GENERALIZED RAMSEY THEORY AND DECOMPOSABLE PROPERTIES OF GRAPHS

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Abstract

In this paper we translate Ramsey-type problems into the language of decomposable hereditary properties of graphs. We prove a distributive law for reducible and decomposable properties of graphs. Using it we establish some values of graph theoretical invariants of decomposable properties and show their correspondence to generalized Ramsey numbers.

Keywords: hereditary properties, additivity, reducibility, decomposability, Ramsey number, graph invariants.

1991 Mathematics Subject Classification: 05C15, 05C55, 05C75.

1   INTRODUCTION AND MOTIVATION

All graphs considered in this paper are finite and simple (without multiple edges or loops), and we use standard notation of [6] and [7]. In particular, $G \cup H$ denotes the disjoint union of graphs G and H. The symbol $\subseteq$ stands for the relation "to be a subgraph". For the sake of brevity we shall say "a graph G contains the graph H" instead of "a graph G contains a subgraph isomorphic to the graph H". If $G = (V(G), E(G))$ is a graph and $V^* \subseteq V(G)$, then by $G[V^*]$ we denote the subgraph induced by the vertex set V^*, i.e. the graph with vertex set V^* and edge set $E^* = \{uv \in E(G) : u, v \in V^*\}$. Analogously, for an edge set $E^{**} \subseteq E(G)$ we define the subgraph $G[E^{**}]$ of G generated by E^{**} as the graph with vertex set $V(G)$ and edge set E^{**}. A *homomorphism* of a graph G to a graph H is a mapping f of the vertex set $V(G)$ into $V(H)$ which preserves the edges i.e., such that $e = \{u, v\} \in E(G)$ implies $f(e) = \{f(u), f(v)\} \in E(H)$. If a homomorphism of G to H exists, we say that G is *homomorphic* to H and write $G \to H$.

Let us denote the class of all finite simple graphs by $\mathcal{I}$. A *graph property* is a non-empty proper isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph G has property $\mathcal{P}$ whenever $G \in \mathcal{P}$. The complementary set $\overline{\mathcal{P}} = \mathcal{I} \backslash \mathcal{P}$ of a property $\mathcal{P}$ will be called a *co-property*.

A property $\mathcal{P}$ of graphs is called *hereditary* if it is closed with respect to the relation $\subseteq$ to be a subgraph, i.e., if $H \subseteq G$ and $G \in \mathcal{P}$ then $H \in \mathcal{P}$. (An overview of hereditary properties can be found in [1]). It is not difficult to see that co-properties of hereditary properties of graphs are closed under taking supergraphs and we shall call them *co-hereditary*. A property $\mathcal{P}$ is called *additive* if it is closed under the disjoint union of graphs, i.e., if every graph has a property $\mathcal{P}$ provided all of its connected components have this property. The set of all additive hereditary properties of graphs forms a complete distributive lattice (for details see [1]).

We list some important additive hereditary properties.

$\mathcal{O} = \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\}$,

$\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k + 1 \text{ vertices}\}$,

$\mathcal{S}_k = \{G \in \mathcal{I} : \text{the maximum degree } \Delta(G) \leq k\}$,

$\mathcal{D}_k = \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e., the minimum degree } \delta(H) \leq k \text{ for } \\ \text{each } H \subseteq G\}$,

$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}$,

$\mathcal{O}^k = \{G \in \mathcal{I} : G \text{ is } k\text{-colorable}\}$,

$\to H = \{G \in \mathcal{I} : G \text{ is homomorphic to the graph } H\}$.

Let $\mathcal{P}$ be a hereditary property, $\mathcal{P} \neq \mathcal{I}$. Then there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$ – it is called the *completeness* of $\mathcal{P}$. Obviously

$$c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{D}_k) = c(\mathcal{I}_k) = k$$

and for additive properties $c(\mathcal{P}) = 0$ if and only if $\mathcal{P} = \mathcal{O}$.

It is rather easy to see that any hereditary property $\mathcal{P}$ and its associated co-hereditary property $\overline{\mathcal{P}}$ are uniquely determined by the set of *minimal forbidden subgraphs* of the property $\mathcal{P}$ defined in the following way:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} \setminus \mathcal{P} : \text{ each proper subgraph of } G \text{ belongs to } \mathcal{P}\}.$$

Note that $\boldsymbol{F}(\mathcal{P})$ may be finite or infinite. For an arbitrary graph theoretical invariant ρ and hereditary property $\mathcal{P}$ we can establish the following invariant (see also [13]):

$$\rho(\mathcal{P}) = \min\{\rho(F) : F \in \boldsymbol{F}(\mathcal{P})\}.$$

For the chromatic number χ the invariant $\psi(\mathcal{P}) = \chi(\mathcal{P}) - 1$ is known as *subchromatic number* (see e.g. [14]) or *index* of property (cf. [4]). It is easy to see that for a nonempty hereditary property $\mathcal{P}$ the invariant $\chi(\mathcal{P})$ has value at least two. The properties with $\chi(\mathcal{P}) = 2$ are called *degenerate*, the properties with $\chi(\mathcal{P})$ greater than two are called *non-degenerate*. A graph theoretical invariant ρ is called *monotone* whenever for any pair G_1, G_2 of graphs satisfying $G_1 \subseteq G_2$ holds $\rho(G_1) \leq \rho(G_2)$.

The properties $\mathcal{I}_k$, $\mathcal{O}^k$, $\mathcal{D}_k$, $\mathcal{S}_k$ and $\mathcal{O}_k$ mentioned above can be uniquely determined by the graph theoretical invariants $\omega(G)$ – the clique number, $\chi(G)$ – the chromatic number, $\mathrm{col}(G)$ – the coloring number (see [7]), $\Delta(G)$ – maximum degree and $o(G)$ – the order of largest component of G. It is known that for any graph G the following holds:

$$\omega(G) \leq \chi(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1 \leq o(G).$$

Moreover, it is not so difficult to see that some other well-known invariants (like choice number, $\mathcal{P}$-choice number, $\mathcal{P}$-chromatic – see e.g. [1]) can be included into similar chains.

Proposition 11. *Let $\mathcal{P}$ be a hereditary property of graphs. Let ρ_1, ρ_2 be monotone graph theoretical invariants which for any graph G satisfy the inequality $\rho_1(G) \leq \rho_2(G)$. Then $\rho_1(\mathcal{P}) \leq \rho_2(\mathcal{P})$.*

Proof. Let $F \in \boldsymbol{F}(\mathcal{P})$ be a graph with $\rho_2(F) = \rho_2(\mathcal{P})$. Then we immediately have

$$\rho_1(\mathcal{P}) \leq \rho_1(F) \leq \rho_2(F) = \rho_2(\mathcal{P}). \qquad \blacksquare$$

Proposition 12. *Let $\mathcal{P}_1, \mathcal{P}_2$ be hereditary properties of graphs such that $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Let ρ be a monotone graph theoretical invariant. Then $\rho(\mathcal{P}_1) \leq \rho(\mathcal{P}_2)$.*

Proof. Let $F \in \boldsymbol{F}(\mathcal{P}_2)$ be a graph satisfying $\rho(F) = \rho(\mathcal{P}_2)$. Since $F \notin \mathcal{P}_2$, we have that $F \notin \mathcal{P}_1$. Thus there exists a graph $F^* \in \boldsymbol{F}(\mathcal{P}_1)$ such that $F^* \subseteq F$ and we obtain the inequalities

$$\rho(\mathcal{P}_1) \leq \rho(F^*) \leq \rho(F) = \rho(\mathcal{P}_2). \qquad \blacksquare$$

A generalization of vertex coloring concepts could be done in the following way: Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A *vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition* of a graph $G \in \mathcal{I}$ is a partition $V_1, V_2, \ldots, V_n$ of its vertex set $V(G)$ such that for each $i = \{1, 2, \ldots, n\}$ the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$. A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition*. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$ we simply write $\mathcal{P}^n$ instead of $\mathcal{P} \circ \mathcal{P} \circ \cdots \circ \mathcal{P}$. A property $\mathcal{R}$ is called *reducible* whenever there exist properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$. In this notation $\mathcal{O}^k$ and $\overline{O^k}$ are the sets of k-colorable graphs and graphs with chromatic number greater than k, respectively. A *generalized chromatic number* $\chi_{\mathcal{P}}(G)$ is defined as the smallest positive integer n such that $G \in \mathcal{P}^n$. Note that for the property $\mathcal{O}$ we obtain the usual vertex chromatic number χ.

The following two results determine the values of the invariants $\chi(\mathcal{R})$ and $c(\mathcal{R})$ for any reducible hereditary property $\mathcal{R}$.

Theorem 13 [13], [1]**.** *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be hereditary properties of graphs. Then*

$$\chi(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n) = \chi(\mathcal{P}_1) + \chi(\mathcal{P}_2) + \cdots + \chi(\mathcal{P}_n) - (n - 1).$$

Theorem 14 [1]**.** *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be hereditary properties of graphs. Then*

$$c(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n) = c(\mathcal{P}_1) + c(\mathcal{P}_2) + \cdots + c(\mathcal{P}_n) + (n - 1).$$

For two properties $\mathcal{P}$ and $\mathcal{Q}$ we define the invariant

$$\Phi(\mathcal{P}, \mathcal{Q}) = \max\{\chi_{\mathcal{P}}(G) : G \in \mathcal{Q}\},$$

if such a values exists (see also [4], [5]). If it does not exist, we say $\Phi(\mathcal{P}, \mathcal{Q}) = \infty$. The function Φ is significant because of the following theorem.

Theorem 15 [5]. *Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. Then $\chi_{\mathcal{P}}(G) \leq \Phi(\mathcal{P}, \mathcal{Q}).\chi_{\mathcal{Q}}(G)$ for all graphs G. Moreover, if Q is in fact degenerate, then for every integer $k \geq 1$, the equality holds for some G satisfying $\chi_{\mathcal{Q}}(G) = k$.*

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be hereditary properties. An edge $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*decomposition* of a graph $G \in \mathcal{I}$ is the decomposition $E_1, E_2, \ldots, E_n$ of its edge set $E(G)$ satisfying that for each $i = \{1, 2, \ldots, n\}$ the induced subgraph $G[E_i]$ has property $\mathcal{P}_i$. A property $\mathcal{Q} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ is defined as the set of all graphs having a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*decomposition*. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots \mathcal{P}_n = \mathcal{P}$ we simply write $n \times \mathcal{P}$ instead of $\mathcal{P} \oplus \mathcal{P} \oplus \cdots \oplus \mathcal{P}$. Note that the property $\mathcal{O}$ is the neutral element for the operation $\oplus$. More precisely, if $\mathcal{P}$ is a hereditary property then $\mathcal{P} \oplus \mathcal{O} = \mathcal{P}$. A *generalized edge-chromatic number* $\chi'_{\mathcal{P}}(G)$ is defined as the smallest positive integer n such that $G \in n \mathrm{x} \mathcal{P}$. A property $\mathcal{Q}$ is called *decomposable* whenever there exist properties $\mathcal{Q}_1, \mathcal{Q}_2 \neq \mathcal{O}$ such that $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$. We remark that many problems from Ramsey Theory can be generalized in terms of decomposable properties of graphs. For example the well-known Ramsey number $r(k, l)$ is equal to the completeness of the property $\mathcal{I}_{k-2} \oplus \mathcal{I}_{l-2}$ plus two.

Proposition 16 [1]. *Let n be a positive integer. If $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are hereditary properties of graphs then the properties $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ and $\mathcal{Q} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ are also hereditary. Moreover, if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are also additive then $\mathcal{R}$ and $\mathcal{Q}$ are additive too.*

In connection with the study of Ramsey numbers and Ramsey arrows it is useful to introduce new symbols. If $G, H_1, H_2, \ldots, H_n$ are arbitrary graphs from $\mathcal{I}$ then the symbol

$$G \rightarrow (H_1, H_2, \ldots, H_n)$$

means that for any decomposition $E_1, E_2, \ldots, E_n$ of the edge set $E(G)$ of G there exists an i such that the induced subgraph $G[E_i]$ contains the graph H_i.

Let G be a graph and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. The notation

$$G \to (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$$

is used for the fact that for every coloring $E(G) = E_1 \cup E_2 \cup \ldots \cup E_n$ of the edges of G there exists a color i and a graph $H_i \in \mathcal{P}_i$ such that $H_i \subseteq G[E_i]$.

Motivated by [3], for a monotone graph theoretical invariant ρ, let us consider the next two parameters:

$$r_\rho(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) = \min\{\rho(G) : G \in \mathcal{I}, G \to (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)\},$$

$$R_\rho(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) = \min\{R : G \to (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) \text{ whenever } \rho(G) = R\}.$$

For an arbitrary invariant ρ and for any properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ the definitions immediately lead to the following inequality

$$r_\rho(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) \le R_\rho(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n).$$

The first result concerning the parameters r_χ and R_χ we can state in our language in the following way:

Theorem 17 [3]. *If $k_1, k_2, \ldots, k_n$ are positive integers, then*

$$r_\chi(\overline{\mathcal{O}^{k_1}}, \overline{\mathcal{O}^{k_2}}, \ldots, \overline{\mathcal{O}^{k_n}}) = R_\chi(\overline{\mathcal{O}^{k_1}}, \overline{\mathcal{O}^{k_2}}, \ldots, \overline{\mathcal{O}^{k_n}}) = \prod_{i=1}^{n} k_i + 1.$$

2   Distributive Law for the Operations $\circ$ and $\oplus$

The next lemma provides a very important identity – a kind of distributive law.

Lemma 21. *Let $\mathcal{P}, \mathcal{Q}_1$ and $\mathcal{Q}_2$ be additive hereditary properties of graphs. Then*

$$\mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2) = (\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2).$$

Proof. Suppose G belongs to $\mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2)$. Then there exists a decomposition (E_1, E_2) of its edge set $E(G)$ such that $G[E_1] \in \mathcal{P}$ and $G[E_2] \in \mathcal{Q}_1 \circ \mathcal{Q}_2$. Since $G[E_2]$ has the property $\mathcal{Q}_1 \circ \mathcal{Q}_2$, there exists a vertex partition (V_1, V_2) of $V(G)$ such that $G[E_2][V_1] \in \mathcal{Q}_1$ and $G[E_2][V_2] \in \mathcal{Q}_2$.

Now, it suffices to verify that $G[V_1] \in \mathcal{P} \oplus \mathcal{Q}_1$ and $G[V_2] \in \mathcal{P} \oplus \mathcal{Q}_2$. So, let us put $E_1^* = \{uv \in E(G) : u, v \in V_1\}$ and $E_2^* = \{uv \in E(G) : u, v \in V_2\}$. It is easy to check that $(E_1^* \cap E_1, E_1^* \cap E_2)$, $(E_2^* \cap E_1, E_2^* \cap E_2)$ are decompositions of the edge sets E_1^* and E_2^* respectively and we immediately have $G[V_1][E_1^* \cap E_1] \subseteq G[E_1] \in \mathcal{P}$, $G[V_1][E_1^* \cap E_2] = G[E_2][V_1] \in \mathcal{Q}_1$, $G[V_2][E_2^* \cap E_1] \subseteq G[E_1] \in \mathcal{P}$ and $G[V_2][E_2^* \cap E_2] = G[E_2][V_2] \in \mathcal{Q}_2$. Therefore $\mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2) \subseteq (\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2)$.

To prove the opposite inclusion suppose that $G \in (\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2)$. It implies that there is a partition (V_1, V_2) of its vertex set which satisfies $G[V_1] \in \mathcal{P} \oplus \mathcal{Q}_1$ and $G[V_2] \in \mathcal{P} \oplus \mathcal{Q}_2$.

Since $G[V_1]$ belongs to $\mathcal{P} \oplus \mathcal{Q}_1$, there exists a decomposition (E_{11}, E_{12}) of $E(G[V_1])$ such that $G[V_1][E_{11}] \in \mathcal{P}$ and $G[V_1][E_{12}] \in \mathcal{Q}_1$. Due to similar reasons there is a decomposition (E_{21}, E_{22}) of $E(G[V_2])$ such that $G[V_2][E_{21}] \in \mathcal{P}$ and $G[V_2][E_{22}] \in \mathcal{Q}_2$. Let us put $E_1' = E_{11} \cup E_{21}$ and $E_2' = E_{12} \cup E_{22} \cup F$ where F is the set of edges $\{uv \in E(G) : u \in V_1, v \in V_2\}$. Then $G[E_1'] = G[E_{11} \cup E_{21}]$. But this is a subgraph of the disjoint union of the graphs $G[V_1][E_{11}]$, $G[V_2][E_{21}]$ and some extra isolated vertices. Because of additivity of $\mathcal{P}$ we have that $G[E_1'] \in \mathcal{P}$. In addition, $G[E_2'] = G[E_{12} \cup E_{22}]$, $G[E_2'][V_1] = G[V_1][E_{12}] \in \mathcal{Q}_1$ and $G[E_2'][V_2] = G[V_2][E_{22}] \in \mathcal{Q}_1 \circ \mathcal{Q}_2$. Thus, we have $(\mathcal{P} \oplus \mathcal{Q}_1) \circ (\mathcal{P} \oplus \mathcal{Q}_2) \subseteq \mathcal{P} \oplus (\mathcal{Q}_1 \circ \mathcal{Q}_2)$. ∎

Remark 22. It is quite easy to see that if we interchange the operations $\circ$ and $\oplus$ in the statement of Lemma 2.1, then the identity so obtained is not true.

Corollary 23. *Let* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *be additive hereditary properties. If at least one of them is reducible then the property* $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ *is reducible too.*

By a repeated application of Lemma 21 we can obtain the following interesting and useful identities.

Corollary 24. *Let* $k_1, k_2, \ldots, k_n$ *be positive integers and let* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *be additive hereditary properties of graphs. Then*

$$\mathcal{P}_1^{k_1} \oplus \mathcal{P}_2^{k_2} \oplus \cdots \oplus \mathcal{P}_n^{k_n} = (\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n)^{o_{i=1}^n k_i}.$$

Corollary 25. *Let* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *be additive hereditary properties. Then for any graph G the following three statements are equivalent:*
(1) $G \in \mathcal{P}_1^{k_1} \oplus \mathcal{P}_2^{k_2} \oplus \cdots \oplus \mathcal{P}_n^{k_n}$;

(2) $G \in \mathcal{O}^{k_1.k_2....k_n} \oplus (\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n)$;

(3) $\chi_{\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n}(G) \leq \prod_{i=1}^n k_i$.

Proof. The equivalence of (1) and (2) can be proved by double induction with respect to the number of hereditary properties and their powers. The equivalence of statements (1) and (3) follows from the definition of generalized chromatic number and Corollary 24. ∎

3 Estimations of Some Invariants of Hereditary Properties

In this section we shall study the invariants of additive hereditary properties and establish some exact values and some bounds of them.

We start with the graph theoretical invariant $o(G)$ – the order of the largest component of G. Of course, its values is also the order of the largest tree which is contained in G.

Lemma 31. *Let $\mathcal{P}$ be an additive hereditary property of graphs. Then*

$$o(\mathcal{P}) = c(\mathcal{P}) + 2.$$

Proof. From the definition of the completeness it follows that $K_{c(\mathcal{P})+1}$ belongs to $\mathcal{P}$ but $K_{c(\mathcal{P})+2}$ does not. It means that every tree on at most $c(\mathcal{P}) + 1$ vertices has property $\mathcal{P}$ and $o(\mathcal{P}) \geq c(\mathcal{P}) + 2$. On the other hand, some connected graph on $c(\mathcal{P}) + 2$ vertices is in F($\mathcal{P}$) and therefore $o(\mathcal{P}) \leq c(\mathcal{P}) + 2$. ∎

A straightforward application of Proposition 11 provides us the following result.

Corollary 32. *Let $\mathcal{P}$ be an additive hereditary property. Let ρ be a monotone invariant which satisfies $\rho(G) \leq o(G)$ for every graph G. Then*

$$\rho(\mathcal{P}) \leq c(\mathcal{P}) + 2.$$

The invariant $\chi(\mathcal{P})$ is very important for the study of hereditary properties of graphs because it determines the asymptotic behaviour of the maximum number of edges of the graphs belonging to $\mathcal{P}$ (see e.g. [14]). The calculation of the chromatic number of reducible hereditary properties is determined by Theorem 13. The next lemma provides lower and upper bounds for the chromatic number of decomposable properties.

Lemma 33. *Let* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *be additive hereditary properties of graphs. Then*

$$\prod_{i=1}^{n}(\chi(\mathcal{P}_i) - 1) + 1 \leq \chi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n) \leq c(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n) + 2.$$

Proof. The inequality $\chi(\mathcal{P}) \geq k$ is equivalent to the statement that every $(k-1)$-colorable graph has the property $\mathcal{P}$. In the language of hereditary properties this can be expressed as the inclusion $\mathcal{O}^{k-1} \subseteq \mathcal{P}$. Therefore we have

$$\mathcal{O}^{\chi(\mathcal{P}_1)-1} \oplus \mathcal{O}^{\chi(\mathcal{P}_2)-1} \oplus \cdots \oplus \mathcal{O}^{\chi(\mathcal{P}_n)-1} \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n.$$

On the other hand, from Corollary 24 it follows that

$$\mathcal{O}^{\chi(\mathcal{P}_1)-1} \oplus \mathcal{O}^{\chi(\mathcal{P}_2)-1} \oplus \cdots \oplus \mathcal{O}^{\chi(\mathcal{P}_n)-1} = \mathcal{O}^{\Pi_{i=1}^{n}(\chi(\mathcal{P}_i)-1)}.$$

Hence

$$\mathcal{O}^{\Pi_{i=1}^{n}(\chi(\mathcal{P}_i)-1)} \subseteq \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$$

and the invariant $\chi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n)$ must have value at least $\prod_{i=1}^{n}(\chi(\mathcal{P}_i) - 1) + 1$.

The upper bound follows immediately from Corollary 32. ∎

The next lemma states that a degenerate property of graphs does not increase the value of the invariant χ of a decomposable property of graphs.

Lemma 34. *If* $\mathcal{P}, \mathcal{Q}$ *are additive hereditary properties of graphs and* $\mathcal{P}$ *is degenerate, then*

$$\chi(\mathcal{P} \oplus \mathcal{Q}) = \chi(\mathcal{Q}).$$

Proof. By Lemma 33 we have

$$\chi(\mathcal{P} \oplus \mathcal{Q}) \geq (\chi(\mathcal{P}) - 1)(\chi(\mathcal{Q}) - 1) + 1 = \chi(\mathcal{Q}) - 1 + 1 = \chi(\mathcal{Q}).$$

In order to prove the opposite inequality we first introduce the following notation: $ex(n, \mathcal{P}) = \{|E(G)| : |V(G)| = n \text{ and } G \in \mathcal{P}\}$. The well-known Erdös-Simonovits Theorem (see e.g. [14]) states that

$$ex(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1)}\right) \binom{n}{2} + o(n^2).$$

Moreover, it is evident that for decomposable properties holds

$$ex(n, \mathcal{P} \oplus \mathcal{Q}) \leq ex(n, \mathcal{P}) + ex(n, \mathcal{Q}).$$

By an easy calculation of the coefficient of $\binom{n}{2}$ in $ex(n, \mathcal{P} \oplus \mathcal{Q})$ we obtain the inequality:

$$\chi(\mathcal{P} \oplus \mathcal{Q}) \leq 1 + \frac{(\chi(\mathcal{P}) - 1)(\chi(\mathcal{Q}) - 1)}{1 - (\chi(\mathcal{P}) - 2)(\chi(\mathcal{Q}) - 2)} = 1 + \frac{\chi(\mathcal{Q}) - 1}{1} = \chi(\mathcal{Q}). \qquad \blacksquare$$

Theorem 35. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be additive hereditary properties of graphs. Then $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are degenerate if and only if the hereditary property $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ is degenerate.*

Proof. If $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are degenerate, then by an application of Lemma 34 we obtain that $\chi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n) = 2$, i.e., $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ is degenerate.

If at least one property $\mathcal{P}_i$, $i \in \{1, 2, \ldots, n\}$ is not degenerate then by Lemma 33 the invariant $\chi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n)$ must have value at least three and $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ is not degenerate. $\qquad \blacksquare$

As a consequence of the previous theorem we obtain the following classical result of Ramsey Theory (see [8]).

Corollary 36. *For any collection $B_1, B_2, \ldots, B_n$ of bipartite graphs there exists a bipartite graph B such that $B \to (B_1, B_2, \ldots, B_n)$.*

Proof. If we have a collection $B_1, B_2, \ldots, B_n$ of bipartite graphs we can define degenerate properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ such that $F(\mathcal{P}_i) = \{B_i\}$ for $i = 1, 2, \ldots, n$. By Theorem 35 we have that the property $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n$ is degenerate too. Thus, it has at least one bipartite graph B forbidden. But this means exactly that for any edge coloring $(E_1, E_2, \ldots, E_n)$ of $E(B)$ there must exist a color i such that the graph B_i appears as a subgraph of $B[E_i]$. Hence, $B \to (B_1, B_2, \ldots, B_n)$. $\qquad \blacksquare$

Remark 37. The reader can easily verify that also the assertion of Corollary 36 implies the assertion of Theorem 35.

The following result will allow us to determine the chromatic number of the properties of the type $\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}$. In order to simplify the proof we introduce some concepts from [1] and [9].

For any graph $G \in \mathcal{I}$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, we define a *multiplication* $G^{\ddot{}}$ of G in the following way:

1. $V(G^{\ddot{}}) = W_1 \cup W_2 \cup \ldots \cup W_n$,
2. for each $1 \le i \le n : |W_i| \ge 1$,
3. for any pair $1 \le i < j \le n$: $W_i \cap W_j = \emptyset$,
4. for any $1 \le i \le j \le n$, $u \in W_i, v \in W_j$: $u, v \in E(G^{\ddot{}})$ if and only if $v_i, v_j \in E(G)$.

The sets $W_1, W_2, \ldots, W_n$ are called the *multivertices* corresponding to vertices $v_1, v_2, \ldots, v_n$, respectively. The condition 4 immediately yields that $W_1, W_2, \ldots, W_n$ are independent sets and any two vertices belonging to the same multivertex have identical neighborhoods. Furthermore, it is not difficult to see that $G^{\ddot{}}$ is homomorphic to G. In order to emphasize the structure of $G^{\ddot{}}$, we also use the notation $G^{\ddot{}}(W_1, W_2, \ldots, W_n)$.

Lemma 38. *Let H be an arbitrary graph and let n be a positive integer. If H belongs to $\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}$ then $\to H \subseteq \mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}$.*

Proof. According to the characterization of graphs having property $\to H$ in [9] we can assume that H is not homomorphic to any of its proper subgraphs. Let us denote by $v_1, v_2, \ldots, v_n$ the vertices of H and let $W_1, W_2, \ldots, W_n$ stand for the corresponding multivertices of a multiplication $H^{\ddot{}}$.

Since H belongs to $\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}$ there exists and edge decomposition $(E_1, E_2, \ldots, E_n)$ such that for each $i = 1, 2, \ldots, n$ the graph $H[E_i]$ belongs to $\mathcal{I}_{k_i}$. We extend this coloring of H to the coloring of $H^{\ddot{}}$ in such a way that for any edge e^* between distinct multivertices W_p, W_q we shall use the same color as was used for the edge $e = v_p v_q$.

Suppose now, that in some color class E_l^* of $H^{\ddot{}}$ we have a subgraph isomorphic to K_{k_l+2}. Then it is easy to see that this subgraph has no two vertices from the same multivertex of $H^{\ddot{}}$ (because each multivertex is an independent set of vertices). But this implies that in the original coloring of the graph H we can also find K_{k_l+2} in the color E_l and we have a contradiction.

According to the results in [9], the property $\to H$ contains only subgraphs of multiplications of H. Hence $\to H \subseteq \mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}$. ∎

Theorem 39. $\chi(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}) = c(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}) + 2$

Proof. Lemma 33 yields that
$\chi(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}) \le c(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}) + 2.$

On the other hand, from the definition of the completeness it follows that the complete graph $K_{c(\mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \cdots \oplus \mathcal{I}_n)+1}$ belongs to $\mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \cdots \oplus \mathcal{I}_n$. Hence, by Lemma 38, the property $\rightarrow K_{c(\mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \cdots \oplus \mathcal{I}_n)+1}$ is contained in the property $\mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \cdots \oplus \mathcal{I}_n$. But the property $\rightarrow K_n$ is in fact the set of all n-colorable graphs – see e.g. [1].

Hence, $\chi(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}) \geq c(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n}) + 2.$ ■

This result shows that the determination of $\chi(\mathcal{I}_{k_1} \oplus \mathcal{I}_{k_2} \oplus \cdots \oplus \mathcal{I}_{k_n})$ is equivalent to the determination of Ramsey number.

4   Ramsey Arrows and Decomposable Properties

In this section we show that the study of the defined arrow relation between a graph and given properties of graphs is interesting mainly for co-hereditary properties of graphs. For those properties we are moreover able to determine some values of r_ρ and R_ρ.

The following three assertions follow immediately from the definitions.

Proposition 41. *Let G be a graph and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be hereditary properties of graphs. Then for an arbitrary graph G the relation $G \rightarrow (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ holds.*

Proposition 42. *Let G be a graph and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be co-hereditary properties of graphs. Then $G \rightarrow (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ if and only if $G \notin \overline{\mathcal{P}_1} \oplus \overline{\mathcal{P}_2} \oplus \cdots \oplus \overline{\mathcal{P}_n}$.*

Corollary 43. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be co-hereditary properties of graphs. The property*

$$\mathcal{T} = \{G \in \mathcal{I} : G \rightarrow (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)\}$$

is co-hereditary (i.e., it is closed under taking supergraphs).

According to the Proposition 41 it is not interesting to study the parameters r_ρ and R_ρ for hereditary properties of graphs. On the other hand, it is very difficult to handle graph properties in general settings. Hence we shall concentrate mainly on co-hereditary properties.

In the language of hereditary properties the parameters r_ρ and R_ρ can for co-hereditary properties be expressed in another manner. Indeed, by Proposition 42 we have for arbitrary hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ that

$$r_\rho(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \ldots, \overline{\mathcal{P}_n}) = \rho(\mathcal{P}_1 \oplus \mathcal{P}_2 \cdots \oplus \mathcal{P}_n),$$

$$R_\rho(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \dots, \overline{\mathcal{P}_n}) = \sup\{\rho(G) : G \in \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n\} + 1.$$

Remark 44. In the light of the previous two identities and Corollary 24 we can now prove Theorem 17 in the following manner:

$$
\begin{aligned}
r_\chi(\overline{\mathcal{O}^{k_1}}, \overline{\mathcal{O}^{k_2}}, \dots, \overline{\mathcal{O}^{k_n}}) &= \chi(\overline{\mathcal{O}^{k_1}} \oplus \overline{\mathcal{O}^{k_2}} \oplus \cdots \oplus \overline{\mathcal{O}^{k_n}}) = \chi((n\mathrm{x}\mathcal{O})^{\Pi_{i=1}^n k_i}) \\
&= \chi(\mathcal{O}^{\Pi_{i=1}^n k_i}) = \prod_{i=1}^n k_i + 1.
\end{aligned}
$$

On the other hand, if for some graph G holds $\chi(G) = \Pi_{i=1}^n k_i + 1$ then obviously $G \notin \mathcal{O}^{\Pi_{i=1}^n k_i}$. This implies that

$$R_\chi(\overline{\mathcal{O}^{k_1}}, \overline{\mathcal{O}^{k_2}}, \dots, \overline{\mathcal{O}^{k_n}}) = \prod_{i=1}^n k_i + 1$$

and the proof is complete.

Proposition 11 immediately yields the following lemma.

Lemma 45. *If $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ are additive hereditary properties, ρ_1, ρ_2 are two graph theoretical invariants satisfying $\rho_1(G) \le \rho_2(G)$ for any graph G, then*

$$r_{\rho_1}(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n) \le r_{\rho_2}(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n).$$

Combining Lemma 45, Proposition 42 and Lemma 31 we obtain the next assertion.

Corollary 46. *Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ be additive hereditary properties of graphs and let ρ be a graph theoretical invariant which satisfies the inequality $\rho(G) \le o(G)$ for an arbitrary graph G. Then*

$$r_\rho(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \dots, \overline{\mathcal{P}_n}) \le c(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n) + 2.$$

5   A RESULT RELATED TO THE GENERALIZED CHROMATIC NUMBER

The next result provides a generalization of Theorem 17 in such a manner that the usual chromatic number χ is replaced by the generalized chromatic number $\chi_\mathcal{P}$ and the property $\mathcal{O}$ by arbitrary degenerate hereditary properties. The symbol $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x.

Theorem 51. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ and $\mathcal{Q}$ be degenerate properties of graphs and let $k_1, k_2, \ldots, k_n$ be positive integers. Then*

$$r_{\chi_{\mathcal{Q}}}(\overline{\mathcal{P}_1^{k_1}}, \overline{\mathcal{P}_2^{k_2}}, \ldots, \overline{\mathcal{P}_n^{k_n}}) = \left\lfloor \frac{\prod_{i=1}^n k_i}{\Phi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \mathcal{Q})} \right\rfloor + 1,$$

provided $\Phi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \mathcal{Q})$ is finite, and

$$R_{\chi_{\mathcal{Q}}}(\overline{\mathcal{P}_1^{k_1}}, \overline{\mathcal{P}_2^{k_2}}, \ldots, \overline{\mathcal{P}_n^{k_n}}) = \Phi(\mathcal{Q}, \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n) \prod_{i=1}^n k_i + 1,$$

provided $\Phi(\mathcal{Q}, \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n)$ is finite.

Proof. First, consider $r_{\chi_{\mathcal{Q}}}(\overline{\mathcal{P}_1^{k_1}}, \overline{\mathcal{P}_2^{k_2}}, \ldots, \overline{\mathcal{P}_n^{k_n}})$. In view of Proposition 42 and Corollary 25, this is the smallest value of $\chi_{\mathcal{Q}}(G)$ for any G for which $\chi_{\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n}(G) > \prod_{i=1}^n k_i$. But by Theorem 15, $\chi_{\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n}(G) \leq \Phi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \mathcal{Q}) \cdot \chi_{\mathcal{Q}}(G)$. Hence,

$$\chi_{\mathcal{Q}}(G) > \frac{\prod_{i=1}^n k_i}{\Phi(\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \mathcal{Q})}.$$

Observing that, by Theorem 15, this can be made as sharp as possible, subject to the constraint that $\chi_{\mathcal{Q}}$ is an integer, leads to the formula for $r_{\chi_{\mathcal{Q}}}$.

Similarly, consider $R_{\chi_{\mathcal{Q}}}(\overline{\mathcal{P}_1^{k_1}}, \overline{\mathcal{P}_2^{k_2}}, \ldots, \overline{\mathcal{P}_n^{k_n}})$. This is the smallest value of $\chi_{\mathcal{Q}}(G)$ such that every G with this value satisfies $\chi_{\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n} > \prod_{i=1}^n k_i$. Again, by Theorem 15, $\chi_{\mathcal{Q}}(G) \leq \Phi(\mathcal{Q}, \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n) \cdot \chi_{\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n}(G)$. And a straightforward application of Corollary 25 leads to the formula for $R_{\chi_{\mathcal{Q}}}$. ∎

6   Some Applications

In this section we present some exact values and bounds on the parameters r_ρ and R_ρ which we can obtain in terms of results which were established in the previous sections.

It is not so difficult to see that the parameter r_ρ is always finite. The next lemma determines for which properties the parameter R_ρ is finite.

Lemma 61. *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be hereditary properties of graphs. Then*

(a) *$R_o(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \ldots, \overline{\mathcal{P}_n})$ is finite if and only if there exists a nonnegative integer k such that $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n \subseteq \mathcal{O}_k$.*

(b) $R_{\mathrm{col}}(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \ldots, \overline{\mathcal{P}_n})$ *is finite if and only if there exists a nonnegative integer* k *such that* $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n \subseteq \mathcal{D}_k$.

(c) $R_\Delta(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \ldots, \overline{\mathcal{P}_n})$ *is finite if and only if there exists a nonnegative integer* k *such that* $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n \subseteq \mathcal{S}_k$.

(d) $R_\chi(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \ldots, \overline{\mathcal{P}_n})$ *is finite if and only if there exists a nonnegative integer* k *such that* $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n \subseteq \mathcal{O}^k$.

(e) $R_\omega(\overline{\mathcal{P}_1}, \overline{\mathcal{P}_2}, \ldots, \overline{\mathcal{P}_n})$ *is finite if and only if there exists a nonnegative integer* k *such that* $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n \subseteq \mathcal{I}_k$.

The proof is based on straightforward applications of the definitions.

By a combination of Lemma 45, Corollary 32 and Theorem 39 we have the next result. It states that the computation of $r_\rho(\overline{\mathcal{I}_{m_1}^{k_1}}, \overline{\mathcal{I}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{I}_{m_n}^{k_n}})$ is in many cases equivalent to the determination of a Ramsey number.

Theorem 62. *Let* $m_1, m_2, \ldots, m_n$ *be nonnegative integers and let* $k_1, k_2, \ldots, k_n$ *be positive integers. If* ρ *is a monotone graph theoretical invariant satisfying* $\chi(G) \leq \rho(G) \leq o(G)$ *for any graph* G, *then*

$$r_\rho(\overline{\mathcal{I}_{m_1}^{k_1}}, \overline{\mathcal{I}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{I}_{m_n}^{k_n}}) = \rho(\mathcal{I}_{m_1}^{k_1} \oplus \mathcal{I}_{m_2}^{k_n} \oplus \cdots \oplus \mathcal{I}_{m_n}^{k_n})$$

$$= \left(\prod_{i=1}^{n} k_i \right) (c(\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n}) + 1) + 1.$$

Proof. Applying Proposition 42 and Corollary 24 we get:

$$r_\chi(\overline{\mathcal{I}_{m_1}^{k_1}}, \overline{\mathcal{I}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{I}_{m_n}^{k_n}}) = \chi(\mathcal{I}_{m_1}^{k_1} \oplus \mathcal{I}_{m_2}^{k_2} \oplus \cdots \oplus \mathcal{I}_{m_n}^{k_n})$$

$$= \chi\left((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^{n} k_i} \right).$$

Applying Theorem 13 we obtain

$$\chi\left((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^{n} k_i} \right)$$

$$= \left(\prod_{i=1}^{n} k_i \right) \chi(\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n}) - \left(\prod_{i=1}^{n} k_i - 1 \right),$$

and by Theorem 39 we have

$$\chi\left((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^n k_i}\right)$$

$$= \left(\prod_{i=1}^n k_i\right)(c(\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n}) + 2) - \prod_{i=1}^n k_i + 1,$$

which is equal to

$$\left(\prod_{i=1}^n k_i\right)(c(\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n}) + 1) + 1.$$

On the other hand,

$$r_o(\overline{\mathcal{I}_{m_1}^{k_1}}, \overline{\mathcal{I}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{I}_{m_n}^{k_n}}) = o(\mathcal{I}_{m_1}^{k_1} \oplus \mathcal{I}_{m_2}^{k_2} \oplus \cdots \oplus \mathcal{I}_{m_n}^{k_n})$$

$$= o\left((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^n k_i}\right).$$

Lemma 31 enables us to write:

$$o\left((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^n k_i}\right) = c((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^n k_i}) + 2.$$

Applying Theorem 14 we obtain

$$o\left((\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})^{\Pi_{i=1}^n k_i}\right) = \left(\prod_{i=1}^n k_i\right) c(\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n})$$

$$+ \left(\prod_{i=1}^n k_i - 1\right) + 2,$$

which is equal to

$$\left(\prod_{i=1}^n k_i\right)(c(\mathcal{I}_{m_1} \oplus \mathcal{I}_{m_2} \oplus \cdots \oplus \mathcal{I}_{m_n}) + 1) + 1.$$

And now, the assertion of the theorem follows immediately from Lemma 45. ∎

The following result is proved in [2].

Theorem 63. *Let n be a positive integer and let $m_1, m_2, \ldots, m_n$ be nonnegative integers. Then*

$$\mathcal{D}_{\Sigma_{i=1}^n m_i} \subseteq \mathcal{D}_{m_1} \oplus \mathcal{D}_{m_2} \oplus \cdots \mathcal{D}_{m_n} \subseteq \mathcal{D}_{2\Sigma_{i=1}^n m_i - 1}.$$

Since $\chi(\mathcal{D}_k) = k + 1$ (cf. [10], [12]) we can prove the following bounds.

Theorem 64.   *Let* n *and* $k_1, k_2, \ldots, k_n$ *be positive integers and let* $m_1, m_2, \ldots, m_n$ *be nonnegative integers. Then*

$$\left(\prod_{i=1}^{n} k_i\right)\left(\sum_{i=1}^{n} m_i\right) + 1 \leq r_\chi(\overline{\mathcal{D}_{m_1}^{k_1}}, \overline{\mathcal{D}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{D}_{m_n}^{k_n}})$$

$$\leq R_\chi(\overline{\mathcal{D}_{m_1}^{k_1}}, \overline{\mathcal{D}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{D}_{m_n}^{k_n}}) \leq 2\left(\sum_{i=1}^{n} m_i\right)\left(\prod_{i=1}^{n} k_i\right) + 1.$$

Proof. The inequality $r_\chi(\overline{\mathcal{D}_{m_1}^{k_1}}, \overline{\mathcal{D}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{D}_{m_n}^{k_n}}) \leq R_\chi(\overline{\mathcal{D}_{m_1}^{k_1}}, \overline{\mathcal{D}_{m_2}^{k_2}}, \ldots \overline{\mathcal{D}_{m_n}^{k_n}})$ follows immediately from the definitions.

Applying Proposition 42 and Corollary 24 we get

$$\begin{aligned}
r_\chi(\overline{\mathcal{D}_{m_1}^{k_1}}, \overline{\mathcal{D}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{D}_{m_n}^{k_n}}) &= \chi(\mathcal{D}_{m_1}^{k_1} \oplus \mathcal{D}_{m_2}^{k_2} \oplus \cdots \oplus \mathcal{D}_{m_n}^{k_n}) \\
&= \chi\left((\mathcal{D}_{m_1} \oplus \mathcal{D}_{m_2} \oplus \cdots \oplus \mathcal{D}_{m_n})^{\Pi_{i=1}^{n} k_i}\right).
\end{aligned}$$

Since $\mathcal{D}_{\Sigma_{i=1}^{n} m_i} \subseteq \mathcal{D}_{m_1} \oplus \mathcal{D}_{m_2} \oplus \cdots \mathcal{D}_{m_n}$ by Theorem 63, we have the following inequality

$$\chi\left((\mathcal{D}_{m_1} \oplus \mathcal{D}_{m_2} \oplus \cdots \oplus \mathcal{D}_{m_n})^{\Pi_{i=1}^{n} k_i}\right) \geq \chi\left(\mathcal{D}_{\Sigma_{i=1}^{n} m_i}^{\Pi_{i=1}^{n} k_i}\right).$$

And by an application of Theorem 13 we obtain

$$\begin{aligned}
\chi\left(\mathcal{D}_{\Sigma_{i=1}^{n} m_i}^{\Pi_{i=1}^{n} k_i}\right) &= \left(\prod_{i=1}^{n} k_i\right)\chi(\mathcal{D}_{\Sigma_{i=1}^{n} m_i}) - \left(\prod_{i=1}^{n} k_i - 1\right) \\
&= \left(\prod_{i=1}^{n} k_i\right)\left(1 + \sum_{i=1}^{n} m_i\right) - \prod_{i=1}^{n} k_i + 1,
\end{aligned}$$

which is exactly

$$\left(\prod_{i=1}^{n} k_i\right)\left(\sum_{i=1}^{n} m_i\right) + 1.$$

Since

$$R_\chi(\overline{\mathcal{D}_{m_1}^{k_1}}, \overline{\mathcal{D}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{D}_{m_n}^{k_n}}) = \sup\{\chi(G) : G \in \mathcal{D}_{m_1}^{k_1} \oplus \mathcal{D}_{m_2}^{k_2} \oplus \cdots \oplus \mathcal{D}_{m_n}^{k_n}\} + 1,$$

using Corollary 24 we have

$$\sup\{\chi(G) : G \in \mathcal{D}_{m_1}^{k_1} \oplus \mathcal{D}_{m_2}^{k_2} \oplus \cdots \oplus \mathcal{D}_{m_n}^{k_n}\} + 1$$

$$= \sup\{\chi(G) : G \in (\mathcal{D}_{m_1} \oplus \mathcal{D}_{m_2} \oplus \cdots \oplus \mathcal{D}_{m_n})^{\Pi_{i=1}^{n} k_i}\} + 1.$$

And now, by arguments similar to those used in the previous case we obtain

$$\sup\{\chi(G) : G \in (\mathcal{D}_{m_1} \oplus \mathcal{D}_{m_2} \oplus \cdots \oplus \mathcal{D}_{m_n})^{\Pi_{i=1}^{n} k_i}\} + 1$$

$$\leq \sup\{\chi(G) : G \in (\mathcal{D}_{2(\Sigma_{i=1}^{n} m_i)-1})^{\Pi_{i=1}^{n} k_i}\} + 1.$$

As any k-degenerate graph is $(k+1)$-colorable we finally have

$$\sup\{\chi(G) : G \in (\mathcal{D}_{2(\Sigma_{i=1}^{n} m_i)-1})^{\Pi_{i=1}^{n} k_i}\} + 1$$

$$\leq \sup\{\chi(G) : G \in (\mathcal{O}^{2(\Sigma_{i=1}^{n} m_i)(\Pi_{i=1}^{n} k_i)}\} + 1,$$

which is exactly $2\left(\Sigma_{i=1}^{n} m_i\right)\left(\Pi_{i=1}^{n} k_i\right) + 1$. ∎

The following result will allow us to utilize Theorem 51 and establish Theorem 68.

Theorem 65 [11]. *Let k and m be positive integers. Then $\Phi(\mathcal{S}_k, \mathcal{S}_m) = 1 + \lfloor \frac{m}{k+1} \rfloor$.*

Corollary 66. *Let $k, m_1, m_2, \ldots, m_n$ be positive integers. Then*

$$\Phi(\mathcal{S}_k, \mathcal{S}_{m_1} \oplus \mathcal{S}_{m_2} \oplus \cdots \oplus \mathcal{S}_{m_n}) \leq 1 + \left\lfloor \frac{\sum_{i=1}^{n} m_i}{k+1} \right\rfloor.$$

Corollary 67. *Let $k, m_1, m_2, \ldots, m_n$ be positive integers. Then*

$$\Phi(\mathcal{S}_{m_1} \oplus \mathcal{S}_{m_2} \oplus \cdots \oplus \mathcal{S}_{m_n}, \mathcal{S}_k) \geq 1 + \left\lfloor \frac{k}{\sum_{i=1}^{n} m_i + 1} \right\rfloor.$$

Now, by Theorem 5.1 we have:

Theorem 68. *Let $l, k_1, k_2, \ldots, k_n$ be positive integers and let $m_1, m_2, \ldots, m_n$ be nonnegative integers. Then*

$$R_{\chi \mathcal{S}_l}(\overline{\mathcal{S}_{m_1}^{k_1}}, \overline{\mathcal{S}_{m_2}^{k_2}}, \ldots, \overline{\mathcal{S}_{m_n}^{k_n}}) \leq \left(1 + \left\lfloor \frac{\sum_{i=1}^{n} m_i}{l+1} \right\rfloor\right)\left(\prod_{i=1}^{n} k_i\right) + 1.$$

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