ and thus $f_v^*(x) \simeq f_v^*(y)$ so f_v^* is edge-preserving. Now, f_v^* maps G onto P_v^* and $f_v^* \circ g$ is the identity map on P_v^*. Therefore P_v^* is a retract.

If $d(v, x) = j \le k$ then $d(f_v^*(v), f_v^*(x)) = j$ and thus v and x are separated by (P_v^*, f_v^*). ∎

It is necessary to separate all pairs of vertices to find the strong isometric dimension.

Lemma 22. *If every pair of vertices in G is separated by at least one of (P_1, f_1), $(P_2, f_2), \ldots, (P_k, f_k)$ then $idim(G) \le k$.*

Proof. Let $H = \boxtimes_{i=1}^k P_i$ and define the map $F : V(G) \to V(H)$ by $F(x) = (f_i(x))_{i=1}^k$.

We claim that $F(G)$ is an isometric subgraph of H. Consider vertices v and w in $V(G)$ with $d_G(v, w) = d$. Since each f_i is edge preserving, $d_{P_i}(f_i(v), f_i(w)) \le d$ for all $i = 1, 2, \ldots, k$. Furthermore, since v and w are separated by at least one path, $d_{P_i}(f_i(v), f_i(w)) = d$ for some i. Therefore $d_H(F(v), F(w)) = d$ and $F(G)$ is an isometric subgraph of H. ∎

The next result not only shows that the strong isometric dimension exists for every finite, connected, reflexive graph but also gives the first upper bound.

Theorem 23. *Let G be a finite, connected, reflexive graph, then $idim(G) \le |V(G)|$.*

Proof. For each $v \in V(G)$ let v' be a vertex such that $d(v, v')$ is maximum. Let P_v be a shortest path in G from v to v'. Clearly, P_v is isometric. Now consider a pair of vertices v and w in $V(G)$ with $d_G(v, w) = d$. Since P_v is a longest isometric path starting at v, $l(P_v) \ge d_G(v, w)$. Hence, with the distance-retraction map $f_v^* : G \to P_v^*$ to $P_v^* = \{0, 1, \ldots, l(P_v)\}$ we have $d_G(v, w) = d_{P_v^*}(f_v^*(v), f_v^*(w))$, and v and w are separated by (P_v^*, f_v^*). Hence, $\{(P_v^*, f_v^*) : v \in V(G)\}$ separate every pair of vertices in $V(G)$ and, by Lemma 22, $idim(G) \le |V(G)|$. ∎

The construction in the previous result is inefficient. A slightly better result is:

Corollary 24. *Let G be a finite, connected, reflexive graph, then $idim(G) \leq |V(G)| - diam(G)$.*

Proof. Find $\{P_v^* : v \in V(G)\}$ as in the previous theorem. Now choose a vertex x such that $|P_x^*| = diam(G) + 1$. If $v \neq x$ and $v \in V(P_x)$ then eliminate P_v^* from the collection of paths. This new collection of paths also separates every pair of vertices in G. This follows since for any v, P_v^* separates v from $V(G) \setminus \{v\}$ and also separates a and b for all $a, b \in P_v$. Thus P_x^* separates all pairs of vertices on P_x; also if $y \in P_x$ and $z \notin P_x$ then y and z are separated on P_z^*. Thus the paths P_y^*, $y \in P_x \setminus \{x\}$ are unnecessary. Hence, by Lemma 22, $idim(G) \leq |V(G)| - diam(G)$. ∎

This is best possible since $idim(C_n) = \lceil n/2 \rceil = |V(C_n)| - diam(C_n)$ where the first equality is found in Lemma 28.

The idea of direction in the distance-retraction maps is generalized in the next result which will be used later in evaluating the strong isometric dimension of hypercubes and trees.

For any graph G we can obtain a directed graph from G by specifying a direction on each edge of $E(G)$. Such a directed graph is called an *orientation* of G. If an orientation is placed on a subset of the edges of $E(G)$ this is called a *sub-orientation* of G.

Suppose we have a walk $W = \{v_0, v_1, \ldots, v_n\}$. We say an edge $(v_{i-1}v_i)$ is *forward directed* on W if $v_{i-1} \to v_i$, *backward directed* on W if $v_{i-1} \leftarrow v_i$, and *undirected* otherwise. Forward and backward directed edges on the closed walk $X = \{v_0, v_1, \ldots, v_n = v_0\}$ are defined similarly. Define the *edge-sum* of a (closed) walk to be the number of forward edges minus the number of backward edges on that (closed) walk.

Lemma 25. *Suppose G is a finite connected graph. Then $idim(G) \leq k$ if and only if there is a set of k sub-orientations of G, $\{G_1, G_2, \ldots, G_k\}$, such that for every pair of vertices in $V(G)$ there is a directed isometric path between them in at least one of the k sub-orientations, and for each $i \in \{1, 2, \ldots, k\}$ the edge-sum of every cycle in G_i is zero.*

Proof. Let $idim(G) = k$ and let $\{P_1, P_2, \ldots, P_k\}$ be a realizer for G. For $1 \leq i \leq k$, construct G_i as follows: for each edge $ab \in E(G)$ let $a \to b$ if $\pi_i(\bar{b}) - \pi_i(\bar{a}) = 1$, let $b \to a$ if $\pi_i(\bar{b}) - \pi_i(\bar{a}) = -1$, and leave ab undirected otherwise.

Now consider a pair of vertices $\{x, y\}$ in $V(G)$. Let $P(x, y) = \{x = x_0, x_1, \ldots, x_d = y\}$ be an isometric path from x to y in G. Suppose that x and y are separated in the i^{th} coordinate and that $\pi_i(\bar{y}) > \pi_i(\bar{x})$. Then, for

all $j = 1, 2, \ldots, d$, $\pi_i(\bar{x}_j) - \pi_i(\bar{x}_{j-1}) = 1$ and so $x_{j-1} \rightarrow x_j$. Hence, $P(x, y)$ is a directed isometric path from x to y in G_i.
Let $C = \{v_0, v_1, \ldots, v_n = v_0\}$ be a cycle in G_i for some $i = 1, 2, \ldots, k$. Obviously, $\pi_i(\bar{v}_n) - \pi_i(\bar{v}_0) = (\pi_i(\bar{v}_n) - \pi_i(\bar{v}_{n-1})) + (\pi_i(\bar{v}_{n-1}) - \pi_i(\bar{v}_{n-2})) + \cdots + (\pi_i(\bar{v}_1) - \pi_i(\bar{v}_0)) = 0$. That is, the edge-sum of C is zero.

To prove the other direction, suppose that $\{G_1, G_2, \ldots, G_k\}$ is a set of sub-orientations of G such that for every pair of vertices in G there is a directed path between them in at least one of the k sub-orientations and for each $i = 1, 2, \ldots, k$ every cycle in G_i has edge-sum zero. This latter fact implies that every closed walk in G_i has an edge-sum of zero. Therefore, for any pair of vertices $x, y \in V(G_i)$ the edge-sums of all walks from x to y are equal.

Let $d = diam(G)$ and let $\{P_1, P_2, \ldots, P_k\}$ be a set of disjoint paths where for each $i = 1, 2, \ldots, k$, $P_i = \{-d, -d+1, \ldots, d\}$. We now define a set of maps $\{f_1, f_2, \ldots, f_k\}$ where $f_i : G \rightarrow P_i$. Choose a vertex $v \in V(G)$ and set $f_i(v) = 0$ for $1 \leq i \leq k$. Now for each vertex $x \in V(G)$ and $1 \leq i \leq k$, let $f_i(x)$ equal the edge sum of any path from v to x in G_i. Choose any pair of vertices $x, y \in V(G_i)$. Then $f_i(y) - f_i(x)$ is the edge sum of any path from x to y. Since there is a path of length $d(x, y)$ we have $|f_i(y) - f_i(x)| \leq d(x, y)$ for all $i = 1, 2, \ldots, k$. Since there is a directed path between x and y in G_i for some i then $|f_i(y) - f_i(x)| = d(x, y)$ for at least one value of i.

Finally, let $F(x) = (f_i(x))_{i=1}^{k}$. Then $F(x)$ maps $V(G)$ into $\boxtimes_{i=1}^{k} P_i$. Note that F is edge preserving and any pair of vertices x, y is separated on P_i for those i in which there is a directed path between them in G_i. Hence, by Lemma 22, $idim(G) \leq k$. ∎

In the case that G is a tree, the cycle condition is not required.

Corollary 26. *Suppose T is a tree. Then $idim(T) \leq k$ if and only if there is a set of k orientations of T, $\{T_1, T_2, \ldots, T_k\}$, such that for every pair of vertices in $V(T)$ there is a directed path between them in at least one of the k orientations of T.*

Proof. This follows from the proof above except that when defining each T_i let $a \rightarrow b$ if $\pi_i(\bar{b}) - \pi_i(\bar{a}) \geq 0$ and let $a \leftarrow b$ otherwise. ∎

We say that G has a *diameter n-tuple* if there exist n distinct vertices $a_1, a_2, \ldots, a_n$ such that $d(a_i, a_{i+1}) = diam(G)$, where addition is done modulo n. For convenience, a 2-tuple will be called a pair.

The next result gives a good lower bound in many cases and allows us to find the strong isometric dimension of cycles and hypercubes exactly. These are given in the subsequent two lemmas.

Lemma 27. *Let G be a graph which contains no diameter 4-tuples. If G contains p disjoint diameter pairs then $idim(G) \geq p$.*

Proof. Let $\{P_i : i = 1, 2, \ldots, k\}$ be a realizer for G. Let the diameter pairs be (a_i, b_i) for $1 \leq i \leq p$. If two diameter pairs (a_i, b_i) and (a_j, b_j) are separated in the same coordinate then they produce a diameter 4-tuple. Thus, all pairs must be separated in distinct coordinates and so at least p paths are required and $idim(G) = k \geq p$. ∎

Lemma 28. *For $n \geq 4$, $idim(C_n) = \lceil n/2 \rceil$.*

Proof. A cycle has no diameter 4-tuples. Let $C = \{c_0, c_1, \ldots, c_{n-1}\}$. There are $\lfloor n/2 \rfloor$ distinct diameter pairs, specifically $c_i, c_{i+\lfloor n/2 \rfloor}$, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. Hence, by the previous lemma $idim(C_n) \geq \lfloor n/2 \rfloor$ for all $n \geq 3$. So for n even we have $idim(C_n) \geq \lceil n/2 \rceil$.

In the case of odd cycles we can improve the lower bound by one. We label the vertices as $\{c_0, c_1, \ldots, c_{2m}\}$. We may assume that c_0 and c_m are separated by the first coordinate with c_0 being mapped to 0 and c_m to m. Now c_i, $0 \leq i \leq m$, is mapped to i. Consider c_1 and c_{m+1}. They can not be separated in the first coordinate and thus require a second coordinate. Again, in the second coordinate the vertex c_i, $1 \leq i \leq m+1$ is mapped to $i - 1$. Inductively, consider c_j and c_{m+j}. These vertices can not be separated in the first $j - 1$ coordinates, since c_j is not mapped to 0 or m in any of these coordinates. Thus they must be separated in say the j^{th} coordinate and c_{j+i} is mapped to i for $0 \leq i \leq m$.

For $0 \leq j \leq m-1$, this is just making specific the proof of the preceding lemma. We now continue. Consider c_m and c_{2m}. The m coordinates of c_m are completely specified and are $m, m-1, \ldots, 1$. If only m coordinates were to be used then this pair must be separated in the first coordinate and the first coordinate of c_{m+i} is $m - i$. Inductively again, consider c_{m+j} and c_{j-1}, $j < m$. For c_{m+j} only the $j + 1^{st}$ coordinate is m and none is 0. Thus, this pair must be separated in the $j + 1^{st}$ coordinate. Therefore, the $j + 1^{st}$ coordinate of c_{m+j+i} is $m - i$. Finally, consider c_{2m}. Now all its coordinates are specified and are $0, 1, \ldots, m - 1$, and so are the coordinates of c_{m-1} specifically $m - 1, m - 2, \ldots, 0$. But then the distance of the image of c_{2m} from c_{m-1} is less than $m-1$ which is impossible. Hence, another coordinate is required to separate this pair and $idim(C_n) \geq \lceil n/2 \rceil$ when n is odd.

To show that $idim(C_n) \leq \lceil n/2 \rceil$, let $C = \{c_1, c_2, \ldots, c_n\}$ and also let $\{P_1, P_2, \ldots, P_{\lceil n/2 \rceil}\}$ be the set of paths of C such that $P_i = \{c_i, c_{i+1}, \ldots, c_{i+\lfloor n/2 \rfloor}\}$ for $i = 1, 2, \ldots, \lceil n/2 \rceil$. Also let f_i^* be the distance retraction map of G onto P_i^*. We will now show that every pair of vertices is separated on at least one of these paths.

Choose any two vertices c_a and c_b on C where $1 \leq a < b \leq n$. If $a \leq \lceil n/2 \rceil$ then c_a and c_b are separated on the path P_a^*. If $\lceil n/2 \rceil < a < b$ then both c_a and c_b lie on the path $P_{\lceil n/2 \rceil}$ and are therefore separated on $P^*_{\lceil n/2 \rceil}$. Hence, by Lemma 22, $idim(C_n) \leq \lceil n/2 \rceil$ and we have $idim(C_n) = \lceil n/2 \rceil$. ■

Let the *upper girth* of a graph G, denoted by $ug(G)$, be the cardinality of the longest isometric cycle in G. Since the strong isometric dimension of a graph is at least as big as the strong isometric dimension of any isometric subgraph the preceding result can be used to show:

Corollary 29. *Let G be a finite, connected, reflexive graph, then $idim(G) \geq \lceil ug(G)/2 \rceil$.*

Lemma 210. *Let Q_k be the hypercube with 2^k vertices. Then $idim(Q_k) = 2^{k-1}$.*

Proof. Since $Q_2 = C_4$ by Lemma 28 we have $idim(Q_2) = 2$. Furthermore, Q_2 has two diameter pairs.

Now, inductively, assume that Q_{k-1} has 2^{k-2} diameter pairs and that $idim(Q_{k-1}) = 2^{k-2}$. Let $Q_k = Q_{k-1} \square P_2$ where $P_2 = \{a, b\}$. Then $V(Q_k) = A \cup B$ where $A = \{(v, a) = v_a : v \in V(Q_{k-1})\}$ and $B = \{(v, b) = v_b : v \in V(Q_{k-1})\}$. Note that $d_{Q_k}(x_a, y_b) = d_{Q_{k-1}}(x, y) + 1$ and $d_{Q_k}(x_a, y_a) = d_{Q_k}(x_b, y_b) = d_{Q_{k-1}}(x, y)$.

If x and y are diameter pairs in Q_{k-1} then x_a and y_b are diameter pairs in Q_k, as are x_b and y_a. Hence, Q_k has 2^{k-1} distinct diameter pairs. Since Q_k has no diameter 4-tuples then by Lemma 27, $idim(Q_k) \geq 2^{k-1}$.

For each $v_a \in A$ let P_{v_a} be the longest isometric path starting at v_a and let $f^*_{v_a} : Q_k \to P^*_{v_a}$ be the distance retraction map. Then each vertex in A is separated from all vertices in $A \cup B$ by at least one of $(P^*_{v_a}, f^*_{v_a})$. Furthermore, for any $x_b, y_b \in B$ we have $f_{x_a}(y_b) - f_{x_a}(x_b) = d(x_a, y_b) - d(x_a, x_b) = d(x, y) + 1 - 1 = d(x_b, y_b)$. Hence, every pair of vertices in B are separated on at least one path. Since every pair of vertices in $V(Q_k)$ are separated on at least one P_{v_a}, by Lemma 22 we have $idim(Q_k) \leq 2^{k-1}$ and thus $idim(Q_k) = 2^{k-1}$. ■

The last few lower bounds are based mainly on neighbourhood considerations. They are in terms of the maximum degree $\Delta(G)$, the chromatic number $\chi(G)$ and the independence number of the neighbourhood of a vertex $\beta(N(v))$. For our purposes we define $\beta^N(G) = \max\{\beta(N(v)) : v \in V(G)\}$.

Theorem 211. *Let G be a finite, connected, reflexive graph. Then*
(a) $idim(G) \geq \lceil \log_3(\Delta(G) + 1) \rceil$;
(b) $idim(G) \geq \lceil \log_2 \beta^N(G) \rceil$;
(c) $idim(G) \geq \lceil \log_2(\chi(G)) \rceil$.

Throughout this proof let $\{P_1, P_2, \ldots, P_k\}$ be a realizer for G. Let $F : G \to \boxtimes_{i=1}^k P_i$ be an isometric embedding of G in $\boxtimes_{i=1}^k P_i$. Recall that we denote by $\bar{v} = (v_1, v_2, \ldots, v_k)$ the vertices of $\boxtimes_{i=1}^k P_i$.

Proof of (a). Consider a vertex $a \in F(G)$ such that $\bar{a} = (a_1, a_2, \ldots, a_k)$. For any $\bar{x} \in N(\bar{a})$ we have $|\pi_i(\bar{x}) - \pi_i(\bar{a})| \leq 1$ for every $i = 1, 2, \ldots, k$. Hence, $\pi_i(\bar{x}) \in \{a_i - 1, a_i, a_i + 1\}$. Since $\bar{x} \neq \bar{a}$, there are $3^k - 1$ possible vertices onto which x may be mapped. Since each vertex in $N(a)$ must be mapped to a unique vertex in $\boxtimes_{i=1}^k P_i$ then $|N(a)| \leq 3^k - 1$. Therefore, $\Delta(G) \leq 3^k - 1$ and thus $k \geq \log_3(\Delta(G) + 1)$.

Proof of (b). If $idim(G) = 2$ then it is easy to see that $\beta^N(G) \leq 4$.

Choose any $\bar{a} \in F(G)$ and let $\bar{a} = (a_1, a_2, \ldots a_k)$. Then for each $\bar{x} \in N(\bar{a})$, $d_{P_i}(\pi_i(\bar{x}), \pi_i(\bar{a})) \leq 1$ for each $i = 1, 2, \ldots, k$. If $I \subseteq N(\bar{a})$ is an independent set then each pair of vertices in I must be separated by at least two on some P_i.

We proceed by induction on k assuming that in a product of $k-1$ paths $\beta^N(G) \leq 2^{k-1}$.

Let I be an independent set in $N(\bar{a})$ with $I = A \cup B \cup C$ where $A = \{\bar{x} : \pi_1(\bar{x}) = a_1 - 1\}$, $B = \{\bar{x} : \pi_1(\bar{x}) = a_1\}$, and $C = \{\bar{x} : \pi_1(\bar{x}) = a_1 + 1\}$. Since any two vertices of $A \cup B$ are not separated on P_1 they must be separated on at least one of the other $k - 1$ paths. Let R be the set of vertices in $\boxtimes_{i=2}^k P_i$ obtained from $A \cup B$ by dropping the first coordinate. Thus R is an independent set and $|R| = |A \cup B|$. By induction we have $|A \cup B| = |R| \leq 2^{k-1}$. Similarly, we have $|B \cup C| \leq 2^{k-1}$. Thus, $|A \cup B \cup C| = |I| \leq 2^k$.

Proof of (c). For each path P_i, $i = 1, 2, \ldots, k$, in the realizer, let v_i be an end vertex of P_i. Consider vertices $x, y \in \boxtimes_{i=1}^k P_i$. We wish to place x and y in the same colour class if $d_{P_i}(v_i, \pi_i(x)) \equiv d_{P_i}(v_i, \pi_i(y)) \pmod 2$ for every $i = 1, 2, \ldots k$. Hence, there are at most 2^k colour classes. We must now show that no two adjacent vertices have been placed in the same colour class.

Suppose x and y are two adjacent vertices in $\boxtimes_{i=1}^{k} P_i$. Then $\pi_i(x)$ and $\pi_i(y)$ are adjacent in P_i for at least one i. Therefore, $d_{P_i}(v_i, \pi_i(x)) - d_{P_i}(v_i, \pi_i(y)) \equiv 1 \pmod 2$ for some i. Hence, x and y are in different colour classes and $\chi(\boxtimes_{i=1}^{k} P_i) \leq 2^k$. In fact, since $\boxtimes_{i=1}^{k} P_i$ contains the complete graph K_{2^k} as a subgraph, $\chi(\boxtimes_{i=1}^{k} P_i) = 2^k$.
Since G is a subgraph of $\boxtimes_{i=1}^{k} P_i$ then $\chi(G) \leq \chi(\boxtimes_{i=1}^{k} P_i)$ and $\log_2(\chi(G)) \leq \log_2(\chi(\boxtimes_{i=1}^{k} P_i)) = k = idim(G)$. ■

Note that in the proof of part (b), we have that $|A| + |B| \leq 2^{k-1}$ and, similarly, $|B| + |C| \leq 2^{k-1}$. Thus $|A| + 2|B| + |C| \leq 2^k$. Thus, for $|I| = |A| + |B| + |C| = 2^k$ we must have $|B| = 0$. Therefore, an independent set of this size is unique and is $\{(a_i + \epsilon_i)_{i=1}^{k} : \epsilon_i = 1 \text{ or } -1\}$.

3 The Strong Isometric Dimension of Trees

In this section, the strong isometric dimension of trees is bounded in terms of the number of leaves. The main result gives the bounds and the proof follows by showing that for every tree T there are two associated trees T_1 and T_2, obtained from T by contraction or subdivision of edges such that $idim(T_1) \leq idim(T) \leq idim(T_2)$.

Theorem 31. *Let T be a tree with k leaves. Then*

$$\lceil \log_2 k \rceil \leq idim(T) \leq 2\lceil \log_2 k \rceil.$$

The theorem is proved by a series of lemmas. The first result is the basic manipulation technique giving us a means of associating with a tree two other trees whose strong isometric dimension is easier to calculate.

If ab is an edge of T then $T \bullet ab$ denotes the tree after the edge has been contracted.

Lemma 32. *Let T be a tree and ab be any edge of T. Then*
(a) *$idim(T \bullet ab) \leq idim(T)$; and*
(b) *$idim(T^*) = idim(T)$ where T^* is the graph obtained by subdividing the edge ab.*

The proof follows directly from Lemma 25. This result also holds for any graph G and any *cut edge* ab. The proof, however, is more involved.

***Proof of first inequality of Theorem* 31.** Let $T = T_0$ be a tree with k leaves. We apply Lemma 32 (a) to an interior edge of T and call the result T_1. We continue this and produce a sequence of trees ending with T_j, a star with the same number of leaves as T_0. From Theorem 211 (b), we know that $idim(T_j) = \lceil \log_2 k \rceil$. Since T_j has the same number of leaves as T, we therefore have what we require, i.e.,

$$idim(T) \geq idim(T_1) \geq \cdots \geq idim(T_k) = \lceil \log_2 k \rceil$$

***Proof of second inequality of Theorem* 31.** We first construct from T the required associated tree which has maximum degree three and no vertices of degree two and whose isometric dimension is at least that of T.

Suppose T is a tree with maximum degree at least 4. Suppose there is a vertex v such that $N(v) = \{v_1, v_2, \ldots, v_n\}$ and $n \geq 4$. Let T_1 be the graph obtained by removing the vertex v and adding the vertices x, y and the edges $\{xv_1, xv_2, xy, yv_3, \ldots, yv_n\}$. Note that $deg(x) = 3$ and $deg(y) = n-1$. Since $T = T_1 \bullet xy$ then from Lemma 32 (a), we have $idim(T) \leq idim(T_1)$. We continue this producing a sequence of trees ending with T_j, a tree with maximum degree three and thus

$$idim(T) \leq idim(T_1) \leq \cdots \leq idim(T_j).$$

Now suppose that T_j has a degree two vertex, v. Let $T_{j+1} = T_j \bullet (vw)$ for some $vw \in E(T)$. Note that by Lemma 32 (b), we have that $idim(T_{j+1}) = idim(T_j)$. We can continue this thereby obtaining a sequence of trees ending with T_{j+r} such that

$$idim(T_j) = idim(T_{j+1}) = \cdots = idim(T_{j+r}).$$

Hence, there is a tree $T' = T_{j+r}$ with the same number of leaves as T such that all vertices of T' are degree one or three and $idim(T') \geq idim(T)$.

Suppose S is a tree such that all vertices have degree one or three. Obviously, if S has only two leaves it is an edge and $idim(S) = 1$. Suppose that S has four leaves. There is only one case to consider. We can find two orientations of S such that there is a directed path between every pair of vertices. Hence, $idim(S) \leq 2$. (In fact, $idim(S) = 2$.)

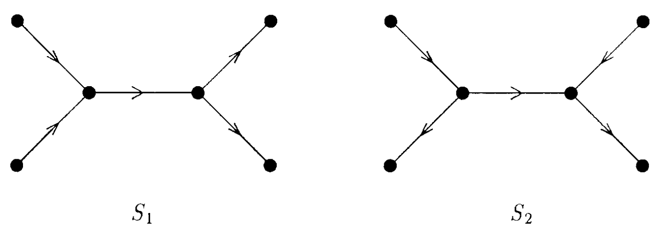

Figure 5. The Orientations of the Associated Four-leaved Tree

Lemma 33. *Let S be a tree with 2^n leaves, $n \geq 1$ and all vertices of degree one or three. Then there exists a degree three vertex, v, such that each connected component of $S \setminus \{v\}$ has at most 2^{n-1} leaves.*

Proof. Suppose for every vertex, $v \in V(S)$ there is one component in $S \setminus \{v\}$ with more than 2^{n-1} leaves. Let x be a vertex such that the number of leaves in that component is minimized. Let A, B, and C be the three components of $S \setminus \{x\}$ where A has more than 2^{n-1} leaves. Let y be the vertex in A which is adjacent to x. Obviously, $deg(y) = 3$ since A contains more than one vertex. Consider the components of $S \setminus \{y\}$. The component containing x is simply $B \cup C \cup \{x\}$. Since A has greater than 2^{n-1} leaves, $B \cup C \cup \{x\}$ has less than 2^{n-1} leaves. Furthermore, the other two components of $S \setminus \{y\}$, together with y form A. Hence, each of these two components has fewer leaves than A. This contradicts the choice of x. Hence, there exists a vertex v such that the components of $S \setminus \{v\}$ each contain at most 2^{n-1} leaves. ■

We now continue the proof and assume that if S is a tree with 2^m leaves where $m \leq n-1$ and all vertices of degree one or three, then $idim(S) \leq 2m$.

Now consider a tree S with 2^n leaves and all vertices of degree one or three. Let v be a vertex such that all components of $S \setminus \{v\}$ have at most 2^{n-1} leaves. Each component has an associated tree with all vertices of degree one or three. By induction, each of these associated trees has strong isometric dimension at most $2n-2$. Hence, by construction of the associated tree, each component also has strong isometric dimension at most $2n-2$. Then, by Corollary 26, each component has a set of $2n-2$ orientations such that each pair of vertices in that component has a directed path between them in at least one of the orientations. Let A, B, and C be the three components. The component A has orientations $\{A_1, A_2, \ldots, A_{2n-2}\}$. Let the orientations of B and C be denoted similarly.

For $i = 1, 2, \ldots, 2n-2$ we can define the orientation S_i as follows: if e is an edge in A (respectively, B, C) assign e the same direction in S_i as it has in A_i (B_i, C_i). Direct the three edges incident with v arbitrarily.

For S_{2n-1} we wish to have directed paths from all the vertices in A to v and from v to all the vertices in $B \cup C$. This can be accomplished by directing the edge connecting v with A toward v, and then for each edge (xy) in $E(A)$ direct $x \to y$ if $d(v,x) > d(v,y)$ and $y \to x$, otherwise. Then direct the other two edges incident with v away from v and for each edge (xy) in $B \cup C$ direct $x \to y$.

For S_{2n} we wish to have directed paths from all the vertices in $A \cup B$

to v and from v to all the vertices in C. This orientation is achieved in a manner similar to the construction of S_{2n-1}.

We now verify that there is a directed path between every pair of vertices in at least one of these $2n$ orientations of S. Suppose we have two vertices x, y such that both are in A (respectively, B, C). Obviously there is a path between the two in one of the first $2n-2$ orientations of S. Suppose that $x \in A$ and $y \in B \cup C$. Then there is a path from x to y in S_{2n-1}. Suppose $x \in B$ and $y \in C$. Then there is a path from x to y in S_{2n}. Finally, there is a directed path between $x \in A \cup B \cup C$ and v in both S_{2n-1} and S_{2n}. Hence, there is a directed path between every pair of vertices. Therefore, by Corollary 26 $idim(S) \leq 2n$.

To complete the proof, note that the given tree T with k leaves, where $2^{m-1} < k \leq 2^m$, can be isometrically embedded in a tree S with 2^m leaves by adding the extra leaves at any interior vertex. There is also a tree S' associated with S which only has vertices of degree one and degree three. We know that $idim(T) \leq idim(S)$, Lemma 32 part (b), gives that $idim(S) \leq idim(S')$ and the preceding argument shows that $idim(S') \leq 2m$. Putting this together we obtain the desired result:

$$idim(T) \leq 2\lceil \log_2 k \rceil. \qquad \blacksquare$$

4 Problems

Both cycles and hypercubes have an strong isometric dimension of $\lceil |V(G)|/2 \rceil$.

Problem 41. *Is there a graph G such that $idim(G) > \lceil |V(G)|/2 \rceil$?*

For a graph G it is easy to recognize when $idim(G) \leq 1$. There is also a polynomial time algorithm [4] which recognizes graphs for which $idim(G) = 2$ and it also constructs the embedding. For C_n, it was easy to determine that $idim(C_n) = \lceil n/2 \rceil$ even though product of paths has $\lceil n/2 \rceil^{\lceil n/2 \rceil}$ many vertices.

Problem 42. *For a given k, what is the complexity of recognizing graphs for which $idim(G) \leq k$?*

Is the upper bound for trees given in Theorem 31 the correct one? Both binary trees and caterpillars have a strong isometric dimension of $\lceil \log_2 t \rceil$ where t is the number of leaves.

Problem 43. *Is there a tree T such that $idim(T) > \lceil \log_2 t \rceil$?*

References

[1] M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Appl. Math. **8** (1984) 1–12. MR 85f:90124.

[2] T. Andreae, *On a pursuit game played on graphs for which the minor is excluded*, J. Combin. Theory (B) **41** (1986) 37–47. MR 87i:05179.

[3] G. Chartrand and L. Lesniak, Graphs and Digraphs (second edition, Wadsworth, 1986).

[4] S.L. Fitzpatrick, *A polynomial-time algorithm for determining if* $idim(G) \leq 2$, preprint 1998.

[5] S.L. Fitzpatrick and R.J. Nowakowski, *Copnumber of graphs with strong isometric dimension two*, to appear in Ars Combinatoria.

[6] J.R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964) 65–76. MR 32#431.

[7] E.M. Jawhari, D. Misane and M. Pouzet, *Retracts: graphs and ordered sets from the metric point of view*, Contemp. Math. **57** (1986) 175–226. MR 88i:54022.

[8] E.M. Jawhari, M. Pouzet and I. Rival, *A classification of reflexive graphs: the use of "holes"*, Canad. J. Math. **38** (1986) 1299–1328. MR 88j:05038.

[9] S. Neufeld, The Game of Cops and Robber, M.Sc Thesis, Dalhousie University, 1990.

[10] R. Nowakowski and I. Rival, *The smallest graph variety containing all paths*, Discrete Math. **43** (1983) 223–234. MR 84k:05057.

[11] R. Nowakowski and I. Rival, *A fixed edge theorem for graphs with loops*, J. Graph Theory **3** (1979) 339–350. MR 80j:05098.

[12] R. Nowakowski and P. Winkler, *Vertex to vertex pursuit in a graph*, Discrete Math. **43** (1983) 235–239. MR 84d:05138.

[13] E. Pesch, *Minimal extensions of graphs to absolute retracts*, J. Graph Theory **11** (1987) 585–598. MR 89g:05102

[14] A. Quilliot, Thèse d'Etat (Université de Paris VI, 1983).

[15] P. Winkler, The metric structure of graphs: theory and applications (London Math. Soc. Lecture Note Ser., 123, Cambridge Univ. Press, Cambridge-New York, 1987). MR 88h:05090.

Received 30 November 1998
Revised 13 December 1999