

PROBLEMS ON FULLY IRREGULAR DIGRAPHS

ZDZISŁAW SKUPIEŃ

Faculty of Applied Mathematics
University of Mining and Metallurgy AGH
al. Mickiewicza 30, 30-059 Kraków, Poland

e-mail: skupien@uci.agh.edu.pl

A simple graph with more than one vertex is well-known to have two vertices of the same degree. This amounts to saying that no simple nontrivial graph can be fully irregular. Recall that directing each edge of a simple graph results in an *oriented graph* (which is a digraph without 2-cycles $\vec{C}_2$).

A digraph D is called *fully irregular* if distinct vertices of D have distinct degree pairs. The *degree pair* of a vertex is the outdegree followed by the indegree of the vertex. The notion of fully irregular digraphs—introduced by the present author in 1995—is investigated in [1, 2, 3, 5]. Some results on fully irregular digraphs were presented at international conferences held in Poland at Lubiatów '96, Gronów '97, '98, and at Kazimierz Dolny '97.

Theorem 1. *Let D be a digraph of order n . There exists an injection $D \mapsto D'$ which associates with D a fully irregular digraph D' of order $n + 2\lceil\sqrt{n}\rceil$ such that D is an induced subdigraph of D' and such that deleting all arcs of D from D' results in an oriented graph.*

Proof. Let $V = \{v_1, \dots, v_n\}$ be the vertex set of D . Let $t = \lceil\sqrt{n}\rceil - 1$. Consider two disjoint linearly ordered sets U and W which comprise altogether $2(t + 1)$ new vertices respectively u_i and w_i , which are ordered by increasing subscripts i , $i = 0, 1, \dots, t$. Let B be the bipartite digraph whose vertex set is $U \cup W$ and all arcs are of the form (w_j, u_i) for each $i \in \{0, 1, \dots, t - j\}$ where $j = 0, 1, \dots, t$. Let D' be a digraph of order $n + 2t + 2$ which includes disjoint digraphs D and B , all arcs both from V to u_0 and from w_0 to V , and possibly arcs (v, u_i) and/or (w_i, v) where $v \in V$ and, moreover, the neighbours of any such v both in U and W make up precisely initial segments of U and W , respectively. Hence the outdegrees

and indegrees of vertices from U and W , respectively, are all zero. Therefore the two obligatory arcs (v, u_0) and (w_0, v) for each vertex v of D enable us to identify D as the subdigraph of D' induced by all vertices whose outdegrees and indegrees are all positive.

For any vertex v of D the optional arcs (possibly none) from v to a segment of U can be chosen in $t + 1$ ways. The same is the number of choices for remaining optional arcs from a segment of W to v . Thus the degree pair in D' of each vertex v can be any of some $(t + 1)^2$ ($\geq n$) points in the plane integral lattice. Therefore distinct degree pairs in D' for all n vertices of D can be designed and realized. The construction associates mutually distinct degree pairs with all remaining vertices, too. Therefore a required injection exists. ■

Corollary 2. *There are at least as many fully irregular digraphs (oriented graphs) of order $n + 2\lceil\sqrt{n}\rceil$ as there are digraphs (oriented graphs) of order n .* ■

It seems likely that fully irregular digraphs can contribute to finding a constructive proof (which is lacking) of the fact (cf. [4]) that almost all digraphs have trivial automorphism group. Given a digraph D on n vertices, let $f(D)$ (and $f'(D)$) be the smallest integer t such that a fully irregular digraph $\tilde{D}$ on $n + t$ vertices includes D as an induced subdigraph (and such that deleting the arc set $A(D)$ of D from $\tilde{D}$ results in an oriented graph). Name $f(D)$ and $f'(D)$ respectively the *irregularity deficit* and *irregularity o-deficit* of D . Clearly, $f(D) \leq f'(D)$. Let $f(n)$ (and $f'(n)$) be the *largest irregularity deficit* (resp. *largest irregularity o-deficit*) among n -vertex digraphs.

Corollary 3. *The irregularity o-deficit among n -vertex digraphs is bounded by $2\lceil\sqrt{n}\rceil$. Thus*

$$f(n) \leq f'(n) \leq 2\lceil\sqrt{n}\rceil. \quad \blacksquare$$

Problem 1 (Problem 1'). Characterize n -vertex digraphs D with the largest possible irregularity deficit $f(D)$ (o-deficit $f'(D)$), i.e., with $f(D) = f(n)$ (resp. $f'(D) = f'(n)$).

Given a nonnegative integer r , a digraph D is called *r -diregular* if degree pairs in D are all (r, r) .

Theorem 4 (Górska et al. [2]). *If D is an r -diregular oriented graph on n vertices then $f'(D) = \lfloor\sqrt{2n} - \frac{1}{2}\rfloor$ for $n \geq 1$ unless $n = 3$, $r = 1$, and then $f'(D) = 2$.* ■

Theorem 5 (Górska et al. [3]). *If D is an r -diregular digraph on n vertices then $f(D) = \lfloor \sqrt{n-1} \rfloor (= \lceil \sqrt{n} \rceil - 1)$ for $n \geq 1$ unless $n = 4$, $r \in \{1, 2\}$, and then $f(D) = 2$. ■*

REFERENCES

- [1] Z. Dziechcińska-Halamoda, Z. Majcher, J. Michael, and Z. Skupień, *Sets of degree pairs in the extremum fully irregular digraphs*, in preparation.
- [2] J. Górska, Z. Skupień, Z. Majcher, and J. Michael, *A smallest fully irregular oriented graph containing a given diregular one*, submitted.
- [3] J. Górska and Z. Skupień, *A smallest fully irregular digraph containing a given diregular one*, in preparation.
- [4] F. Harary and E.M. Palmer, *Graphical Enumeration* (Academic Press, New York, 1973).
- [5] Z. Majcher, J. Michael, J. Górska, and Z. Skupień, *The minimum size of fully irregular oriented graphs*, in: Proc. Kazimierz Dolny '97 Conf., Discrete Math., to appear.

Received 8 April 1999

Revised 14 September 1999