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NOTE ON CYCLIC DECOMPOSITIONS OF COMPLETE BIPARTITE GRAPHS INTO CUBES

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Abstract

So far, the smallest complete bipartite graph which was known to have a cyclic decomposition into cubes Q_d of a given dimension d was $K_{d2^{d-1},d2^{d-2}}$. We improve this result and show that also $K_{d2^{d-2},d2^{d-2}}$ allows a cyclic decomposition into Q_d. We also present a cyclic factorization of $K_{8,8}$ into Q_4.

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1. Introduction

The 1-dimensional cube Q_1 is the graph K_2 while the 2-dimensional cube Q_2 is isomorphic to the cycle C_4. In general, the *d-dimensional hypercube* Q_d is defined recursively as the product $Q_{d-1}\Box K_2$. Obviously, such a hypercube has 2^d vertices and $d2^{d-1}$ edges. Another nice definition of a hypercube Q_d (often called just a *cube*) can be stated as follows: Take all binary numbers of length d and assign them to the vertices $v_1, v_2, \ldots, v_{2^d}$. Then join two vertices by an edge if and only if their binary labels differ exactly at one position. We present in Figure 1 the bipartite adjacency matrices of the cubes Q_d for $d = 1, 2, 3, 4$ as we shall need them later.

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$$\begin{bmatrix} 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Figure 1

One can notice that the $2^{d-1} \times 2^{d-1}$ bipartite adjacency matrix $A(Q_d)$ of a cube Q_d can be easily recursively constructed from the $2^{d-2} \times 2^{d-2}$ matrix $A(Q_{d-1})$ of Q_{d-1} in such a way that we put to both left upper and right lower $2^{d-2} \times 2^{d-2}$ submatrix of $A(Q_d)$ a copy of $A(Q_{d-1})$. Then we fill the secondary diagonal with 1s and all other entries with 0s.

As the hypercubes are bipartite graphs, it is natural to ask a question which complete bipartite graphs can be decomposed or even factorized into hypercubes. The necessary condition for factorization of a complete bipartite graph $K_{n,m}$ into d-dimensional hypercubes is that the parts have to be both of the same order 2^{d-1} and d itself must be a power of 2. If it is not so, then the number of edges (or a *size*) of the hypercube does not divide the size of $K_{n,m}$. It was proved by El-Zanati and Vanden Eynden [1] that the necessary condition is also sufficient. In fact, they also proved that for other dimensions than powers of 2 the hypercubes Q_d can be packed into $K_{2^{d-1},2^{d-1}}$, the smallest complete bipartite graph that allows embedding of Q_d. Their result follows.

Theorem A (El-Zanati, Vanden Eynden [1]). *Let d be a positive integer with $t = 2^{d-1} = dq + r$, $0 \leq r < d$. Then $K_{t,t}$ can be decomposed into q cubes Q_d and an r-factor. If $r \neq 0$ this r-factor itself decomposes into 2^{d-r} cubes Q_r.*

However, in this note we are interested in cyclic decompositions and the decompositions used in the proof of Theorem A are not cyclic. Cyclic decompositions were studied by Vanden Eynden [3]. We shall follow the notation used in [3]. Let $K_{n,m}$ be a complete bipartite graph and G a bipartite graph such that $nm = q|E(G)|$. We denote edges of $K_{n,m}$ as (i,j), where $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$. We say that $K_{n,m}$

has an (r, s)-cyclic decomposition into G if we can assign labels to vertices of G such that for any edge (i, j) belonging to $G_0 \cong G$ all edges $(i + lr, j + ls), l = 1, 2, \ldots, q - 1$ belong to different copies $G_1, G_2, \ldots, G_{q-1}$ of G, where the set $\{G_0, G_1, \ldots, G_{q-1}\}$ forms a decomposition of $K_{n,m}$. Vanden Eynden generalized earlier results of Rosa [2] (concerning decompositions of complete graphs) to prove the following.

Theorem B (Vanden Eynden [3])**.** *Let G be a bipartite graph with parts V_1, V_2 and edge set E. Suppose that n and m are positive integers and r and s are integers such that $r|m$, $s|n$, and $|E| = \gcd(ms, nr)$. Let $t = \gcd(r, s)$, $R = r/t$, $S = s/t$, and $k = \gcd(Sm, Rn)$. Define $\psi : Z_m \times Z_n \to Z_k \times Z_t$ by $\psi(i, j) = (Si - Rj, \lfloor i/R \rfloor)$. Then there exists an (r, s)-cyclic decomposition of $K_{m,n}$ into copies of G if and only if there exist one-to-one functions N_1 and N_2 from V_1 and V_2 into Z_m and Z_n, respectively, such that the function $\theta : E \to Z_k \times Z_t$ defined by $\theta(v_1, v_2) = \psi(N_1(v_1), N_2(v_2))$ is one-to-one.*

It was proved by Vanden Eynden that for a given $d \geq 2$, the graph $K_{d2^{d-2},d2^{d-1}}$ can be cyclically decomposed into hypercubes Q_d. It is conjectured that if d is a power of 2, then the graph $K_{2^{d-1},2^{d-1}}$ can be cyclically factorized into copies of Q_d. In this note we improve Vanden Eynden's result and show that for a given $d \geq 2$ the graph $K_{d2^{d-2},d2^{d-2}}$ can be cyclically decomposed into hypercubes Q_d.

2. Decomposition of $K_{d2^{d-2},d2^{d-2}}$ Into Hypercubes Q_d

We start with a cyclic decomposition of the graph $K_{6,6}$ with partite sets $V_1 = \{v_1, v_2, \ldots, v_6\}$ and $V_2 = \{u_1, u_2, \ldots, u_6\}$. To find such a decomposition, we have to label the vertices of each partite set by labels from the set $\{0, 1, \ldots, 5\}$. Then we choose numbers r, s and define a cube Q_3 in such a way that for any given edge (i, j) of Q_3 neither $(i+r, j+s)$ nor $(i+2r, j+2s)$ belongs to the Q_3, where the labels are taken mod 4. Then the other two copies of Q_3, namely Q_3^1 and Q_3^2, are defined exactly by the sets of edges $\{(i + r, j + s)|(i, j) \in Q_3\}$ and $\{(i + 2r, j + 2s)|(i, j) \in Q_3\}$, respectively.

As $|E(Q_3)| = 12$, the divisibility condition is clearly satisfied. We set parameters r, s defined in Theorem B as $r = -2$ and $s = 2$. Then $t = \gcd(r, s) = 2$. Indeed it holds that $r|m, s|n$ and $12|\gcd(ms, nr)$. It follows that $R = r/t = -1, S = s/t = 1$ and $k = \gcd(Sm, Rn) = 6$. The function $\psi : Z_m \times Z_n \to Z_k \times Z_t$ from Theorem B appears to be $\psi(i, j) = (i + j, i)$. It only remains to find the functions N_1 and N_2 from V_1 and V_2 both into

Z_6 such that the function $\theta : E(Q_3) \to Z_6 \times Z_2$ defined by $\theta(v_a, u_b) = (N_1(v_a) + N_2(u_b), N_1(v_a))$ will be one-to-one. We define the cube Q_3 by the bipartite adjacency matrix presented in Figure 1 and label the vertices of each partite set of the cube Q_3 (that means, define the functions θ, N_1, N_2) with labels from the set $\{0, 1, 2, 3\}$. We assign vertices v_1, v_2, v_3, v_4 to the rows and u_1, u_2, u_3, u_4 to the columns. Now we define the functions N_1, N_2 as follows: $N_1(v_1) = 0, N_1(v_2) = 1, N_1(v_3) = 3, N_1(v_4) = 2$ and $N_2(u_j) = j - 1$ for $j = 1, 2, 3, 4$. The values of the function θ are presented in the "labeling array" shown in Figure 2. Notice that the asterisks correspond to zeros in the bipartite adjacency matrix of Q_3. A non-blank entry in a row v_a and a column u_b denotes the value of the first entry of the function $\theta(v_a, u_b) = (N_1(v_a) + N_2(u_b), N_1(v_a))$, that means, the sum $N_1(v_a) + N_2(u_b)$ taken mod 6 (because $k = 6$). The second entry, $N_1(v_a)$, is taken mod 2, as the parameter t equals 2.

	N_2	0	1	2	3
N_1					
0		0	1	$*$	3
1		1	2	3	$*$
3		$*$	4	5	0
2		2	$*$	4	5

Figure 2

One can check now that the function θ is really one-to-one: The entries in the rows v_1 and v_4 (recall that $N_1(v_1) \equiv N_1(v_4) \equiv 0 (\text{mod } 2)$) are exactly the elements of Z_6. The same holds for the entries of the rows v_2 and v_3 (here $N_1(v_2) \equiv N_1(v_3) \equiv 1 (\text{mod } 2)$).

For $d = 4$ we want to decompose $K_{d2^{d-2}, d2^{d-2}} = K_{16,16}$ into eight copies of Q_4. We use the recursive definition of Q_4. This means that we take two copies of Q_3 and join them by eight independent edges. Then we label the vertices of one copy of Q_3 as in the previous case. To label the vertices of the other copy of Q_3, we use the same "pattern". While the edges of the first copy have now labels $(0, 0), (1, 0), \ldots, (5, 0)$ and $(1, 1), (2, 1) \ldots, (6, 1)$ (notice that there is now an edge labeled $(6, 1)$ rather than $(0, 1)$ as the values of $N_1(v_a) + N_2(u_b)$ are not taken mod 6, but mod 16), we want the edges of the second copy to have labels $(10, 0), (11, 0), \ldots, (15, 0)$ and $(11, 1), (12, 1) \ldots, (15, 1), (0, 1)$. The remaining values, namely $(6, 0), (7, 0), (8, 0), (9, 0)$ and $(7, 1), (8, 1), (9, 1), (10, 1)$ are to be assigned to the edges joining the two copies of Q_3. Notice that the edges of the two copies of Q_3 appear in the left upper and right lower submatrix of the incidence matrix of Q_4 while the joining edges appear on the secondary diagonal. It means that we want to label the vertices $v_5, \dots v_8, u_5, \dots u_8$ such that $N_1(v_{a+4}) + N_2(u_{b+4}) = N_1(v_a) + N_2(u_b) + 10$ for $a, b = 1, 2, 3, 4$. Moreover, we have to guarantee the correct values of the joining edges. One can check that the labeling defined as $N_1(v_{a+4}) = N_1(v_a) + 4$ and $N_2(u_{b+4}) = N_2(u_b) + 6$ satisfies our requirements. The corresponding array is shown in Figure 3. Notice that not only the right lower subarray has now the same structure, but the same holds for the secondary diagonal: its left lower and right upper parts repeat the structure of the secondary diagonal of the array of Q_3.

N_2 / N_1	0	1	2	3	6	7	8	9
0	0	1	$*$	3	$*$	$*$	$*$	9
1	1	2	3	$*$	$*$	$*$	9	$*$
3	$*$	4	5	6	$*$	10	$*$	$*$
2	2	$*$	4	5	8	$*$	$*$	$*$
4	$*$	$*$	$*$	7	10	11	$*$	13
5	$*$	$*$	7	$*$	11	12	13	$*$
7	$*$	8	$*$	$*$	$*$	14	15	0
6	6	$*$	$*$	$*$	12	$*$	14	15

Figure 3

Here again the assumptions of Theorem B are satisfied: $m = n = 16, r = -2, s = 2$ and $t = \gcd(r, s) = 2$. Indeed it holds that $r|m, s|n$ and $|E(Q_4)| = 32|\gcd(ms, nr)$. It follows that $R = r/t = -1, S = s/t = 1$ and $k = \gcd(Sm, Rn) = 16$ and the function $\psi : Z_{16} \times Z_{16} \to Z_{16} \times Z_2$ appears to be $\psi(i, j) = (i + j, i)$. Hence the function $\theta : E(Q_4) \to Z_{16} \times Z_2$ is defined by $\theta(v_a, u_b) = (N_1(v_a) + N_2(u_b), N_1(v_a))$.

We now use the same idea to label recursively any cube Q_d using a labeling of Q_{d-1}. First we prove a lemma.

Lemma 1. *Let $d \geq 3$ and let $N_j^d : \{1, 2, \dots, 2^{d-1}\} \to Z_{d2^{d-2}}$ for $j = 1, 2$ be defined recursively as follows: $N_1^3(1) = 0, N_1^3(2) = 1, N_1^3(3) = 3, N_1^3(4) = 2$ and $N_2^3(i) = i - 1$ for $i = 1, 2, 3, 4$. Furthermore, for i with $1 \leq i \leq 2^{d-1}$, $N_1^{d+1}(i) = N_1^d(i)$ and $N_1^{d+1}(2^{d-1} + i) = N_1^d(i) + (d + 1)2^{d-3}$. Similarly, for*

$1 \le i \le 2^{d-1}$, $N_2^{d+1}(i) = N_2^d(i)$ and $N_2^{d+1}(2^{d-1}+i) = N_2^d(i) + (d+3)2^{d-3}$. Then N_1^d and N_2^d are one-to-one.

Proof. We denote the maximal value of $N_j^d(i), i \in \{1, 2, \ldots, 2^{d-1}\}$ by $h_j(d)$. First we prove that $h_1(d) < (d+1)2^{d-3}$. It is indeed true for $d = 3$. We suppose that $h_1(d-1) < d2^{d-4}$ for every $d \le d_0$, and want to show that then it follows that $h_1(d_0) < (d_0+1)2^{d_0-3}$. From the definition of N_1^d it is clear that $h_1(d_0) = h_1(d_0-1) + d_0 2^{d_0-4}$ and from our assumption it follows that $h_1(d_0-1) < d_0 2^{d_0-4}$. Therefore $h_1(d_0) < 2d_0 2^{d_0-4} = d_0 2^{d_0-3}$. On the other hand, $d_0 2^{d_0-3} < (d_0+1)2^{d_0-3}$ and hence $h_1(d_0) < (d_0+1)2^{d_0-3}$. By the same manner we show that $h_2(d) < (d+3)2^{d-3}$.

Now we can prove that N_1^d and N_2^d are one-to-one. It is obviously true for N_1^3 and N_2^3. Then we suppose that N_1^d is one-to-one for any $d \le d_0$ and want to show that then it holds that $N_1^{d_0+1}$ is one-to-one. If $1 \le i < j \le 2^{d_0-1}$, then $N_1^{d_0+1}(i) = N_1^{d_0}(i) \ne N_1^{d_0}(j) = N_1^{d_0+1}(j)$. Similarly, if $2^{d_0-1}+1 \le i < j \le 2^{d_0}$, then $N_1^{d_0+1}(i) = N_1^{d_0}(i) + (d_0+1)2^{d_0-3} \ne N_1^{d_0}(j) + (d_0+1)2^{d_0-3} = N_1^{d_0+1}(j)$. It remains to show that $N_1^{d_0+1}(i) \ne N_1^{d_0+1}(j)$ even when $1 \le i \le 2^{d_0-1} < j \le 2^{d_0}$. But for j with $2^{d_0-1} < j \le 2^{d_0}$ it holds that $N_1^{d_0+1}(j) \ge (d_0+1)2^{d_0-3}$. On the other hand, for i with $1 \le i \le 2^{d_0-1}$ it holds that $N_1^{d_0+1}(i) = N_1^{d_0}(i) \le h_1(d_0) < (d_0+1)2^{d_0-3}$ and therefore the inequality above holds as well. For N_2^d the considerations are essentially similar and therefore can be left to the reader.

To complete the proof, we have to show that $h_1(d)$ and $h_2(d)$ do not exceed $d2^{d-2} - 1$. To do this, we observe that for $d \ge 3$ it holds that $d+3 \le 2d$ and therefore $(d+3)2^{d-3} \le 2d2^{d-3} = d2^{d-2}$. Because for $i = 1, 2$ we have $h_i(d) \le (d+3)2^{d-3} - 1$, it obviously holds that $h_i(d) \le d2^{d-2} - 1$. ■

Theorem 2. *For a given $d \ge 2$, the complete bipartite graph $K_{d2^{d-2},d2^{d-2}}$ is (r,s)-cyclically decomposable into hypercubes Q_d. In particular, such a decomposition always exists for $r = -2, s = 2$.*

Proof. We suppose that $d > 2$, as the case $d = 2$ is trivial. Let $r = -2$ and $s = 2$. Then $r|m$ and $s|n$, as $m = n = d2^{d-2}$ and $t = \gcd(r,s) = 2$. It holds that $|E(Q_d)| = d2^{d-1} = \gcd(ms, nr)$. It follows that $R = r/t = -1, S = s/t = 1$ and $k = \gcd(Sm, Rn) = d2^{d-2}$. The function $\psi : Z_{d2^{d-2}} \times Z_{d2^{d-2}} \to Z_{d2^{d-2}} \times Z_2$ from Theorem B is then $\psi(i,j) = (i+j, i)$.

We define the functions N_1^d and N_2^d from V_1^d and V_2^d both into $Z_{d2^{d-2}}$ recursively, similarly as in our example for $d = 4$. Let $N_1^3(v_1) = 0, N_1^3(v_2) = 1, N_1^3(v_3) = 3, N_1^3(v_4) = 2$ and $N_2^3(u_j) = j - 1$ for $j = 1, 2, 3, 4$. Now suppose that we have labeled the vertices $v_1, v_2, \ldots, v_{2^{d-1}}$ and $u_1, u_2, \ldots, u_{2^{d-1}}$ of a cube Q_d using functions N_1^d and N_2^d such that the function $\theta^d : E(Q_d) \to Z_{d2^{d-2}} \times Z_2$ defined by $\theta^d(v_i, u_j) = (N_1^d(v_i) + N_2^d(u_j), N_1^d(v_i))$ is one-to-one. We moreover suppose that the secondary diagonal of the array defining the labeling of the cube Q_d consists of 2^{d-1} entries corresponding to the edges joining in Q_d the vertices of two copies of the cube Q_{d-1}. These edges are labeled $((d-1)2^{d-3}, 0), ((d-1)2^{d-3}+1, 0), \ldots, ((d-1)2^{d-3}+2^{d-2}-1), 0)$ and $((d-1)2^{d-3}+1, 1), ((d-1)2^{d-3}+2, 1), \ldots, ((d-1)2^{d-3}+2^{d-2}, 1)$.

We now define the functions N_1^{d+1}, N_2^{d+1} and θ^{d+1} as follows. Similarly as before, $\theta^{d+1}(v_i, u_j) = (N_1^{d+1}(v_i) + N_2^{d+1}(u_j), N_1^{d+1}(v_i))$. For $v_1, v_2, \ldots, v_{2^{d-1}}$ and $u_1, u_2, \ldots, u_{2^{d-1}}$ we define $N_1^{d+1}(v_i) = N_1^d(v_i)$ and $N_2^{d+1}(u_j) = N_2^d(u_j)$. Recall that the function θ^{d+1} is taking $E(Q_{d+1})$ into $Z_{(d+1)2^{d-1}} \times Z_2$ while θ^d was taking $E(Q_d)$ into $Z_{d2^{d-2}} \times Z_2$. Hence the entries of the labeling array of Q^{d+1} are taken mod $(d+1)2^{d-1}$ and the entry in the row $v_{2^{d-1}-1}$ and the column $u_{2^{d-1}}$ is now $d2^{d-2}$ rather than 0. Notice that $d2^{d-2}+(d+2)2^{d-2} = (d+1)2^{d-1}$. In all other cases $\theta^{d+1}(v_i, u_j) = \theta^d(v_i, u_j)$. For $v_{2^{d-1}+i}$ and $u_{2^{d-1}+j}$ we define $N_1^{d+1}(v_{2^{d-1}+i}) = N_1^d(v_i) + (d+1)2^{d-3}$ and $N_2^{d+1}(u_{2^{d-1}+j}) = N_2^d(u_j) + (d+3)2^{d-3}$ for each $i, j = 1, 2, \ldots, 2^{d-1}$. It follows from Lemma 1 that both N_1^{d+1} and N_2^{d+2} are one-to-one. Then the right lower subarray of the labeling array of Q^{d+1} (corresponding to one copy of Q_d contained in Q_{d+1}) repeats the structure of the left upper subarray (corresponding to the other copy of Q_d): For every pair $(v_{2^{d-1}+i}, u_{2^{d-1}+j})$ inducing an edge of Q_{d+1} we get

$$\begin{aligned}
&\theta^{d+1}(v_{2^{d-1}+i}, u_{2^{d-1}+j}) \\
&= (N_1^{d+1}(v_{2^{d-1}+i}) + N_2^{d+1}(u_{2^{d-1}+j}), N_1^{d+1}(v_{2^{d-1}+i})) \\
&= (N_1^d(v_i) + (d+1)2^{d-3} + N_2^d(u_j) + (d+3)2^{d-3}, N_1^d(v_i) + (d+1)2^{d-3}) \\
&= (N_1^d(v_i) + N_2^d(u_j) + (d+2)2^{d-2}, N_1^d(v_i)).
\end{aligned}$$

Thus the left upper subarray contains the edges labeled $(0,0), (1,0), \ldots, (d2^{d-2}-1, 0)$ and $(1,1), (2,1) \ldots, (d2^{d-2}, 1)$ while the right lower subarray contains the edges labeled $((d+2)2^{d-2}, 0), ((d+2)2^{d-2}+1, 0), \ldots, ((d+1)2^{d-1}-1, 0)$ and $((d+2)2^{d-2}+1, 1), ((d+2)2^{d-2}+2, 1), \ldots, ((d+1)2^{d-1}-1, 1), (0, 1)$ (notice that $(d+2)2^{d-2}+d2^{d-2}-1 = (d+1)2^{d-1}-1$). The remaining labels are assigned to the edges joining the copies of Q_d and appear on the secondary diagonal. For every pair $(v_{2^{d-1}+i}, u_i)$, $i =$

$1, 2, \ldots, 2^{d-1}$ inducing an edge appearing in the left lower part of the diagonal we have

$$\begin{aligned}\theta^{d+1}(v_{2^{d-1}+i}, u_i) &= (N_1^{d+1}(v_{2^{d-1}+i}) + N_2^{d+1}(u_i), N_1^{d+1}(v_{2^{d-1}+i})) \\ &= (N_1^d(v_i) + (d+1)2^{d-3} + N_2^d(u_i), N_1^d(v_i) + (d+1)2^{d-3}) \\ &= (N_1^d(v_i) + N_2^d(u_i) + (d+1)2^{d-3}, N_1^d(v_i)).\end{aligned}$$

Therefore this part of the diagonal repeats the structure of the secondary diagonal of the left upper subarray (and hence the structure of the secondary diagonal of Q_d) and contains the labels $((d-1)2^{d-3}+(d+1)2^{d-3}, 0)$, $((d-1)2^{d-3}+(d+1)2^{d-3}+1, 0), \ldots, ((d-1)2^{d-3}+(d+1)2^{d-3}+2^{d-2}-1), 0)$ or, more conveniently, $(d2^{d-2}, 0), (d2^{d-2}+1, 0), \ldots, ((d+1)2^{d-2}-1, 0)$, and $(d2^{d-2}+1, 1), (d2^{d-2}+2, 1), \ldots, ((d+1)2^{d-2}, 1)$. Similarly, the right upper part of the secondary diagonal contains the labels $((d+1)2^{d-2}, 0)$, $((d+1)2^{d-2}+1, 0), \ldots, ((d+2)2^{d-2}-1, 0)$ and $((d+1)2^{d-2}+1, 1)$, $((d+1)2^{d-2}+2, 1), \ldots, ((d+2)2^{d-2}, 1)$. This is so because for every pair $(v_i, u_{2^{d-1}+i})$, $i = 1, 2, \ldots, 2^{d-1}$ inducing an edge appearing in the right upper part of the secondary diagonal we have

$$\begin{aligned}\theta^{d+1}(v_i, u_{2^{d-1}+i}) &= (N_1^{d+1}(v_i) + N_2^{d+1}(u_{2^{d-1}+i}), N_1^{d+1}(v_i)) \\ &= (N_1^d(v_i) + N_2^d(u_i) + (d+3)2^{d-3}, N_1^d(v_i)).\end{aligned}$$

Hence we have checked that the function $\theta^{d+1} : E(Q_{d+1}) \to Z_{(d+1)2^{d-1}} \times Z_2$ is one-to-one and the proof is complete. ■

Although we are not able to cyclically factorize the graphs $K_{2^{d-1},2^{d-1}}$ into cubes Q_d for $d = 2^c > 4$, we present an example of such factorization for the smallest non-trivial case with $c = 2$. Thus we factorize $K_{8,8}$ into cubes Q_4 as follows: We set $r = -4, s = 8$. This yields $t = 4, R = -1, S = 2$ and $k = 8$.

The function $\psi : Z_8 \times Z_8 \to Z_8 \times Z_4$ is then $\psi(i, j) = (2i + j, -i)$. We define the functions N_1^d and N_2^d from $V_1 = \{v_1, v_2, \ldots, v_8\}$ and $V_2 = \{u_1, u_2, \ldots, u_8\}$ both into Z_8 as $N_1(v_a) = a - 1$ for $a = 1, 2, 3, 4, N_1(v_5) = 5$, $N_1(v_6) = 4, N_1(v_7) = 7, N_1(v_8) = 6$ and $N_1(u_b) = b - 1$ for $b = 1, 2, \ldots, 8$. The function θ defined in Theorem B is one-to-one, as can be observed from the labeling array shown in Figure 4.

N_1 \ N_2	0	1	2	3	4	5	6	7
0	0	1	$*$	3	$*$	$*$	$*$	7
1	2	3	4	$*$	$*$	$*$	0	$*$
2	$*$	5	6	7	$*$	1	$*$	$*$
3	6	$*$	0	1	2	$*$	$*$	$*$
5	$*$	$*$	$*$	5	6	7	$*$	1
4	$*$	$*$	2	$*$	4	5	6	$*$
7	$*$	7	$*$	$*$	$*$	3	4	5
6	4	$*$	$*$	$*$	0	$*$	2	3

Figure 4

References

[1] S. El-Zanati and C. Vanden Eynden, *Decompositions of $K_{m,n}$ into cubes*, J. Comb. Designs **4** (1) (1996) 51–57.

[2] A. Rosa, *On certain valuations of the vertices of a graph*, Internat. Sympos. ICC Rome, Dunod, Paris, 1967, 349–355.

[3] C. Vanden Eynden, *Decompositions of complete bipartite graphs*, Ars Combinatoria, to appear.

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