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ON THE COMPLETENESS OF DECOMPOSABLE PROPERTIES OF GRAPHS

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Abstract

Let $\mathcal{P}_1, \mathcal{P}_2$ be additive hereditary properties of graphs. A $(\mathcal{P}_1, \mathcal{P}_2)$-*decomposition* of a graph G is a partition of $E(G)$ into sets E_1, E_2 such that induced subgraph $G[E_i]$ has the property $\mathcal{P}_i$, $i = 1, 2$. Let us define a property $\mathcal{P}_1 \oplus \mathcal{P}_2$ by $\{G : G \text{ has a } (\mathcal{P}_1, \mathcal{P}_2)\text{-decomposition}\}$.

A property D is said to be *decomposable* if there exists nontrivial additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $D = \mathcal{P}_1 \oplus \mathcal{P}_2$. In this paper we determine the completeness of some decomposable properties and we characterize the decomposable properties of completeness 2.

Keywords: decomposition, hereditary property, completeness.

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1 Introduction and Notation

We consider finite undirected simple graphs. In general, we follow the notation and terminology of [4, 6]. Let us denote by $\mathcal{I}$ the class of all simple finite graphs. A *graph property* $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}$. $\mathcal{P}$ will also denote the property that a graph is a member of $\mathcal{P}$. A property $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ (H is a subgraph of G) implies $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if for each graph G all

of whose components have property $\mathcal{P}$ it follows that $G \in \mathcal{P}$, too. The set $\mathbb{L}^a$ of all hereditary and additive properties of graphs, partially ordered by set inclusion forms a complete distributive lattice. We will denote by $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ the interval between $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in the lattice $\mathbb{L}^a$.
Every hereditary property $\mathcal{P}$ is uniquely determined by the set

$$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph of } G \text{ belongs to } \mathcal{P}\}$$

of its *minimal forbidden subgraphs*. By the property $-\{H_1, \cdots, H_k\}$ we mean the property $\mathcal{P}$ with $\mathbf{F}(\mathcal{P}) = \{H_1, \cdots, H_k\}$.

Example. We list some important additive hereditary properties, using partially the notation of [2, 4].

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},\\
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \text{the maximum degree } \Delta(G) \le k\},\\
\mathcal{W}_k &= \{G \in \mathcal{I} : \text{the length of the longest path in } G \text{ is at most } k\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate},\\
&\qquad \text{i.e., the minimum degree } \delta(H) \le k \text{ for each } H \subseteq G\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or}\\
&\qquad K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\},\\
\mathcal{E}_k &= \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k \text{ edges}\},\\
\mathcal{LF} &= \{G \in \mathcal{I} : G \in \mathcal{D}_1 \wedge G \in \mathcal{S}_2\},\\
\mathcal{SF} &= \{G \in \mathcal{I} : \text{each component of } G \text{ is a star}\}.
\end{aligned}$$

An additive hereditary property $\mathcal{P}$ is said to be nontrivial if $\mathcal{P} \neq \mathcal{O}$ and $\mathcal{P} \neq \mathcal{I}$. Let $\mathcal{P}$ be a nontrivial additive hereditary property. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$; it is called the *completeness* of $\mathcal{P}$. Obviously

$$c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{W}_k) = c(\mathcal{D}_k) = c(\mathcal{T}_k) = c(\mathcal{I}_k) = k,$$

$$c(\mathcal{E}_k) = \left\lfloor \frac{1}{2}(-1 + \sqrt{1+8k}) \right\rfloor$$

and for additive properties $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathcal{O}$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be arbitrary hereditary properties of graphs. A vertex *$(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition* of a graph G is a partition of $V(G)$ into sets $V_1, V_2, \ldots, V_n$ such that for each $i = 1, 2, \ldots, n$, the induced subgraph $G[V_i]$

has the property $\mathcal{P}_i$ (for convenience, the empty set $\emptyset$ will be regarded as the set inducing the subgraph with any property $\mathcal{P}$).

A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. It is easy to see that if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is additive and hereditary, too. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $\mathcal{P}^n = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$. Thus, e.g., $\mathcal{O}^k,\ k \geq 2$ denotes the class of all k-colourable graphs.
An hereditary property $\mathcal{R}$ is said to be *reducible* if there exist hereditary properties $\mathcal{P}, \mathcal{Q}$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and *irreducible*, otherwise.

A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*decomposition* of a graph G is a partition of $E(G)$ into sets $E_1, E_2, \ldots, E_n$ such that for each $i = 1, 2, \ldots, n$, the subgraph $G[E_i]$ has the property $\mathcal{P}_i$ (for convenience, the empty set $\emptyset$ will be regarded as the set inducing the subgraph with any property $\mathcal{P}$).

A property $\mathcal{D} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_n$ is defined as the set of all graphs having a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-decomposition. It is easy to see that if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are additive and hereditary, then $\mathcal{D} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_n$ is additive and hereditary, too. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $n\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_n$.

A hereditary property $\mathcal{D}$ is said to be *decomposable* if there exist non-trivial hereditary properties $\mathcal{P}, \mathcal{Q}$ such that $\mathcal{D} = \mathcal{P} \oplus \mathcal{Q}$ and *indecomposable*, otherwise.

The Ramsey number $r(m, n)$ is the smallest integer for which every graph of order $r(m, n)$ contains either a clique of size m or an independent set of size n.

Throughout this article, all properties we deal with are hereditary and additive.

2 Completeness

There is an easy formula to determine the completeness of any reducible property $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$, namely, $c(\mathcal{R}) = c(\mathcal{P}) + c(\mathcal{Q}) + 1$ (see [8]). The calculation of the completeness of decomposable properties is much more difficult. It is easy to see that:

$$\max\{c(\mathcal{P}), c(\mathcal{Q})\} \leq c(\mathcal{P} \oplus \mathcal{Q}) \leq c(\mathcal{I}_{c(\mathcal{P})} + \mathcal{I}_{c(\mathcal{Q})}) = r(c(\mathcal{P}) + 2, c(\mathcal{Q}) + 2) - 2,$$

and hence the problem is related to the problem of determining the Ramsey numbers.

Obviously, there is only one decomposable property of completeness 1, the property $\mathcal{O}_1 \oplus \mathcal{O}_1$. The next result characterize the decomposable properties of completeness equals 2.

Theorem 1. *Let $\mathcal{P}, \mathcal{Q}$ be nontrivial additive hereditary properties. Then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ if and only if $\mathcal{P}$ and $\mathcal{Q}$ satisfy at least one of the following conditions:*

(i) $\mathcal{P} = \mathcal{O}_1$ *and* $\mathcal{Q} \in \langle \mathcal{E}_2, -\{C_4\}\rangle$,

(ii) $\mathcal{P} = \mathcal{E}_2$ *and* $\mathcal{Q} \in \langle \mathcal{O}_1, -\{\curlyvee, C_4\}\rangle$,

(iii) $\mathcal{P} = \mathcal{O}_2$ *and* $\mathcal{Q} \in \langle \mathcal{O}_1, -\{C_4\} \wedge \mathcal{S}_2\rangle$,

(iv) $\mathcal{P} \in \langle \mathcal{E}_2, \mathcal{S}_2 \wedge -\{C_3, C_4\}\rangle$ *and* $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{W}_2\rangle$,

(v) $\mathcal{P} \in \langle \mathcal{E}_2, \mathcal{SF}\rangle$ *and* $\mathcal{Q} \in \langle \mathcal{O}_1, -\{C_3, C_4\}\rangle$.

Proof. By the definition of the completeness if $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ then $K_4 \notin \mathcal{P} \oplus \mathcal{Q}$. Let $c(\mathcal{P} \oplus \mathcal{Q}) = 2$. Since $O_1 \subseteq \mathcal{P}, \mathcal{Q}$, then $C_4 \notin \mathcal{P}$ and $C_4 \notin \mathcal{Q}$ (because $K_4 \in (K_2 \cup K_2) \oplus C_4$).
To prove the theorem let us consider the following cases:

Case 1. Let $K_2 \in \mathcal{P}$ and $P_3 \notin \mathcal{P}$. Then $C_4 \notin \mathcal{Q}$.
Conversely, if $\mathcal{P} = \mathcal{O}_1$ and $\mathcal{Q} \in \langle \mathcal{E}_2, -\{C_4\}\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (i).

Case 2. Let $P_3 \in \mathcal{P}$, $K_3 \notin \mathcal{P}$, $K_{1,3} \notin \mathcal{P}$ and $P_4 \notin \mathcal{P}$.
Then $C_4 \notin \mathcal{Q}$ and $\curlyvee \notin \mathcal{Q}$. Conversely, if $\mathcal{P} = \mathcal{E}_2$ and $\mathcal{Q} \in \langle \mathcal{O}_1, -\{C_4, \curlyvee\}\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (ii).

Case 3. Let $P_3 \in \mathcal{P}$, $K_3 \in \mathcal{P}$, $K_{1,3} \notin \mathcal{P}$ and $P_4 \notin \mathcal{P}$.
Then $C_4 \notin \mathcal{Q}$ and $K_{1,3} \notin \mathcal{Q}$. Conversely, if $\mathcal{P} = \mathcal{O}_2$ and $\mathcal{Q} \in \langle \mathcal{O}_1, -\{C_4, K_{1,3}\}\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (iii).

Case 4. Let $P_3 \in \mathcal{P}$, $K_3 \notin \mathcal{P}$, $K_{1,3} \in \mathcal{P}$ and $P_4 \notin \mathcal{P}$.
Then $C_4 \notin \mathcal{Q}$ and $C_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \mathcal{SF}\rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, -\{C_4, C_3\}\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (v).

Case 5. Let $P_3 \in \mathcal{P}$, $K_3 \in \mathcal{P}$, $K_{1,3} \in \mathcal{P}$ and $P_4 \notin \mathcal{P}$.
Then $C_4 \notin \mathcal{Q}, C_3 \notin \mathcal{Q}$ and $K_{1,3} \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \mathcal{W}_2\rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{S}_2 \wedge -\{C_4, C_3\}\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (iv).

Case 6. Let $P_4 \in \mathcal{P}$, $K_3 \notin \mathcal{P}$, $K_{1,3} \notin \mathcal{P}$. Then $P_4 \notin \mathcal{Q}$.
Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \mathcal{S}_2 \wedge -\{C_4, C_3\}\rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{W}_2\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (iv).

Case 7. Let $P_4 \in \mathcal{P}$, $K_3 \in \mathcal{P}$, $K_{1,3} \notin \mathcal{P}$. Then $P_4 \notin \mathcal{Q}$ and $K_{1,3} \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, \mathcal{S}_2 \wedge -\{C_4\}\rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{O}_2\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (iii).

Case 8. Let $P_4 \in \mathcal{P}$, $K_3 \notin \mathcal{P}$, $K_{1,3} \in \mathcal{P}$. Then $P_4 \notin \mathcal{Q}$ and $K_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, -\{C_4, C_3\}\rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{SF}\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (v).

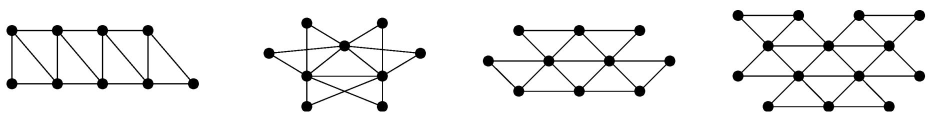
Case 9. Let $P_4 \in \mathcal{P}$, $K_3 \in \mathcal{P}$, $K_{1,3} \in \mathcal{P}$ and $\boldsymbol{\Upsilon} \notin \mathcal{P}$.
Then $P_4 \notin \mathcal{Q}, K_{1,3} \notin \mathcal{Q}$ and $K_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, -\{C_4, \boldsymbol{\Upsilon}\}\rangle$ and $\mathcal{Q} \in \langle \mathcal{O}_1, \mathcal{E}_2\rangle$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (ii).

Case 10. Let $\boldsymbol{\Upsilon} \in \mathcal{P}$.
Then $P_3 \notin \mathcal{Q}$. Conversely, if $\mathcal{P} \in \langle \mathcal{E}_2, -\{C_4\}\rangle$ and $\mathcal{Q} = \mathcal{O}_1$, then $c(\mathcal{P} \oplus \mathcal{Q}) = 2$ and we have (i).

Because all possible $(\mathcal{P}, \mathcal{Q})$-decomposition were considered and taking into consideration fact that $K_4 \notin \mathcal{P} \oplus \mathcal{Q}$, the proof is complete. ■

Theorem 2. *$\mathcal{D}_2$ is indecomposable.*

Proof. It is easy to check that the graphs G_i in Figure 1, belongs to $\mathcal{D}_2$, for $i = 1, ..., 4$ and $G_1 \notin \mathcal{O}_1 \oplus -\{C_4\}$, $G_2 \notin \mathcal{E}_2 \oplus -\{\boldsymbol{\Upsilon}, C_4\}$, $G_3 \notin \mathcal{O}_2 \oplus -\{C_4\} \wedge \mathcal{S}_2$, $G_2 \notin \mathcal{W}_2 \oplus -\{C_3, C_4\} \wedge \mathcal{S}_2$ and $G_4 \notin \mathcal{SF} \oplus -\{C_3, C_4\}$.



G_1 G_2 G_3 G_4

Figure 1

Hence, it follows: $\mathcal{D}_2 \not\subseteq \mathcal{O}_1 \oplus -\{C_4\}$, $\mathcal{D}_2 \not\subseteq \mathcal{E}_2 \oplus -\{\boldsymbol{\Upsilon}, C_4\}$, $\mathcal{D}_2 \not\subseteq \mathcal{O}_2 \oplus -\{C_4\} \wedge \mathcal{S}_2$, $\mathcal{D}_2 \not\subseteq \mathcal{W}_2 \oplus -\{C_3, C_4\} \wedge \mathcal{S}_2$ and $\mathcal{D}_2 \not\subseteq \mathcal{SF} \oplus -\{C_3, C_4\}$ and by Theorem 1 $\mathcal{D}_2$ is indecomposable. ■

Theorem 3. *Every reducible property of completeness* 2 *is indecomposable.*

Proof. It is easy to see that the graph G in Figure 2, belongs to $\mathcal{O} \circ \mathcal{O}_1$ and $G \notin \mathcal{O}_1 \oplus -\{C_4\}$, $G \notin \mathcal{E}_2 \oplus -\{\boldsymbol{\Upsilon}, C_4\}$, $G \notin \mathcal{O}_2 \oplus -\{C_4\} \wedge \mathcal{S}_2$, $G \notin \mathcal{W}_2 \oplus -\{C_3, C_4\} \wedge \mathcal{S}_2$, and $G \notin \mathcal{SF} \oplus -\{C_3, C_4\}$.

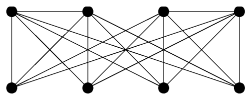


Figure 2

Hence, it follows: $\mathcal{O} \circ \mathcal{O}_1 \not\subseteq \mathcal{O}_1 \oplus -\{C_4\}$, $\mathcal{O} \circ \mathcal{O}_1 \not\subseteq \mathcal{E}_2 \oplus -\{\mathbf{Y}, C_4\}$, $\mathcal{O} \circ \mathcal{O}_1 \not\subseteq \mathcal{O}_2 \oplus - \{C_4\} \wedge \mathcal{S}_2$, $\mathcal{O} \circ \mathcal{O}_1 \not\subseteq \mathcal{W}_2 \oplus -\{C_3, C_4\} \wedge \mathcal{S}_2$ and $\mathcal{O} \circ \mathcal{O}_1 \not\subseteq \mathcal{SF} \oplus -\{C_3, C_4\}$. Thus, since $\mathcal{O} \circ \mathcal{O}_1$ is the smallest reducible property of completeness 2, any reducible property $\mathcal{R}$ of completeness 2 is indecomposable. ■

Now we can reformulate as examples some well-known results in Ramsey Theory using our notations.

Theorem 4 [10]. *$c(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_n) \leq \frac{(\sum_{i=1}^{n} c(\mathcal{P}_i)+n)!}{\prod_{i=1}^{n}(c(\mathcal{P}_i)+1)!} - 2$.*

Theorem 5 [7]. *$c(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \ldots \oplus \mathcal{I}_{k_n}) \leq c(\mathcal{I}_{k_1-1} \oplus \mathcal{I}_{k_2} \oplus \ldots \oplus \mathcal{I}_{k_n}) + c(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2-1} \oplus \ldots \oplus \mathcal{I}_{k_n}) + \ldots + c(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \ldots \oplus \mathcal{I}_{k_n-1}) + n$.*

Proposition 6. *$c(\mathcal{I}_1 \oplus \mathcal{I}_1) = 4$, $c(\mathcal{I}_1 \oplus \mathcal{I}_1 \oplus \mathcal{I}_1) = 15$.*

Theorem 7 [5].

$$c(\mathcal{S}_{k_1} \oplus \mathcal{S}_{k_2} \oplus \ldots \oplus \mathcal{S}_{k_n}) = \begin{cases} \sum_{i=1}^{n} k_i, & \textit{when} \quad \sum_{i=1}^{n} k_i \textit{ is odd or } \forall_{i=1}^{n} k_i \textit{ is even}, \\ \sum_{i=1}^{n} k_i - 1, & \textit{otherwise.} \end{cases}$$

We found an upper bound for $c(\mathcal{D}_p \oplus \mathcal{D}_q)$.

Theorem 8. *$c(\mathcal{D}_p \oplus \mathcal{D}_q) \leq p + q - 1 + \frac{1+\sqrt{1+8pq}}{2}$.*

Proof. For any graph $G \in \mathcal{D}_p \oplus \mathcal{D}_q$, if $K_n \subseteq G$ then $K_n \in \mathcal{D}_p \oplus \mathcal{D}_q$. Since the number of edges in a k-degenerate graph of order n is at most $kn - \binom{k+1}{2}$, then $\binom{n}{2} \leq pn - \binom{p+1}{2} + qn - \binom{q+1}{2}$. By an easy computation we have $n \leq p + q + \frac{1+\sqrt{1+8pq}}{2}$. ■

Corollary 9. *$c(k\mathcal{D}_p) \leq kp + \frac{-1+\sqrt{1+4p^2k(k-1)}}{2}$.*

Proof. For any graph $G \in k\mathcal{D}_p$, if $K_n \subseteq G$ then $K_n \in k\mathcal{D}_p$. Then $\binom{n}{2} \leq k\left(pn - \binom{p+1}{2}\right)$. It implies $n \leq kp + 1 + \frac{-1+\sqrt{1+4p^2k(k-1)}}{2}$. ■

But we are expecting that the following conjectures are true.

Conjecture 10. $c(\mathcal{D}_p \oplus \mathcal{D}_q) = p + q - 1 + \left\lfloor \frac{1+\sqrt{1+8pq}}{2} \right\rfloor$.

Conjecture 11. $c(k\mathcal{D}_p) = kp + \left\lfloor \frac{-1+\sqrt{1+4p^2k(k-1)}}{2} \right\rfloor$.

In the paper [3] the following upper bound is found

$$c(\mathcal{D}_{k_1} \oplus \mathcal{D}_{k_2} \oplus \ldots \oplus \mathcal{D}_{k_n}) \leq 2\sum_{i=1}^{n} k_i - 1.$$

In [9] has been proved that $\mathcal{P} \oplus \mathcal{Q}^k = (\mathcal{P} \oplus \mathcal{Q})^k$. From this we have the following equality.

Corollary 12. $c(\mathcal{O}^2 \oplus \mathcal{P}) = 2c(\mathcal{P}) + 1$.

Proposition 13. $c(k\mathcal{LF}) = c(k\mathcal{D}_1) = 2k - 1$.

Proof. Beineke [1] proved that a complete graph K_{2k} can be decomposed into k spanning paths. Hence $c(k\mathcal{LF}) \geq 2k - 1$. Because $|E(K_{2k+1})| > |E(G)|$, for any graph $G \in k\mathcal{D}_1$, then $c(k\mathcal{D}_1) \leq 2k - 1$. This establishes the formula $c(k\mathcal{LF}) = c(k\mathcal{D}_1) = 2k - 1$. ∎

Theorem 14. $c(2\mathcal{I}_1 \oplus \mathcal{P}) \geq 5c(\mathcal{P}) + 4$.

Theorem 15. *Let $\mathcal{P}, \mathcal{Q}$ be nontrivial additive hereditary properties. Then $c(\mathcal{P} \oplus \mathcal{Q}) = 1$ if and only if $\mathcal{P} = \mathcal{O}_1$ and $\mathcal{Q} = \mathcal{O}_1$.*

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