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A NOTE ON KERNELS AND SOLUTIONS IN DIGRAPHS

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Abstract

For given nonnegative integers k, s an upper bound on the minimum number of vertices of a strongly connected digraph with exactly k kernels and s solutions is presented.

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Kernels (solutions) are vertex subsets of digraphs that are studied in [2, 3, 5, 8]. The decision problem of the existence of a kernel in a digraph is known to be NP-complete (see e.g. the book [4]). The number of kernels (solutions) was investigated in the papers [1, 7, 9]. In [6] the first author has shown that for given nonnegative integers k, s there are infinitely many pairwise nonisomorphic strongly connected digraphs with no pair of opposite arcs that have exactly k kernels and s solutions. An upper bound on the minimum number of vertices of those digraphs is also presented there. In the following a better upper bound is established.

1. Preliminaries

An ordered pair $D = (V, A)$ is said to be *a digraph* whenever V is a non-empty set (vertices of D) and A (arcs of D) is a subset of the set of ordered pairs of elements V such that $\overrightarrow{vv} \notin A$ for each $v \in V$, and if $u, v \in V$ then $\overrightarrow{uv} \in A$ implies $\overrightarrow{vu} \notin A$.

A set of vertices $W \subseteq V$ is called *independent* if for every pair of vertices $u, v \in W$ neither $\overrightarrow{uv}$ nor $\overrightarrow{vu}$ is present in the digraph. $W \subseteq V$ is *absorbent* if for each $u \in V - W$ there exists $\overrightarrow{uv} \in A$ with $v \in W$ and *dominant* if for each $v \in V - W$ there exists $\overrightarrow{uv} \in A$ with $u \in W$. A set $W \subseteq V$ is *a kernel of D* if W is independent and absorbent and W is *a solution of D* if W is independent and dominant.

As usual, a digraph is *strongly connected*, if for every $u, v \in V$ there exists a sequence $\overrightarrow{ua_1}, \overrightarrow{a_1a_2}, \overrightarrow{a_2a_3}, \dots, \overrightarrow{a_kv}$ in A. Let $\mathcal{G}$ denote the class of all finite strongly connected digraphs.

2. Results

Let $\mathcal{G}_{(k,s)}$ denote the set of all strongly connected digraphs with k kernels and s solutions. It is known (see [6], Theorem 2.6) that the set $\mathcal{G}_{(k,s)}$ is infinite whenever k and s are nonnegative integers. A digraph belonging to $\mathcal{G}_{(k,s)}$ with the minimum number of vertices is called *a minimum digraph* of $\mathcal{G}_{(k,s)}$. The number of vertices of a minimum digraph of $\mathcal{G}_{(k,s)}$ will be denoted by $k \star s$. The following assertions were proved:

Proposition 1 ([6] 1.1, 1.2, 2.7)**.** *Let k, s be nonnegative integers. Then*

(i) $k \star s = s \star k$,
(ii) $0 \star 0 = 3, \qquad 0 \star 1 = 1 \star 0 = 5, \qquad 0 \star 2 = 2 \star 0 = 6,$
(iii) $1 \star 1 = 2 \star 2 = 4, \qquad 1 \star 2 = 2 \star 1 = 5,$
(iv) *if $k > 1$ then $k \star 0 \le 4k$ and $k \star 1 \le 4k + 1$, and*
(v) *$k \star s \le 4(k + s) - 7$ whenever $k > 1$ and $s > 1$.*

Let k, s be positive integers. By part (i) of the previous proposition $k \le s$ can be supposed without loss of generality. Define a digraph $D_{(k,s)}$ as follows. Denote by T and U two disjoint copies of an acyclic tournament with s vertices such that $t_1, t_2, \dots, t_s$ are the vertices of T, $u_1, u_2, \dots, u_s$ are the vertices of U, $\overrightarrow{t_it_j}$ (resp. $\overrightarrow{u_iu_j}$) are the arcs of T (of U) for $i, j \in \{1, 2, \dots, s\}$ whenever $i < j$. Take T, U, two new vertices v, w and add the following arcs: $\overrightarrow{t_iu_j}$ and $\overrightarrow{u_it_j}$ for $i, j \in \{1, 2, \dots, s\}$ whenever $i > j$,

$\overrightarrow{t_i v}, \overrightarrow{v u_i}, \overrightarrow{u_i w}$ for every $i \in \{1, 2, \ldots, s\}$,
$\overrightarrow{w t_i}$ for $i \leq k$, $\overrightarrow{t_i w}$ for $k < i \leq s$ and $\overrightarrow{wv}$.

Proposition 2. *The digraph $D_{(k,s)}$ belongs to $\mathcal{G}_{(k,s)}$ whenever k, s are positive integers.*

Proof. $D_{(k,s)}$ has a hamiltonian cycle (for instance $t_1, t_2, \ldots, t_{s-1}, t_s, v, u_1, u_2, \ldots, u_{s-1}, u_s, w, t_1$), thus it is strongly connected. Since no vertex of the digraph $D_{(k,s)}$ creates absorbent (dominant) set then every kernel (solution) of $D_{(k,s)}$ contains at least two vertices. On the other hand no triple of vertices of $D_{(k,s)}$ is independent. Thus any kernel (solution) must contain exactly two vertices. But if $\{x, y\}$ is an independent subset of the vertex set of $D_{(k,s)}$ then there exists $i \in \{1, 2, \ldots, s\}$ such that $x = t_i, y = u_i$ or $x = u_i, y = t_i$. It is easy to check that S is a solution of $D_{(k,s)}$ if and only if $S = \{t_i, u_i\}$ for $i \in \{1, 2, \ldots, s\}$ and K is a kernel of $D_{(k,s)}$ if and only if $K = \{t_i, u_i\}$ for $i \in \{1, 2, \ldots, k\}$. ■

Corollary. *Let k, s be positive integers. Then $k \star s \leq 2 \cdot \max\{k, s\} + 2$.*

Proof. By the previous proposition it suffices to take the digraph $D_{(k,s)}$ having $\max\{k, s\} + 2$ vertices. Therefore the number of the vertices of a minimum digraph of $\mathcal{G}_{(k,s)}$ is at most $2 \cdot \max\{k, s\} + 2$. ■

Remark. The upper bound of $k \star s$ above is sharp in the case $k = s = 1$ and also if $k = 0, s = 2$. On the contrary it is not attained for $k = 0$ and $s \in \{0, 1\}$. The new bound improved the bound from (v) in Proposition 1 in all cases where $k > 1, s > 1$ or $k = 1, s > 2$ or $k > 2, s = 1$.

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