

ON CYCLICALLY EMBEDDABLE GRAPHS

MARIUSZ WOŹNIAK*

Faculty of Applied Mathematics AGH
Department of Discrete Mathematics
Al. Mickiewicza 30, 30-059 Kraków, Poland

e-mail: mwozniak@uci.agh.edu.pl

Abstract

An *embedding* of a simple graph G into its complement $\overline{G}$ is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In this note we consider some families of embeddable graphs such that the corresponding permutation is cyclic.

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1 INTRODUCTION

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges. If a graph G has order n and size m , we say that G is an (n, m) -graph.

Assume now that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of k (≥ 2) disjoint copies of a graph H , then we write $G = kH$.

An *embedding* of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In others words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G (of order n) into a complete graph K_n . If, additionally, an embedding of G is a cyclic permutation we say that G is

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cyclically embeddable (CE for short). The aim of this note is to study some families of CE graphs.

The following theorem was proved, independently, in [2], [3] and [8].

Theorem 1. *Let $G = (V, E)$ be a graph of order n . If $|E(G)| \leq n - 2$ then G can be embedded in its complement $\overline{G}$.*

The example of the star $K_{1,n-1}$ shows that Theorem 1 cannot be improved by raising the size of G .

The following theorem, proved in [9], has been used in the study of embeddings of $(n, n - 1)$ graphs.

Theorem 2. *Let $G = (V, E)$ be a graph of order n . If $|E(G)| \leq n - 2$ then there exists an embedding σ of G in its complement such that σ has no fixed points, i.e. $\sigma(x) \neq x$ for $x \in V(G)$.*

The above theorem cannot be improved by increasing the number of edges as it is showed by the graph $K_{1,2} \cup K_3$.

However, Theorem 2 can be improved in other direction by specifying the structure of the packing permutation. In particular we have the following result proved first in [10].

Theorem 3. *Let $G = (V, E)$ be a graph of order n . If $|E(G)| \leq n - 2$, then there exists a cyclic embedding of G .*

As we have seen, if $|E(G)| = n - 1$ then there are graphs that are not embeddable and even in the case where a graph is embeddable, a fixed-point-embedding does not necessarily exist. However, if we assume in addition that G is a tree, we have the following result (cf. [11]).

Theorem 4. *Let T be a tree of order n . If $T \neq S_n$ then there exists a cyclic embedding of G .*

The general references for these and other packing problems are [1], [13] and [12] (see also [14]).

We shall need some additional definitions in order to formulate the results. Let G and H be two rooted graphs at u and x , respectively. The graph of order $|V(G)| + |V(H)| - 1$ obtained from G and H by identifying u with x will be called the *touch* of G and H and will be denoted by $G \cdot H$.

A similar operation consisting in the identification of a couple of vertices of G , say (u_1, u_2) with a couple of vertices of H , say (x_1, x_2) will be called the *2-touch* of G and H and will be denoted by $G : H$. The graph $G : H$ is of order $|V(G)| + |V(H)| - 2$. By definition, the edge say (u_1, u_2) belongs to $E(G : H)$ if $u_1 u_2 \in E(G)$ or $x_1 x_2 \in E(H)$.

Let σ be a cyclic permutation defined on $V(G)$. For $u \in V(G)$, we denote often the vertex $\sigma(u)$ by u^+ and $\sigma^{-1}(u)$ by u^- . The edge uu^+ is said to be of *length one* (with respect to σ).

2 SOME LEMMAS

Lemma 5. *Let G be a graph obtained from the graph H by removing a pendent vertex. If G is CE then H is CE.*

Proof. Denote by x the pendent vertex of H and consider the graph $G = H - \{x\}$. By the assumptions, there exists a cyclic permutation σ' of $V(G)$ that is an embedding of G . Let $\sigma' = (a_1 a_2 \dots a_n)$. Without loss of generality we may assume that $a_1 x$ belongs to $E(H)$. Observe that at least one of the edges $a_1 a_2$ or $a_1 a_n$ does not belong to $E(G)$. Suppose that $a_1 a_2 \in E(G)$. Then $a_1 a_n \notin E(G)$ and it is easy to see that the cyclic permutation on $V(H)$ defined by $\sigma = (a_1 x a_2 \dots a_n)$ is an embedding of H . If $a_1 a_n \in E(G)$ then we put $\sigma = (a_1 a_2 \dots a_n x)$. ■

Lemma 6. *Let H be a graph with at least one isolated vertex v and let $G = H - \{v, x\}$ be a graph obtained from the graph H by removing v and another vertex x . If G has an isolated vertex and is CE then H is CE.*

Proof. Let us consider the graph $G = H - \{x, v\}$. Denote by σ' an cyclic embedding of G and let $\sigma' = (a_1 a_2 \dots a_{n-1})$. Without loss of generality we may assume that a_1 is an isolated vertex of G . It is easy to see that the cyclic permutation on $V(H)$ defined by $\sigma = (a_1 a_2 \dots a_{n-1} x v)$ is an embedding of G . ■

Lemma 7. *Let G and H be two CE graphs. Then $G \cup H$ is CE.*

Proof. Denote by $\alpha = (a_1 a_2 \dots a_n)$ an cyclic embedding of G and by $\beta = (b_1 b_2 \dots b_k)$ an cyclic embedding of H . An cyclic embedding σ of $G \cup H$ can be defined as follows: $\sigma(a_1) = b_2$, $\sigma(b_1) = a_2$ and $\sigma(v) = \alpha(v)$ for $v \in V(G) - a_1$ and $\sigma(v) = \beta(v)$ for $v \in V(H) - b_1$. ■

Lemma 8. *Let G and H be two CE graphs rooted at u and x , respectively. Then the graph $G \cdot H$ is CE.*

Proof. Denote by α and β the cyclic embeddings of G or H , respectively. Assume first that the edge uu^+ does not belong to $E(G)$. A cyclic embedding of $G \cdot H$ can be defined as follows: $\sigma(x) = \sigma(u) = u^+$, $\sigma(u^-) = x^+$ and $\sigma(v) = \alpha(v)$ for $v \in V(G) - u^-$ and $\sigma(v) = \beta(v)$ for $v \in V(H) - x$. If $uu^+ \in E(G)$ then $uu^- \notin E(G)$ and we can repeat the above construction with α replaced by α^{-1} . ■

Remark. A similar result holds also if "cyclically embeddable" is replaced by "embeddable" (see [6]).

Lemma 9. *Let G and H be two CE graphs such that the vertices v, u of G and x, y of H are consecutive with respect to the cyclic embeddings of G and H , respectively. Suppose that the edges uu^+ and xx^- as well as the edges yy^+ and vv^- are not simultaneously present. Then the graph $G : H$ obtained by identifying u with x and v with y is CE.*

Proof. Denote by α and β the cyclic embeddings of G or H , respectively. A cyclic embedding of $G : H$ can be defined as follows: $\sigma(x) = \sigma(u) = u^+$, $\sigma(y) = \sigma(v) = y^+$ and $\sigma(v) = \alpha(v)$ for $v \in V(G) - v$ and $\sigma(v) = \beta(v)$ for $v \in V(H) - x$. ■

Remark. Observe that the condition that the edges uu^+ and xx^- as well as the edges yy^+ and vv^- are not simultaneously present is in particular fulfilled if uv is an edge of G or xy is an edge of H .

3 SOME FAMILIES OF CE GRAPHS

31. Trees and $(n, n - 2)$ -graphs

By Theorems 3 and 4 all $(n, n - 2)$ -graphs as well as all non-star trees are cyclically embeddable.

32. Cycles

It is easy to see that neither C_3 nor C_4 are embeddable. The cycle C_5 is embeddable but not cyclically. The aim of this subsection is to prove that

Theorem 10. *Let C_n be the cycle of order n . If $n \geq 6$ then there exists a cyclic embedding of C_n .*

Proof. The cycles C_6 , C_7 , C_8 , and C_9 are drawn in Figure 1 in such a way that the corresponding cyclic embeddings are easy to guess as a "rotation".

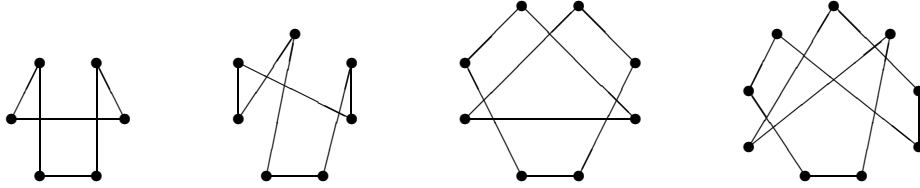


Figure 1. Cyclic embeddings of C_6 , C_7 , C_8 , and C_9 .

Since for $n \geq 10$ the cycle C_n can be considered as a subgraph of the graph $C_{n-4} : C_6$ where the 2-touch is realized by identifying two edges of lengths one (with respect to the corresponding cyclic embeddings), the remaining part of the theorem follows from Lemma 9. Observe that each C_i for $i \geq 6$ has a cyclic embedding with at least three edges of length one, so the above construction can be continued. Figure 2 provides an example of this construction for the cycle C_{10} . ■

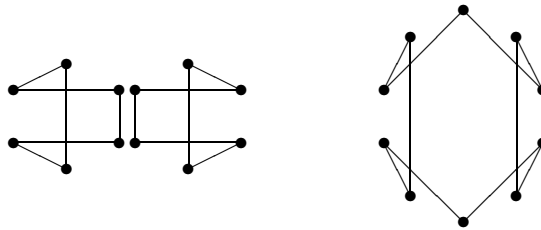


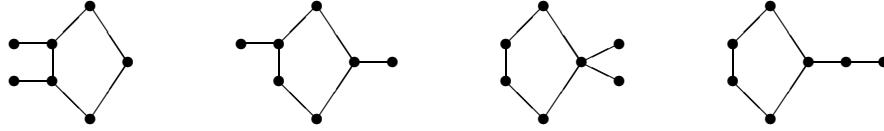
Figure 2. 2-touch of two cycles C_6 and the resulting cyclic embedding of C_{10} .

33. Unicyclic Graphs

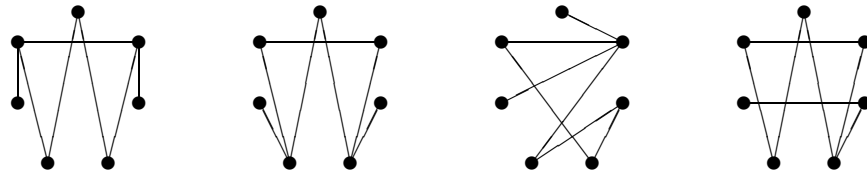
Let now G be a unicyclic graph that is the connected (n, n) -graphs. If the unique cycle of G is of length greater than or equal to six then G is CE because of Lemma 5.

So, consider the case where the unique cycle of G is of length five. Observe first that the graph $C_5 \cup K_1$ is not cyclically embeddable. This

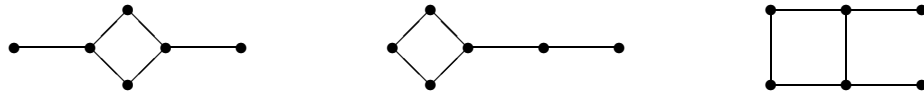
implies that the graph of order six obtained from C_5 by adding one pendent edge is not cyclically embeddable. There are four unicyclic graphs of order seven obtained from C_5 by adding two new edges (see below).



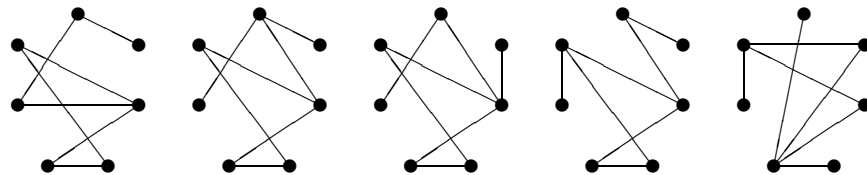
All these graphs are CE as it is showed in the next figure.



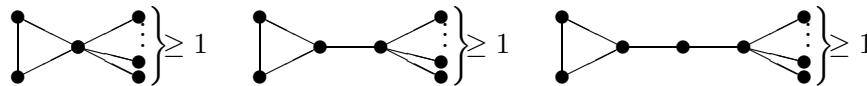
Consider now the unicyclic graphs based on C_4 . It is known (cf. [5]) and easy to verify that the graph of order $4 + k$, $k \geq 1$, obtained by identifying a vertex of the cycle C_4 with the center of a star $K_{1,k}$ is not embeddable. Within three graphs given below, two first graphs are embeddable but not cyclically and the third one is CE.



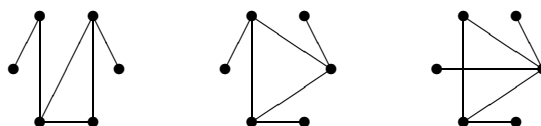
So, we have to verify five $(7, 7)$ -graphs obtained from C_4 by adding three new edges. All these graphs are CE as it follows from the figure below.



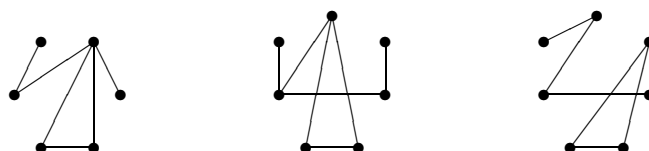
Finally, consider the unicyclic graphs based on C_3 . It is known (cf. [5]) that the following graphs are not embeddable.



It is easy to see that the graph A of order five obtained from the cycle C_3 by adding two independent pendent edges is not CE. There are three unicyclic graphs of order six obtained from A by adding one new edge. All these graphs are CE (see below).

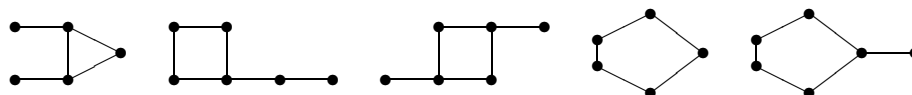


It remains to verify the existence of the cyclic embedding of three graphs obtained from nonembeddable graphs. These embeddings exist as it is shown in the figure below.



The considerations of this subsection can be formulated in the following way.

Theorem 11. *The unicyclic graphs that are embeddable are also cyclically embeddable except for five graphs given below.*



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