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# ON-LINE RANKING NUMBER FOR CYCLES AND PATHS

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#### Abstract

A k-ranking of a graph G is a colouring  $\varphi : V(G) \to \{1, \ldots, k\}$  such that any path in G with endvertices x, y fulfilling  $\varphi(x) = \varphi(y)$  contains an internal vertex z with  $\varphi(z) > \varphi(x)$ . On-line ranking number  $\chi_r^*(G)$  of a graph G is a minimum k such that G has a k-ranking constructed step by step if vertices of G are coming and coloured one by one in an arbitrary order; when colouring a vertex, only edges between already present vertices are known. Schiermeyer, Tuza and Voigt proved that  $\chi_r^*(P_n) < 3 \log_2 n$  for  $n \geq 2$ . Here we show that  $\chi_r^*(P_n) \leq 2\lfloor \log_2 n \rfloor + 1$ . The same upper bound is obtained for  $\chi_r^*(C_n), n \geq 3$ .

Keywords: ranking number, on-line vertex colouring, cycle, path.

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## 1 INTRODUCTION

In this article we deal with simple finite undirected graphs. For formal reasons we also use the empty graph  $K_0 = (\emptyset, \emptyset)$ . A *k*-ranking of a graph *G* is a vertex colouring of *G* which takes as colours integers  $1, \ldots, k$  in such a way that, whenever a path of *G* has endvertices of the same colour, it contains an internal vertex with a greater colour. If *k* is not specified, we speak simply about a ranking. Evidently, a ranking is a proper vertex colouring and a *k*-ranking of a connected graph uses *k* at most once. Rankings are important in the parallel Cholesky factorization of matrices (Liu [3]) and also in VLSI layout (Leiserson [2]).

Ranking number  $\chi_{\mathbf{r}}(G)$  of a graph G is a minimum k such that G has a k-ranking. The problem of finding the ranking number of an arbitrary graph is NP-complete, see Llewelyn et al. [4]. Katchalski et al. [1] proved, among other results on trees, that  $\chi_{\mathbf{r}}(P_n) = \lfloor \log_2 n \rfloor + 1$  for  $n \geq 1$ . They have also an upper bound for the ranking number of a planar graph G, namely  $\chi_{\mathbf{r}}(G) \leq 3(\sqrt{6}+2)\sqrt{|V(G)|}$ .

In an *on-line* version of the problem vertices of a graph G are coming in an arbitrary order. They are coloured one by one in such a way that only a local information concerning edges between already present vertices is known in a moment when a colour for a vertex is to be chosen. Schiermeyer et al. [5] showed that, for  $n \ge 2$ , there is an on-line algorithm providing a ranking of *n*-vertex path, for which the maximum used number is smaller than  $3\log_2 n$ , independently from arriving order of vertices. Our main aim is to show that this number is  $\le 2|\log_2 n| + 1$ .

For a graph G and a set  $W \subseteq V(G)$  let  $G\langle W \rangle$  be the subgraph of G induced by W. The notation  $C_n$  and  $P_n$  is used for *n*-vertex cycle and *n*-vertex path, respectively.

For integers p, q we denote by [p, q] the set of all integers r with  $p \leq r \leq q$ , and by  $[p, \infty)$  the set of all integers r with  $p \leq r$ .

The length of a finite sequence A (i.e., the number of terms of A), is denoted by |A|. For finite sequences  $A = (a_1, \ldots, a_m)$  and  $B = (b_1, \ldots, b_n)$ let  $AB = (a_1, \ldots, a_m, b_1, \ldots, b_n)$  be the concatenation of A and B (in this order); the concatenation can be generalized to any finite number of finite sequences. The concatenation is, clearly, associative, and we will use  $\prod_{i=1}^{k} A_i$ for the concatenation of finite sequences  $A_1, \ldots, A_k$  (in this order).

Now, let us describe our on-line version of the ranking problem more precisely. An *input sequence* for a graph G is any sequence of vertices of G containing all vertices of G exactly once. Let Is(G) be the set of all input sequences for G and let  $Y = \prod_{i=1}^{n} (y_i) \in Is(G)$ . Vertices  $y_1, \ldots, y_n$  are coloured in this order one by one in the following way: We denote by  $G(Y, y_i)$ the graph  $G\langle \{y_j : j \in [1, i]\} \rangle$  induced by all vertices that come in Y not later than  $y_i$  does,  $i \in [1, n]$ . We colour  $y_1$  with an arbitrary positive integer. In the moment when  $y_i, i \in [2, n]$ , is to be coloured, only the graph  $G(Y, y_i)$  and a ranking of  $G(Y, y_{i-1})$  is known; the colour of  $y_i$  has to be chosen in such a way that a ranking of  $G(Y, y_i)$  results (without altering "old" colours).

We would like to analyze all possibilities of forming a ranking of a graph G in the above on-line fashion. To that aim, we denote by Q the set of all quadruples  $(G, H, \varphi, x)$  such that G is a non-empty graph, H is an induced subgraph of G with |V(H)| = |V(G)| - 1,  $\varphi$  is a ranking of H

and  $\{x\} = V(G) - V(H)$ . We say that two quadruples  $(G, H, \varphi, x)$  and  $(G', H', \varphi', x')$  are *equivalent* (and we do not distinguish them in  $\mathcal{Q}$ ) if there is an isomorphism  $\iota$  between G and G' which maps H onto H' (so that  $\iota(x) = x'$ ) and an automorphism  $\alpha'$  of H' such that for any  $y \in V(H)$  it holds  $\varphi(y) = \varphi'(\alpha'(\iota(y)))$ . A ranking algorithm is a mapping  $\mathcal{A} : \mathcal{Q} \to [1, \infty)$  such that, for any  $(G, H, \varphi, x) \in \mathcal{Q}, \varphi \cup \{(x, \mathcal{A}(G, H, \varphi, x))\}$  is a ranking of G.

Let  $\mathcal{A}$  be a ranking algorithm, let G be a graph and let  $Y = \prod_{i=1}^{n} (y_i) \in$ Is(G). The algorithm  $\mathcal{A}$  provides a ranking rank( $\mathcal{A}, G, Y, y_i$ ) of the graph  $G(Y, y_i), i \in [1, n]$ , recurrently as follows:

 $\operatorname{rank}(\mathcal{A}, G, Y, y_1) := \{(y_1, \mathcal{A}(K_1, K_0, \emptyset, y_1))\},$  $\operatorname{rank}(\mathcal{A}, G, Y, y_i) := \operatorname{rank}(\mathcal{A}, G, Y, y_{i-1})$  $\cup \{(y_i, \mathcal{A}(G(Y, y_i), G(Y, y_{i-1}), \operatorname{rank}(\mathcal{A}, G, Y, y_{i-1}), y_i))\}, i \in [2, n].$ 

We denote by rank( $\mathcal{A}, G, Y$ ) the ranking rank( $\mathcal{A}, G, Y, y_n$ ) of the graph  $G(Y, y_n) = G$  provided by the algorithm  $\mathcal{A}$  if the vertices of G are coming in the input sequence Y. Clearly, the ranking rank( $\mathcal{A}, G, Y, y_i$ ) is a restriction of the ranking rank( $\mathcal{A}, G, Y$ ) to the graph  $G(Y, y_i), i \in [1, n]$ . By max ( $\mathcal{A}, G, Y$ ) we will denote the maximum number attributed to a vertex of G by rank( $\mathcal{A}, G, Y$ ) and by max( $\mathcal{A}, G$ ) the maximum of max( $\mathcal{A}, G, Y$ ) over all  $Y \in Is(G)$ . The on-line ranking number  $\chi_r^*(G)$  of the graph G is the minimum of max( $\mathcal{A}, G, G$ ) over all ranking algorithms  $\mathcal{A}$ . Evidently, for any graph G and any ranking algorithm  $\mathcal{A}$  we have

$$\chi_{\mathbf{r}}(G) \le \chi_{\mathbf{r}}^*(G) \le \max(\mathcal{A}, G).$$

**Proposition 1.** If  $G_1$  is an induced subgraph of  $G_2$  and  $\mathcal{A}$  is a ranking algorithm, then  $\max(\mathcal{A}, G_1) \leq \max(\mathcal{A}, G_2)$ .

**Proof.** Consider an input sequence  $Y_1 = \prod_{i=1}^n (y_i) \in \operatorname{Is}(G_1)$  such that  $\max(\mathcal{A}, G_1, Y_1) = \max(\mathcal{A}, G_1)$  and an arbitrary input sequence  $Y_2$  of the graph  $G_2\langle V(G_2) - V(G_1)\rangle$ . Then  $Y_1Y_2 \in \operatorname{Is}(G_2)$ , and we have  $\operatorname{rank}(\mathcal{A}, G_2, Y_1Y_2, y_n) = \operatorname{rank}(\mathcal{A}, G_1, Y_1)$ , so that  $\max(\mathcal{A}, G_2) \geq \max(\mathcal{A}, G_2, Y_1Y_2) \geq \max(\mathcal{A}, G_1, Y_1) = \max(\mathcal{A}, G_1)$ .

**Corollary 2.** If  $G_1$  is an induced subgraph of  $G_2$ , then  $\chi_r^*(G_1) \leq \chi_r^*(G_2)$ .

### 2 REDUCTION

A natural greedy algorithm  $\mathcal{G}$  (called also First Fit Algorithm) is determined by the requirement that, for any  $(G, H, \varphi, x) \in \mathcal{Q}$ ,  $\mathcal{G}(G, H, \varphi, x)$  is the minimum positive integer k such that  $\varphi \cup \{(x, k)\}$  is a ranking of G. In other words, we can describe  $\mathcal{G}$  as follows: A colour  $l \in [1, \infty)$  is forbidden for x if the colouring  $\psi = \varphi \cup \{(x, l)\}$  produces a (u, v)-path P in G with  $\psi(u) = \psi(v) = \max\{\psi(y) : y \in V(P)\}$  (clearly,  $x \in V(P)$ ). The greedy algorithm colours x with the smallest colour that is not forbidden for x. Evidently, the colour  $\max\{\varphi(y) : y \in V(H)\} + 1$  is not forbidden for x. That is why, we know that for any graph G and any input sequence  $Y \in \mathrm{Is}(G)$  the ranking  $\mathrm{rank}(\mathcal{G}, G, Y)$  of G uses every integer from the interval  $[1, \max(\mathcal{G}, G, Y)]$  at least once.

Now we are going to analyze how  $\mathcal{G}$  works for cycles and paths. For that purpose suppose that  $G = C_n$ ,  $n \in [3, \infty)$ , or  $G = P_n$ ,  $n \in [1, \infty)$ , with  $V(G) = \{x_i : i \in [1, n]\}$  and  $E(G) \supseteq \{x_i x_{i+1} : i \in [1, n-1]\}$  (there is an equality in this inclusion if  $G = P_n$ , and, if  $G = C_n$ , there is an additional edge  $x_n x_1$ ). Sometimes it will be necessary to use for indices arithmetics modulo n, i.e.,  $x_{i-n} = x_i = x_{i+n}$  for any  $i \in [1, n]$ .

As an example, consider the input sequence  $Y = (x_6, x_7, x_3, x_5, x_2, x_4, x_1) \in \text{Is}(C_7) = \text{Is}(P_7)$ . We have  $\text{rank}(\mathcal{G}, C_7, Y) = \{(x_6, 1), (x_7, 2), (x_3, 1), (x_5, 3), (x_2, 2), (x_4, 4), (x_1, 5)\}$  and  $\text{rank}(\mathcal{G}, P_7, Y)$  differs from  $\text{rank}(\mathcal{G}, C_7, Y)$  only by attributing 1 to  $x_1$ .

An important role in our analysis is played by the following reduction process: We suppose that  $G = C_n$ ,  $n \in [5, \infty)$ , or  $G = P_n$ ,  $n \in [2, \infty)$ ,  $Y \in \text{Is}(G)$  and  $\varphi = \text{rank}(\mathcal{G}, G, Y)$ . A vertex  $x_i \in V(G)$  is said to be a survivor of G (with respect to the input sequence Y) if  $\varphi(x_i) \geq 2$ ; if  $\varphi(x_i) = 1$ , it is a non-survivor. We transform G into a non-empty graph R(G, Y) homeomorphic to G as follows: We delete from G all non-survivors and we join by a new edge any two survivors having a non-survivor as a common neighbour (i.e., we delete all non-survivors of degree 1 and we "smooth out" all non-survivors of degree 2). We can do this because it is easy to see that the number of survivors is always positive and, in the case  $G = C_n$ , it is  $\geq 3$ . The input sequence Y induces in a natural way an input sequence R(Y, G) for the graph R(G, Y) – we simply delete from Yall non-survivors.

If  $Y \in Is(C_7)$  is as above, then  $R(C_7, Y) = C_5$ ,  $R(Y, C_7) = (x_7, x_5, x_2, x_4, x_1)$  and  $R(P_7, Y) = P_4$ ,  $R(Y, P_7) = (x_7, x_5, x_2, x_4)$ .

**Lemma 3.** Let  $G = C_n$ ,  $n \in [5, \infty)$ , or  $G = P_n$ ,  $n \in [2, \infty)$ , let  $Y \in Is(G)$ ,  $\varphi = \operatorname{rank}(\mathcal{G}, G, Y)$ ,  $\dot{G} = R(G, Y)$ ,  $\dot{Y} = R(Y, G)$  and  $\dot{\varphi} = \operatorname{rank}(\mathcal{G}, \dot{G}, \dot{Y})$ . Then, for any survivor  $x_i$  of G with respect to Y, it holds  $\dot{\varphi}(x_i) = \varphi(x_i) - 1$ .

**Proof.** Consider a sequence  $Y' \in \text{Is}(G)$  in which all non-survivors (with respect to Y) come first (in an arbitrary order) and then all survivors (with respect to Y) come in the order induced by that of Y. It is easy to see that  $\varphi = \text{rank}(\mathcal{G}, G, Y')$ .

Let  $Y' = \prod_{i=1}^{n} (y_i)$  and let  $y_s$  be the first survivor with respect to Y' (and Y as well). We are going to show by induction on i that  $\dot{\varphi}(y_i) = \varphi(y_i) - 1$  for any  $i \in [s, n]$ . Obviously,  $\dot{\varphi}(y_s) = 1 = 2 - 1 = \varphi(y_s) - 1$ .

Now suppose that  $i \in [s + 1, n]$  and that  $\dot{\varphi}(y_j) = \varphi(y_j) - 1$  for every  $j \in [s, i - 1]$ . Note that survivors  $y_j, y_k$  with  $j, k \in [s, i], j \neq k$ , are joined by a path P in  $G(Y', y_i)$  if and only if they are joined in  $\dot{G}(\dot{Y}, y_i)$  by the path  $\dot{P}$  such that  $V(\dot{P}) = V(P) - \{y_l : l \in [1, s - 1]\}$ . Hence, by the induction hypothesis and the fact that  $\varphi(y_l) = 1$  for any  $l \in [1, s - 1]$ , a colour  $a \in [2, \infty)$  is forbidden for  $y_i$  in  $G(Y, y_i)$  by the corresponding path  $\dot{P}$ . Since  $\varphi(y_i) \geq 2$ , we obtain  $\dot{\varphi}(y_i) = \varphi(y_i) - 1$ , as necessary.

We define a section of our graph G as follows: A section of  $P_n$  is any sequence  $\Pi_{i=i}^{k}(x_i)$  of vertices of  $P_n$  with  $j, k \in [1, n]$  and  $j \leq k$ . A section of  $C_n$  is any sequence  $\prod_{i=j}^{k} (x_i)$  of vertices of  $C_n$  with  $j, k \in [1-n, 2n]$  and  $j \leq k \leq j-1+n$ . From the definition we see that a section  $\prod_{i=j}^{k} (x_i)$  consists of  $k+1-j \leq n$ distinct vertices of G and that  $x_i x_{i+1}$  is an edge of G for every  $i \in [j, k-1]$ . An endsection of  $P_n$  is any section of  $P_n$  containing an endvertex of  $P_n$ . The type of a section  $\prod_{i=j}^{k} (x_i)$  (with respect to the ranking  $\varphi = \operatorname{rank}(\mathcal{G}, G, Y)$ ) is the sequence formed from  $\prod_{i=i}^{k}(\varphi(x_i))$  by replacing any term  $\varphi(x_i)$  fulfilling  $\varphi(x_i) \geq 3$  with 3+. The ranking  $\varphi = \operatorname{rank}(\mathcal{G}, \mathcal{G}, Y)$  determines two types of vertices in G: a vertex  $x \in V(G)$  is high (with respect to  $\varphi$ ), if  $\varphi(x) \geq 3$ , otherwise it is *low*. A section of G containing only high [low] vertices, which is maximal (non-extendable with respect to this property), is called a *high* [low] section of G. The defect of a section S of G is the difference def(S)between the number of low vertices in S and the number of high vertices in S. The defect of a graph G is the difference def(G) between the number of low vertices in V(G) and the number of high vertices in V(G), i.e., the defect of (any) section S of G with |S| = |V(G)|.

**Lemma 4.** Let  $G = C_n$ ,  $n \in [3, \infty)$ , or  $G = P_n$ ,  $n \in [1, \infty)$ , let  $Y \in \text{Is}(G)$ ,  $\varphi = \text{rank}(\mathcal{G}, G, Y)$  and  $q \in [1, n]$ .

- 1. If  $\prod_{i=q}^{q+3}(x_i)$  is a section of G, then there are  $j,k \in [q,q+3]$  such that  $\varphi(x_j) = 1$  and  $\varphi(x_k) \geq 3$ .
- 2. If  $\prod_{i=q}^{q+2}(x_i)$  is such a section of G that  $\varphi(x_{q+1}) = 2$ , then  $\min\{\varphi(x_q), \varphi(x_{q+2})\} = 1$ .
- 3. If  $G = P_n$  and  $\varphi(x_1) \ge 2$ , then  $n \ge 2$  and  $\varphi(x_2) = 1$ .
- 4. If  $G = P_n$  and  $\varphi(x_1) \ge 3$ , then  $n \ge 3$ ,  $\varphi(x_2) = 1$  and  $\varphi(x_3) = 2$ .
- 5. If  $G = P_n$  and  $\varphi(x_n) \ge 2$ , then  $n \ge 2$  and  $\varphi(x_{n-1}) = 1$ .
- 6. If  $G = P_n$  and  $\varphi(x_n) \ge 3$ , then  $n \ge 3$ ,  $\varphi(x_{n-1}) = 1$  and  $\varphi(x_{n-2}) = 2$ .
- 7. If  $\prod_{i=q}^{q+2}(x_i)$  is a section of G of type (3+,3+,3+), then  $\prod_{i=q-2}^{q+4}(x_i)$  also is a section of G and it is of type (2,1,3+,3+,3+,1,2).
- 8. If  $\Pi_{i=q}^{q+3}(x_i)$  is a section of G of type (3+,3+,1,3+), then  $\Pi_{i=q-2}^{q+5}(x_i)$  also is a section of G and it is of type (2,1,3+,3+,1,3+,1,2) or (2,1,3+,3+,1,3+,2,1).
- 9. If  $\Pi_{i=q}^{q+3}(x_i)$  is a section of G of type (3+,1,3+,3+), then  $\Pi_{i=q-2}^{q+5}(x_i)$  also is a section of G and it is of type (1,2,3+,1,3+,3+,1,2) or (2,1,3+,1,3+,3+,1,2).
- 10. If  $G = P_n$ ,  $n \ge 3$ ,  $\varphi(x_1) = 1$  and  $\varphi(x_3) \ge 3$ , then  $\varphi(x_2) = 2$ .
- 11. If  $G = P_n$ ,  $n \ge 3$ ,  $\varphi(x_n) = 1$  and  $\varphi(x_{n-2}) \ge 3$ , then  $\varphi(x_{n-1}) = 2$ .
- 12. If  $G = P_n$  and  $\prod_{i=q}^{q+1}(x_i)$  is a section of G of type (3+,3+), then  $n \ge 6$ and  $q \in [3, n-3]$ .
- 13. If  $G = P_n$  and  $\prod_{i=q}^{q+2}(x_i)$  is a section of G of type (3+, 1, 3+), then  $n \ge 7$  and  $q \in [3, n-4]$ .
- 14. If  $G = P_n$  and  $\prod_{i=q}^{q+2}(x_i)$  is a section of G of type (3+,3+,2), then  $n \ge 7$  and  $q \in [3, n-4]$ .
- 15. If  $G = P_n$  and  $\prod_{i=q}^{q+2}(x_i)$  is a section of G of type (2,3+,3+), then  $n \ge 7$  and  $q \in [3, n-4]$ .

**Proof.** 1. The existence of k follows immediately from the definition of a ranking. As concerns the existence of j, we may suppose that  $\min\{\varphi(x_q), \varphi(x_{q+3})\} \geq 2$  – otherwise we are done. Let  $x_j$  be that vertex from among  $x_{q+1}, x_{q+2}$ , which comes sooner in Y. Then, clearly,  $\varphi(x_j) = 1$ .

2. Suppose that  $\varphi(x_q) \geq 3$  and  $\varphi(x_{q+2}) \geq 3$ . We have  $\varphi(x_{q+1}) \neq 1$ , hence the colour 1 is forbidden for  $x_{q+1}$  because of an  $(x_s, x_t)$ -path with  $\varphi(x_s) = \varphi(x_t) = a$  containing  $x_{q+1}$  as an internal vertex. Clearly,  $\min\{\varphi(x_s), \varphi(x_t)\} \geq 3$  implies  $a \geq 3$ . Then, however, the colour 2 is forbidden for  $x_{q+1}$ , too, a contradiction. 3. The inequality  $n \ge 2$  is immediate. Also, we cannot have  $\varphi(x_2) \ge 2$ , because then  $\varphi(x_1) = 1$ .

4. Since  $\varphi$  uses each colour from  $[1, \max(\mathcal{G}, G, Y)]$  at least once, we have  $n \geq 3$ . From 3 we know that  $\varphi(x_2) = 1$ . The assumption  $\varphi(x_3) \geq 3$  then would lead to  $\varphi(x_1) = 2$ .

5,6. The situation is symmetric with that of 3 and 4.

7. Since, clearly,  $n \ge 5$  (1 and 2 are used at least once), the reduction process applies and yields  $\dot{G} = R(G, Y), \dot{Y} = R(Y, G), \dot{\varphi} = \operatorname{rank}(\mathcal{G}, \dot{G}, \dot{Y}).$ 

If  $G = C_n$ , then, by 1,  $\prod_{i=q-1}^{q+3}(x_i)$  is a section of G of type (1,3+, 3+, 3+, 1), hence  $n \ge 6$  ( $\varphi$  as a ranking is a proper vertex colouring of G). If  $n \ge 7$ , then, as in the case  $G = P_n$ , we conclude that S is a section of G of type (2, 1, 3+, 3+, 3+, 1, 2). If n = 6,  $\prod_{i=q-2}^{q+3}(x_i)$  would be a section of G of type (2, 1, 3+, 3+, 3+, 1). Then, however,  $\dot{G} = C_4$  and  $\dot{\varphi} = \operatorname{rank}(\mathcal{G}, C_4, \dot{Y})$  uses 1 exactly once in contradiction with the following fact (which can be easily checked out):

(\*) For any input sequence  $\bar{Y} \in \text{Is}(C_4)$  the ranking  $\text{rank}(\mathcal{G}, C_4, \bar{Y})$  uses 1 exactly twice.

8. As in 7, we use the reduction process leading to  $\dot{G}, \dot{Y}$  and  $\dot{\varphi}$ . In the case  $G = P_n$ , we obtain from 4 and 6 that  $\prod_{i=q-1}^{q+4}(x_i)$  is a section of G. Clearly, because of 7, we have  $\varphi(x_{q-1}) \leq 2$ . Then, the assumption q = 2 would mean  $\varphi(x_q) \leq 2$ , a contradiction. Thus,  $q \geq 3$ . Suppose that  $\varphi(x_{q-1}) = 2$ . If  $x_q$  comes in Y before  $x_{q+1}$ , then  $\varphi(x_q) = 1$ , and, if  $x_{q+1}$ comes in Y before  $x_q$ , then  $\varphi(x_{q+1}) \leq 2$ , in both cases a contradiction. Thus,  $\varphi(x_{q-1}) = 1$ ; we cannot have  $\varphi(x_{q-2}) \geq 3$ , because in such a case, by Lemma 3,  $(x_{q-2}, x_q, x_{q+1}, x_{q+3})$  would be a section of  $\dot{G}$  contradicting 1 (applied to  $\dot{\varphi}$ ). The mentioned contradiction yields  $\varphi(x_{q-2}) = 2$ . If  $\varphi(x_{q+4}) \geq 3$ , considering the section  $(x_q, x_{q+1}, x_{q+3}, x_{q+4})$  of  $\dot{G}$  supplies an analogous contradiction. So, there are two possibilities for  $\varphi(x_{q+4})$ : If  $\varphi(x_{q+4}) = 1$ , then  $n \geq q + 5$ , as n = q + 4 would imply  $\varphi(x_{q+3}) = 2$ , a contradiction; then, by 1 applied to  $\dot{\varphi}$ , we get  $\dot{\varphi}(x_{q+5}) = 1$  and  $\varphi(x_{q+5}) = 2$ . The assumption  $\varphi(x_{q+4}) = 2$  excludes n = q+4, by 5. Then, by 2,  $\varphi(x_{q+5}) \geq 3$  is impossible and  $\varphi(x_{q+5}) = 1$ , as necessary.

Now, consider the case  $G = C_n$ . Since  $\varphi$  must use 2, we have  $n \ge 5$ . However, n = 5 is impossible, because then  $\dot{\varphi}$  would contradict (\*). Thus,  $n \ge 6$  and, just as in the case  $G = P_n$ , we can show that  $\varphi(x_{q-1}) = 1$ and  $\varphi(x_{q-2}) = 2$ . That is why, n = 6 is impossible – use again (\*) for  $\dot{\varphi}$ . We cannot have  $\varphi(x_{q+4}) \ge 3$  from the same reason as applied for  $G = P_n$ . Then the assumption n = 7 would lead to  $\varphi(x_{q+4}) = 1$  ( $\varphi$  is proper) and a contradiction involving once more (\*) for  $\dot{\varphi}$ . Finally, for  $n \ge 8$ , the reasoning for  $G = P_n$  can be repeated, and we are done.

9. Use the symmetry with the situation of 8.

10,11. The proof is immediate.

12. From 4 we see that  $q \ge 2$ . If  $\varphi(x_{q-1}) \ge 2$ , from 3 we obtain  $q \ge 3$ . If  $\varphi(x_{q-1}) = 1$ , then  $q \ge 3$ , since q = 2 would lead to  $\varphi(x_q) = 2$ . Thus,  $q \ge 3$  in any case, and, because of the symmetry of the type (3+,3+), we have  $n \ge q+3$ , too.

13. The proof is analogous to that of 12.

14. By 5 we have  $n \ge q+3$ , so that 1 yields  $\varphi(x_{q+3}) = 1$ . Now, n = q+3 is impossible – this would mean that  $\varphi(x_{q+1}) = 1$ . To show that  $q \ge 3$ , proceed as in 12.

15. Symmetry with 14.

For a ranking algorithm  $\mathcal{A}$ , we will denote by  $f_i(\mathcal{A}, G, Y), i \in [1, \infty)$ , the number of vertices that are coloured with i by rank $(\mathcal{A}, G, Y)$ .

**Lemma 5.** Let  $G = C_n$ ,  $n \in [3, \infty)$ , or  $G = P_n$ ,  $n \in [1, \infty)$ , and let  $Y \in \text{Is}(G)$ . Then the sequence  $\{f_i(\mathcal{G}, G, Y)\}_{i=1}^{\infty}$  is non-increasing.

**Proof.** We proceed by induction on n. First, it is straightforward to see that  $f_1(\mathcal{G}, P_1, Y) = 1$  for (the unique)  $Y \in \text{Is}(P_1)$ ,  $f_i(\mathcal{G}, C_3, Y) = 1, i = 1, 2, 3$ , for any  $Y \in \text{Is}(C_3)$ , and  $f_1(\mathcal{G}, C_4, Y) = 2$  (in fact, this is (\*)),  $f_i(\mathcal{G}, C_4, Y) = 1, i = 2, 3$ , for any  $Y \in \text{Is}(C_4)$ .

Now, suppose that  $n \geq 5$  (if  $G = C_n$ ) or  $n \geq 2$  (if  $G = P_n$ ) and that  $\{f_i(\mathcal{G}, G', Y')\}_{i=1}^{\infty}$  is a non-increasing sequence for any graph G' homeomorphic to G with |V(G')| < n and any input sequence  $Y' \in \mathrm{Is}(G')$ . Let  $\varphi = \mathrm{rank}(\mathcal{G}, G, Y), \ \dot{G} = R(G, Y), \ \dot{Y} = R(Y, G), \ \dot{\varphi} = \mathrm{rank}(\mathcal{G}, \dot{G}, \dot{Y}).$ From Lemma 3 we know that, for any  $i \in [2, \infty)$ , we have  $f_{i-1}(\mathcal{G}, \dot{G}, \dot{Y}) = f_i(\mathcal{G}, G, Y)$  and, since  $|V(\dot{G})| < n$  (there are non-survivors of G with respect to Y, because  $\varphi$  uses 1 at least once), from the induction hypothesis we obtain  $f_i(\mathcal{G}, G, Y) = f_{i-1}(\mathcal{G}, \dot{G}, \dot{Y}) \geq f_i(\mathcal{G}, \dot{G}, \dot{Y}) = f_{i+1}(\mathcal{G}, G, Y).$  Put  $V_i = \{x \in V(G) : \varphi(x) = i\}, i = 1, 2$ , and consider a mapping  $\alpha : V_2 \rightarrow V_1$  defined in such a way that  $x\alpha(x)$  is an edge of G for any  $x \in V_2$ . From Lemmas 4.2, 4.3 and 4.5 it follows that  $\alpha$  is well defined. Moreover, the definition of a ranking implies that  $\alpha$  is an injection; thus,  $f_1(\mathcal{G}, G, Y) = |V_1| \geq |V_2| = f_2(\mathcal{G}, G, Y)$ , which represents the last wanted inequality.  $\blacksquare$ Suppose that  $G \in \{C_n, P_n\}, n \in [4, \infty)$  and let  $\tilde{G}$  be the cycle defined as follows:  $\tilde{G} = G$  if  $G = C_n$ ,  $\tilde{G} = G + x_n x_1$  if  $G = P_n$ . The ranking  $\varphi$  of Gis then also a vertex colouring of  $\tilde{G}$ , which, if  $G = P_n$ , in general is *not* a ranking of  $\tilde{G}$  (it may be even not proper). When working with  $\tilde{G}$ , types of vertices will be always related to this colouring "inherited" from the ranking  $\varphi$  of the "underlying" graph G. With respect to this colouring we define also high and low sections of  $\tilde{G}$ .

By Lemma 4.1, rotating around G we meet alternately high and low sections; their possible lengths are between 1 and 3 if  $G = C_n$ , and between 1 and 6 if  $G = P_n$  (and in this case, due to Lemmas 4.4 and 4.6, only one section, namely low, obtained by joining two low endsections of  $P_n$ , can be of length greater than 3). Let s be the number of high (and low as well) sections of G. We will denote those sections  $S_i, i \in [1, 2s]$ , in such a way that  $S_1$  is that high section of maximum length which contains a vertex  $x_t$ with minimum index t. Consider a (high) section  $S_{2i-1}$ ,  $i \in [1, s]$ . Starting from it and rotating around G in the sense of the orientation of G given by the growing order of sections indices (modulo 2s) we take all sections until we arrive at the first high section not shorter than  $S_{2i-1}$  (maybe  $S_{2i-1}$ itself). The section which arises by the concatenation of those sections (in their natural "rotating" order) is called the *closure* of  $S_{2i-1}$  and is denoted by  $cl(S_{2i-1})$ . Thus,  $cl(S_{2i-1}) = \prod_{k=2i-1}^{2j} S_k$ , where  $j \in [i, s]$  is (uniquely) chosen to fulfill the conditions  $|S_{2k-1}| < |S_{2i-1}|$  for each  $k \in [i+1,j]$  and  $|S_{2j+1}| \ge |S_{2i-1}|$  (note that  $j \le s$  because  $S_1$  is the longest high section).

In our example we have  $S_1 = (x_4, x_5), cl(S_1) = S_1S_2 = (x_4, x_5, x_6, x_7),$  $S_3 = (x_1), cl(S_3) = S_3S_4 = (x_1, x_2, x_3)$  (for  $G = C_7$ ) and  $S_1 = (x_4, x_5),$  $cl(S_1) = S_1S_2 = (x_4, x_5, x_6, x_7, x_1, x_2, x_3)$  (for  $G = P_7$ ).

**Lemma 6.** The closure of any high section of  $\tilde{G}$  has a nonnegative defect. **Proof.** Let  $S_{2i-1}$  be a high section of  $\tilde{G}$  and suppose that  $cl(S_{2i-1}) = \prod_{k=2i-1}^{2j} S_k$ .

 $\begin{array}{l} \prod_{k=2i-1} S_k \\ 1. \text{ If } |S_{2i-1}| = 1, \text{ then } \operatorname{cl}(S_{2i-1}) = S_{2i-1}S_{2i} \text{ and } \operatorname{def}(\operatorname{cl}(S_{2i-1})) = |S_{2i}| - 1 \ge 0. \end{array}$ 

2. Assume that  $|S_{2i-1}| = 2$ . Evidently, we have  $def(cl(S_{2i-1})) = def(S_{2i-1}S_{2i}) + \sum_{k=i+1}^{j} def(S_{2k-1}S_{2k})$ . Since  $2 = |S_{2i-1}| > |S_{2k-1}| = 1$ 

for each  $k \in [i+1, j]$ , the sum consists of nonnegative summands  $|S_{2k}| - 1$ . Thus, we are done if  $def(S_{2i-1}S_{2i}) \ge 0$ .

If def $(S_{2i-1}S_{2i}) = |S_{2i}| - |S_{2i-1}| < 0$ , then, necessarily,  $|S_{2i}| = 1$ . From Lemmas 4.2, 4.3 and 4.5 we then see that  $S_{2i}$  is of type (1). Suppose that  $S_{2i-1}S_{2i} = \prod_{k=q}^{q+2}(x_k), q \in [1, n]$ , and consider the section  $S = \prod_{k=q}^{q+3}(x_k)$  of  $\tilde{G}$ of type (3+, 3+, 1, 3+). If S is also a section of G, then, by Lemma 4.8,  $S_{2i+1}$ is of length 1 (so that  $j \geq i+1$ ) and def $(S_{2i+1}S_{2i+2}) \geq 1$ , which implies def $(\operatorname{cl}(S_{2i-1})) \geq -1+1+\sum_{k=i+2}^{j}(|S_{2k}|-1) \geq 0$ . If S is not a section of G, then  $G = P_n$  and  $n \in [q, q+2]$ . However, n = q is impossible by Lemma 4.4, n = q + 1 by Lemma 4.5 and n = q + 2 by Lemma 4.11.

3. Now, let  $|S_{2i-1}| = 3$ . First we show that, for any  $l \in [i, j]$ , we have  $d_l = def(\prod_{k=2i-1}^{2l} S_k) \geq -1$ , and, if  $d_k = -1$  for every  $k \in [i, l]$ , then either  $S_{2l}$  is of type (1,2) or  $S_{2l-1}S_{2l}$  is of type (3+,1). We proceed by induction on l. If l = i and  $S_{2i-1} = \prod_{k=q}^{q+2} (x_k)$  with  $q \in [1, n]$ , we know that  $S_{2i-1}$  is a section of G (otherwise  $G = P_n$  and  $n \in [q, q+1]$ , which contradicts Lemma 4.3 or Lemma 4.5). Thus, we can use Lemma 4.7, from which it follows that  $d_i \geq -1$  and  $d_i = -1$  only if  $S_{2i}$  is of type (1,2).

Suppose that j > i and that our statement is true for some  $l \in [i, j-1]$ (so that  $|S_{2l+1}| \leq 2$ ). Since  $d_{l+1} = d_l + |S_{2l+2}| - |S_{2l+1}| \geq d_l + 1 - 2 = d_l - 1$ , to prove the statement for l+1 it is sufficient to analyze the case  $d_l = -1$ . (If  $d_l \geq 0$ , then  $d_{l+1} \geq -1$  and it is not true that  $d_k = -1$  for any  $k \in [i, l+1]$ .) By the induction hypothesis, we have two possibilities:

a)  $S_{2l} = \prod_{k=q}^{q+1}(x_k)$ , where  $q \in [1, n]$ , is of type (1,2). If  $|S_{2l+1}| = 2$ , then  $\prod_{k=q}^{q+5}(x_k)$  is the section of the graph G ( $G = P_n$  and  $n \in [q, q+4]$ would be in contradiction with one of Lemmas 4.3, 4.5 and 4.11) and  $S_{2l+2}$ is neither of type (1,1) nor of type (2,2) (this would mean  $G = P_n$  and n = q + 4). Next, by Lemma 4.1,  $S_{2l+2}$  cannot be of type (2) or (2,1), and, by Lemma 4.8, of type (1); thus, either  $d_{l+1} = d_l = -1$  and  $S_{2l+2}$  is of type (1,2) (as necessary) or  $d_{l+1} \ge 0$  (and there is nothing more to prove). Let  $|S_{2l+1}| = 1$ . The only interesting case (in which  $d_{l+1} = -1$ ) is that with  $|S_{2l+2}| = 1$ . Then, because of Lemma 4.2 or 4.5,  $S_{2l+2}$  is not of type (2), and, consequently,  $S_{2l+1}S_{2l+2}$  is of type (3+, 1), as needed.

b)  $S_{2l-1}S_{2l} = (x_q, x_{q+1})$ , where  $q \in [1, n]$ , is of type (3+,1). If  $|S_{2l+1}| = 2$ , then  $\prod_{k=q}^{q+3}(x_k)$  is the section of the graph G  $(G = P_n \text{ and } n \in [q, q+2]$  would be in contradiction with one of Lemmas 4.3, 4.6 and 4.10). Then, by Lemma 4.9,  $\varphi(x_{q+4}) = 1$  and  $\varphi(x_{q+5}) = 2$ , so that either  $d_{l+1} = -1$  and  $S_{2l+2}$  is of type (1,2) or  $d_{l+1} = 0$ ; in both cases we are done. Suppose  $|S_{2l+1}| = 1$ . It is sufficient to deal with the case  $d_{l+1} = -1$ , in which

 $|S_{2l+2}| = 1$ . If  $S_{2l+1}S_{2l+2}$  is of type (3+,1), we are done. On the other hand, by Lemmas 4.2 and 4.5,  $S_{2l+2}$  cannot be of type (2) and our statement is completely proved.

Now, it is clear that we cannot have  $d_k = -1$  for each  $k \in [i, j]$ , because  $|S_{2j+1}| = 3$  and, by Lemma 4.7, the type of  $S_{2j}$  ends up with (2,1). Thus, there exists (uniquely determined)  $l \in [i, j]$  fulfilling  $d_l \ge 0$  and  $d_k = -1$  for any  $k \in [i, l-1]$ . If l = j, then def(cl $(S_{2i-1})) = d_l \ge 0$ . Suppose therefore l < j. If  $|S_{2k-1}| = 1$  for any  $k \in [l+1, j]$ , then def(cl $(S_{2i-1})) = d_l + \sum_{k=l+1}^{j} (|S_{2k}| - 1) \ge 0$ . If  $|S_{2m-1}| = 2$  for some  $m \in [l+1, j]$  and  $|S_{2k-1}| = 1$  for any  $k \in [l+1, m-1]$ , delete from the sequence  $\prod_{k=m}^{j} (2k-1)$  all terms 2k-1 with  $|S_{2k-1}| = 1$  and denote by  $\prod_{k=1}^{q} (p_k)$  the resulting sequence. Then it is easy to see directly from the definitions that  $\prod_{k=2m-1}^{2j} S_k = \prod_{k=1}^{q} \operatorname{cl}(S_{p_k})$  and, as  $S_{p_k}$  is a high section of length 2, by 2 we have def(cl $(S_{p_k})) \ge 0$  for each  $k \in [1, q]$ . That is why, def(cl $(S_{2i-1})) = d_l + \sum_{k=l+1}^{m-1} (|S_{2k}| - 1) + \sum_{k=1}^{q} \operatorname{def}(\operatorname{cl}(S_{p_k})) \ge 0$ .

**Theorem 7.** Let  $G = C_n$ ,  $n \in [3, \infty)$ , or  $G = P_n$ ,  $n \in [1, \infty)$ , and let  $Y \in \text{Is}(G)$ . Then  $\sum_{i=1}^2 f_i(\mathcal{G}, G, Y) \ge \lceil n/2 \rceil$  and  $f_1(\mathcal{G}, G, Y) \ge \lceil n/2 \rceil/2 \rceil$ .

**Proof.** The assertion is immediate if  $n \leq 3$ . If  $n \in [4, \infty)$ , consider the graph  $\tilde{G}$  and its high and low sections  $S_i, i \in [1, 2s]$ , as defined before Lemma 6. Let  $\prod_{i=1}^{m}(l_i)$  be the increasing sequence of indices of all longest high sections of  $\tilde{G}$ . Then, obviously, the section  $\prod_{i=1}^{m} \operatorname{cl}(S_{l_i})$  contains all vertices of  $V(\tilde{G}) = V(G)$ , and so, by Lemma 6,  $\sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) - \sum_{i=3}^{\infty} f_i(\mathcal{G}, G, Y) = \operatorname{def}(G) = \operatorname{def}(\prod_{i=1}^{m} \operatorname{cl}(S_{l_i})) = \sum_{i=1}^{m} \operatorname{def}(\operatorname{cl}(S_{l_i})) \geq 0$ . Thus, we have  $n = \sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) + \sum_{i=3}^{\infty} f_i(\mathcal{G}, G, Y) \leq 2 \sum_{i=1}^{2} f_i(\mathcal{G}, G, Y)$ and the first inequality follows. The remaining one comes from Lemma 5, since  $2f_1(\mathcal{G}, G, Y) \geq \sum_{i=1}^{2} f_i(\mathcal{G}, G, Y) \geq \lfloor n/2 \rfloor$ .

**Proposition 8.** If  $k \in [1, \infty)$  and  $l \in [3, \infty)$ , there exist  $q \in [1, \infty)$  and  $r \in [3, \infty)$  such that  $\max(\mathcal{G}, P_q) = k$  and  $\max(\mathcal{G}, C_r) = l$ .

**Proof.** Suppose that there is no  $q \in [1, \infty)$  such that  $\max(\mathcal{G}, P_q) = k$ . Since, evidently,  $\max(\mathcal{G}, P_n) = n, n = 1, 2$ , we have  $k \ge 3$ . The sequence  $\{\chi_r(P_n)\}_{n=1}^{\infty} = \{\lfloor \log_2 n \rfloor + 1\}_{n=1}^{\infty}$  is unbounded and  $\max(\mathcal{G}, P_n) \ge \chi_r^*(P_n) \ge \chi_r(P_n)$ , hence there exists  $q \in [1, \infty)$  such that  $\max(\mathcal{G}, P_q) \ge k+1$ ; without loss of generality, we may suppose that q is minimum with this property, i.e.,  $\max(\mathcal{G}, P_n) \le k-1$  for any  $n \in [1, q-1]$ . Consider such an input sequence  $Y \in \operatorname{Is}(P_q)$  that  $\max(\mathcal{G}, P_q, Y) = \max(\mathcal{G}, P_q)$ . Clearly,  $q \ge k+1 \ge 4$ , so we may use our reduction process yielding  $\dot{G} = R(P_q, Y), \ \dot{Y} = R(Y, P_q)$ . We have  $|V(\dot{G})| < q$ , which implies  $\max(\mathcal{G}, \dot{G}) \le k - 1$ . On the other hand, by Lemma 3, the maximum number used by  $\dot{\varphi}$  is by 1 smaller than that used by  $\varphi$ , i.e.,  $\max(\mathcal{G}, \dot{G}, \dot{Y}) = \max(\mathcal{G}, P_q, Y) - 1 = \max(\mathcal{G}, P_q) - 1 \ge (k+1) - 1 = k$ , hence  $\max(\mathcal{G}, \dot{G}) \ge \max(\mathcal{G}, \dot{G}, \dot{Y}) \ge k$ , a contradiction.

For cycles we proceed analogously using the fact that  $\max(\mathcal{G}, C_3) = 3$ and that the reduction process applies if the number of vertices of  $C_n$  is at least 5. Note that also the sequence  $\{\chi_r(C_n)\}_{n=1}^{\infty}$  is unbounded, because  $P_{n-1}$  is an induced subgraph of  $C_n$ , and so (as can be easily seen)  $\chi_r(P_{n-1}) \leq \chi_r(C_n)$  for any  $n \in [3, \infty)$ .

From Proposition 8 we conclude that the numbers

$$f(k) := \min\{n \in [1, \infty) : \max(\mathcal{G}, P_n) = k\}, \quad k \in [1, \infty),$$
$$g(k) := \min\{n \in [3, \infty) : \max(\mathcal{G}, C_n) = k\}, \quad k \in [3, \infty)$$

(f(k)) was introduced in [5]) are correctly defined. It is easily seen that f(k) = k for k = 1, 2, 3 and g(3) = 3. Clearly, from Lemma 3 it follows that  $f(k) \neq f(l)$  and  $g(k) \neq g(l)$  for  $k \neq l$ . However, we can say more:

**Proposition 9.** The sequences  $\{f(k)\}_{k=1}^{\infty}$  and  $\{g(k)\}_{k=3}^{\infty}$  are increasing.

**Proof.** In the case of paths use simply Proposition 1 and the fact that  $P_m$  is an induced subgraph of  $P_n$  if m < n.

For cycles suppose that  $\{h(k)\}_{k=3}^{\infty}$  is the increasing sequence created by rearranging  $\{g(k)\}_{k=3}^{\infty}$ , that  $\{h(k)\} \neq \{g(k)\}$  and that k is the minimum index with  $h(k) \neq g(k)$ . Since g(3) = h(3) = 3, we have  $k \geq 4$  and h(k) =g(l) < g(k) with k < l. For n = g(l) take an input sequence  $Y \in Is(C_n)$ fulfilling max $(\mathcal{G}, C_n, Y) = l$ . As  $l \geq 5$ ,  $\dot{G} = R(C_n, Y)$  and  $\dot{Y} = R(Y, C_n)$ are well defined. Then, by Lemma 3, max $(\mathcal{G}, \dot{G}, \dot{Y}) = max(\mathcal{G}, C_n, Y) - 1 =$  $l - 1 \geq k$  and, since  $|V(\dot{G})| < |V(C_n)| = g(l)$ , we have  $g(l - 1) \leq |V(\dot{G})| <$ g(l) < g(k) and l - 1 > k. Now, g(l - 1) > g(k - 1) is in contradiction with h(k) = g(l) and g(l - 1) < g(k - 1) contradicts the minimality of k.

**Corollary 10.** For any  $k, n \in [1, \infty)$  it holds  $\max(\mathcal{G}, P_n) = k$  if and only if  $n \in [f(k), f(k+1) - 1]$ .

**Proof.** A consequence of Propositions 1 and 9.

For cycles the situation is unclear, but we conjecture that, analogously, for any  $k, n \in [3, \infty)$ ,  $\max(\mathcal{G}, C_n) = k$  if and only if  $n \in [g(k), g(k+1) - 1]$ .

Theorem 7 has an important consequence:

**Theorem 11.** Let  $k \in [1, \infty), l \in [3, \infty), q \in [2, \infty)$  and  $r \in [7, \infty)$ .

- 1. If  $f(k) \ge q$ , then  $f(k+2i) \ge q \cdot 2^i$  for any  $i \in [0,\infty)$ .
- 2. If  $g(k) \ge r$ , then  $g(k+2i) \ge r \cdot 2^i$  for any  $i \in [0,\infty)$ .

**Proof.** 1. We proceed by induction on *i*. For i = 0 there is nothing to prove, so we suppose that  $i \in [1, \infty)$  and  $f(k + 2i - 2) \ge q \cdot 2^{i-1}$ . With respect to Proposition 9 it is sufficient to show that  $\max(\mathcal{G}, P_n, Y) \le k + 2i - 1$  for any  $n \in [q \cdot 2^{i-1} + 2, q \cdot 2^i - 1]$  and any  $Y \in \operatorname{Is}(P_n)$ . Since  $n \ge q \cdot 2^{i-1} + 2 \ge q + 2 \ge 4$ , the reduction process applied to  $P_n$  and Y yields  $\dot{G} = R(P_n, Y)$  and  $\dot{Y} = R(Y, P_n)$ . The ranking  $\operatorname{rank}(\mathcal{G}, P_n, Y)$  is a proper vertex colouring of  $P_n$ , hence  $f_1(\mathcal{G}, P_n, Y) \le \lceil n/2 \rceil$ ,  $|V(\dot{G})| = n - f_1(\mathcal{G}, P_n, Y) \ge n - \lceil n/2 \rceil = \lfloor n/2 \rfloor \ge 2$ , so that the reduction process applied to  $\dot{G}$  and  $\dot{Y}$  leads to  $\ddot{G} = R(\dot{G}, \dot{Y})$  and  $\ddot{Y} = R(\dot{Y}, \dot{G})$ . By a repeated use of Lemma 3 we see that  $|V(\ddot{G})| = n - \sum_{i=1}^2 f_i(\mathcal{G}, P_n, Y)$ , hence, by Theorem 7,  $|V(\ddot{G})| \le n - \lceil n/2 \rceil = \lfloor n/2 \rfloor \le q \cdot 2^{i-1} - 1$ , and, by the induction hypothesis,  $\max(\mathcal{G}, \ddot{G}, \ddot{Y}) \le \max(\mathcal{G}, \ddot{G}) \le k + 2i - 3$ . Using Lemma 3 twice then  $\max(\mathcal{G}, P_n, Y) = \max(\mathcal{G}, \dot{G}, \dot{Y}) + 1 = \max(\mathcal{G}, \ddot{G}, \ddot{Y}) + 2 \le k + 2i - 1$ , as needed.

2. We proceed as in 1 and use the fact that  $f_1(\mathcal{G}, C_n, Y) \leq \lfloor n/2 \rfloor$ , so that  $|V(R(C_n, Y))| \geq n - \lfloor n/2 \rfloor = \lceil n/2 \rceil \geq 5$  for any  $n \in [r \cdot 2^{i-1} + 2, r \cdot 2^i - 1], i \in [1, \infty)$  and any  $Y \in \mathrm{Is}(C_n)$ , which enables us to use the reduction process twice, as above.

### **3** INSERTION

Now we are going to show that, in some extent, our reduction process can be inverted. Let  $\mathcal{A}_{m,n}, n \in [1, \infty), m \in [0, n]$ , be the set of all non-empty increasing sequences of integers from [m, n].

We will analyze in detail the case  $G = P_n$ . For  $A = \prod_{i=1}^l (a_i) \in \mathcal{A}_{0,n}$  we denote by  $I(P_n, A)$  the path with n + l vertices constructed as follows: Add to  $V(P_n) = \{x_i : i \in [1, n]\}$  *l* new vertices (called *newcomers*)  $z_i, i \in [1, l]$ . If  $i \in [1, l]$  is such that  $a_i \in [1, n - 1]$ , the newcomer  $z_i$  is inserted between vertices  $x_{a_i}$  and  $x_{a_i+1}$  (i.e., the edge  $x_{a_i}x_{a_i+1}$  is deleted and edges  $x_{a_i}z_i$  and  $z_ix_{a_i+1}$  are added). If  $a_1 = 0$ , the newcomer  $z_1$  is a new endvertex – the edge  $z_1x_1$  is added. Similarly, if  $a_l = n$ , the newcomer  $z_l$  is a new endvertex – the edge  $x_nz_l$  is added. Note that the set of newcomers is an independent set of vertices of  $I(P_n, A)$ . An input sequence  $Y \in Is(P_n)$  for the path  $P_n$ yields in a natural way an input sequence  $I(P_n, A, Y) = [\prod_{i=1}^l (z_i)]Y$  for the path  $I(P_n, A)$  – newcomers are coming first  $(z_i \text{ comes as } i\text{-th}, i \in [1, l])$  and then vertices of  $P_n$  arrive in the order given by Y. Consider the ranking  $\varphi = \operatorname{rank}(\mathcal{G}, P_n, Y)$ . An internal vertex  $x_i$  of  $P_n$ ,  $i \in [2, n-1]$ , is Y-good, if it comes in Y as the last from among  $x_{i-1}, x_i, x_{i+1}$ , and  $\varphi(x_{i-1}) = \varphi(x_{i+1})$ . A sequence  $A \in \mathcal{A}_{0,n}$  is Y-proper, if any vertex of  $P_n$ , that is not Y-good, has in  $I(P_n, A)$  at least one newcomer as a neighbour.

For example, if Y is the input sequence  $(x_3, x_2, x_5, x_6, x_4, x_1) \in Is(P_6)$ , there is only one Y-good vertex in  $P_6$ , namely  $x_4$  – we have rank $(\mathcal{G}, P_6, Y) =$  $\{(x_3, 1), (x_2, 2), (x_5, 1), (x_6, 2), (x_4, 3), (x_1, 1)\}$  ( $x_2$  is not Y-good, because it comes in Y before  $x_1$ ). Thus, the sequence  $A = (1, 2, 5) \in \mathcal{A}_{0,6}$  is Yproper – vertices  $x_i$ ,  $i \in [1, 6] - \{5\}$ , that are not Y-good, are "dominated" by newcomers of the graph  $I(P_6, A) = P_9$  (its vertices are successively  $x_1, z_1, x_2, z_2, x_3, x_4, x_5, z_3, x_6$ ). The input sequence  $I(P_6, A, Y)$  is  $(z_1, z_2, z_3, x_3, x_2, x_5, x_6, x_4, x_1)$ .

**Lemma 12.** Let  $n \in [1, \infty), Y \in \text{Is}(P_n)$ , let a sequence  $A \in \mathcal{A}_{0,n}$  be Yproper and let  $\varphi = \text{rank}(\mathcal{G}, P_n, Y)$ ,  $\hat{G} = I(P_n, A)$ ,  $\hat{Y} = I(P_n, A, Y), \hat{\varphi} = \text{rank}(\mathcal{G}, \hat{G}, \hat{Y})$ . Then  $\hat{\varphi}(z_i) = 1$  for any newcomer  $z_i, i \in [1, |A|]$ , and  $\hat{\varphi}(x_i) = \varphi(x_i) + 1$  for any  $i \in [1, n]$ .

**Proof.** Newcomers of the graph  $\hat{G}$  are attributed 1 by  $\hat{\varphi}$  because they form an independent set of vertices in  $\hat{G}$  and they are coming at the beginning of  $\hat{Y}$ , before all remaining vertices of  $\hat{G}$ .

Let us prove by induction on *i* that  $\hat{\varphi}(y_i) = \varphi(y_i) + 1$  for every  $i \in [1, n]$ . The vertex  $y_1$ , clearly, is not Y-good, hence it has at least one newcomer as a neighbour and  $\hat{\varphi}(y_1) = 2 = \varphi(y_1) + 1$ .

Suppose that  $i \in [2, n]$  and that  $\hat{\varphi}(y_j) = \varphi(y_j) + 1$  for any  $j \in [1, i - 1]$ . Vertices  $y_j, y_k$  with  $j, k \in [1, i], j \neq k$ , are joined by a path  $\hat{P}$  in  $\hat{G}(\hat{Y}, y_i)$  if and only if they are joined in  $G(Y, y_i)$  by the path P with  $V(P) = V(\hat{P}) - \{z_l : l \in [1, |A|]\}$ . Since  $\hat{\varphi}(z_l) = 1$  for any  $l \in [1, |A|]$ , using the induction hypothesis we see that a colour  $a \in [2, \infty)$  is forbidden for  $y_i$  in  $\hat{G}(\hat{Y}, y_i)$ because of a path  $\hat{P}$  if and only if the colour a-1 is forbidden for  $y_i$  in  $G(Y, y_i)$ because of the corresponding path P. Moreover, the colour 1 is forbidden for  $y_i$  in  $\hat{G}(\hat{Y}, y_i)$ , too – either a neighbour of  $y_i$  is a newcomer (and so is coloured with 1 in  $\hat{G}(\hat{Y}, y_i)$ ) or both neighbours of  $y_i$  are coloured in  $\hat{G}(\hat{Y}, y_i)$ and they received the same colour. This means that  $\varphi(y_i) = \hat{\varphi}(y_i) - 1$  and we are done.

In our illustrative example with 
$$n = 6$$
 we have  $\hat{\varphi} = \operatorname{rank}(\mathcal{G}, P_9, I(P_6, A, Y))$   
= { $(z_1, 1), (z_2, 1), (z_3, 1), (x_3, 2), (x_2, 3), (x_5, 2), (x_6, 3), (x_4, 4), (x_1, 2)$ }.  
Put  $e_l := 3 \cdot 2^{l-1} - 1$  and  $o_l := 2^{l+1} - 1, l \in [1, \infty)$ .

**Theorem 13.** For any  $l \in [1, \infty)$  there exists

- 1. an input sequence  $Y_{2l} \in \text{Is}(P_{e_l})$  such that  $\max(\mathcal{G}, P_{e_l}, Y_{2l}) = 2l$  and the set of  $Y_{2l}$ -good vertices of the path  $P_{e_l}$  is  $\{x_{3i} : i \in [1, 2^{l-1} 1]\};$
- 2. an input sequence  $Y_{2l+1} \in \text{Is}(P_{o_l})$  such that  $\max(\mathcal{G}, P_{o_l}, Y_{2l+1}) = 2l + 1$ and the set of  $Y_{2l+1}$ -good vertices of the path  $P_{o_l}$  is  $\{x_{4i} : i \in [1, 2^{l-1}-1]\}$ .

**Proof.** Evidently, for l = 1 any input sequence  $Y_2 \in Is(P_2)$  has all the properties required by 1 (no vertex of  $P_2$  is  $Y_2$ -good). We are going to show that for any  $l \in [1, \infty)$  the existence of  $Y_{2l}$  implies that of  $Y_{2l+1}$  and the existence of  $Y_{2l+1}$  implies that of  $Y_{2l+2}$ . So, suppose that there is an input sequence  $Y_{2l} \in \operatorname{Is}(P_{e_l})$  with properties given by 1. The sequence  $A_{2l} := \prod_{i=1}^{2^{l-1}} (3i-2) \in \mathcal{A}_{0,e_l}$  is  $Y_{2l}$ -proper – note that vertices of  $P_{e_l}$ , that are not  $Y_{2l}$ -good, are in pairs  $x_{3i-2}, x_{3i-1}$ , and an "old" edge  $x_{3i-2}x_{3i-1}$  is subdivided by the newcomer  $z_i, i \in [1, 2^{l-1}]$ . The graph  $I(P_{e_l}, A_{2l})$  is a path with  $e_l + 2^{l-1} = o_l$  vertices and, if we define  $Y_{2l+1} := I(P_{e_l}, A_{2l}, Y_{2l})$ , then, by Lemma 12,  $\max(\mathcal{G}, P_{o_l}, Y_{2l+1}) = \max(\mathcal{G}, P_{e_l}, Y_{2l}) + 1 = 2l + 1$ . Moreover, any  $Y_{2l}$ -good vertex  $x_{3i}, i \in [1, 2^{l-1} - 1]$ , is  $Y_{2l+1}$ -good. There are no other  $Y_{2l+1}$ -good vertices, because, by Lemma 12, any vertex of the path  $P_{e_l}$ , that is  $Y_{2l+1}$ -good and not  $Y_{2l}$ -good, must have two newcomers as neighbours (and the distance between any two newcomers in  $I(P_{o_l}, A_{2l})$  is at least 3). Now, if we rename vertices of  $I(P_{e_l}, A_{2l}) = P_{o_l}$  in our ordinary way (i.e., they will be  $x_i, i \in [1, o_l]$ , then  $x_{3i}$  becomes  $x_{4i}, i \in [1, 2^{l-1} - 1]$ , and the set of  $Y_{2l+1}$ -good vertices of  $P_{o_l}$  is  $\{x_{4i} : i \in [1, 2^{l-1} - 1]\}$ .

The sequence  $A_{2l+1} := \prod_{i=1}^{2^{l-1}} (4i - 3, 4i - 2) \in \mathcal{A}_{0,o_l}$  is  $Y_{2l+1}$ proper, because vertices of  $P_{o_l}$ , that are not  $Y_{2l}$ -good, occur in triples  $x_{4i-3}, x_{4i-2}, x_{4i-1}$ , which are "dominated" by newcomers  $z_{2i-1}$  and  $z_{2i}$ ,  $i \in [1, 2^{l-1}]$ . The graph  $I(P_{o_l}, A_{2l+1})$  is a path with  $o_l + 2 \cdot 2^{l-1} = e_{l+1}$  vertices and, for  $Y_{2l+2} := I(P_{o_l}, A_{2l+1}, Y_{2l+1})$ , we have, by Lemma 12,  $\max(\mathcal{G}, P_{e_{l+1}}, Y_{2l+2}) = \max(\mathcal{G}, P_{o_l}, Y_{2l+1}) + 1 = 2l + 2$ . Any  $Y_{2l+1}$ -good vertex  $x_{4i}, i \in [1, 2^{l-1}]$ , is  $Y_{2l+2}$ -good. Moreover, the vertex  $x_{4i-2}, i \in [1, 2^{l-1}]$ , is  $Y_{2l+2}$ -good vertices, because there are no more pairs of newcomers which are at the distance 2 apart. Thus, after renaming vertices of  $I(P_{o_l}, A_{2l+1}) = P_{e_{l+1}}$  in our ordinary way (so that  $x_{4i}$  becomes  $x_{6i}$ ,  $i \in [1, 2^{l-1} - 1]$ , and  $x_{4i-2}$  becomes  $x_{6i-3}, i \in [1, 2^{l-1}]$ ), the set of  $Y_{2l+2}$ -good vertices of  $P_{e_{l+1}}$  is  $\{x_{3i} : i \in [1, 2^l - 1]\}$ .

**Corollary 14.** For any  $l \in [1, \infty)$ ,  $f(2l) \leq e_l$  and  $f(2l+1) \leq o_l$ .

Evidently, the reduction process can also be (partially) inverted for cycles. In this case the sequence  $A = \prod_{i=1}^{l} (a_i)$ , characterizing positions of newcomers, is from the set  $\mathcal{A}_{1,n}$  (if the original cycle is  $C_n$ ), a newcomer  $z_i$  subdivides the edge  $x_{a_i}x_{a_i+1}$ ,  $i \in [1, l]$ , and there is no restriction on index of a Ygood vertex. (Recall that, for paths, endvertices are not Y-good.) Thus, an analogue of Lemma 12 is presented without proof (no new idea is necessary).

**Lemma 15.** Let  $n \in [3, \infty)$ ,  $Y \in \text{Is}(C_n)$ , let a sequence  $A \in \mathcal{A}_{1,n}$  be Y-proper and let  $\varphi = \text{rank}(\mathcal{G}, C_n, Y)$ ,  $\hat{G} = I(C_n, A)$ ,  $\hat{Y} = I(C_n, A, Y)$ ,  $\hat{\varphi} = \text{rank}(\mathcal{G}, \hat{G}, \hat{Y})$ . Then  $\hat{\varphi}(z_i) = 1$  for any newcomer  $z_i$ ,  $i \in [1, |A|]$  and  $\hat{\varphi}(x_i) = \varphi(x_i) + 1$  for any  $i \in [1, n]$ .

## 4 MAIN RESULTS

Now we are able to analyze First Fit Algorithm for cycles and paths in a detailed way.

**Proposition 16.**  $g(4) \le 5$ ,  $g(5) \le 7$ ,  $g(6) \le 10$  and  $g(7) \le 15$ .

**Proof.** It is easy to check that the sequences  $\hat{A}_3 = (1,2)$ ,  $\hat{A}_4 = (1,4)$ ,  $\hat{A}_5 = (2,5,7)$  and  $\hat{A}_6 = (1,3,5,7,9)$  are such that  $\hat{A}_n$  is  $\hat{Y}_n$ -proper,  $n \in [3,6]$ , if the graph  $\hat{G}_n$  and the input sequence  $\hat{Y}_n$  for  $\hat{G}_n$ ,  $n \in [3,7]$ , are defined by the following recurrence:  $\hat{G}_3 := C_3$ ,  $\hat{Y}_3 := (x_1, x_2, x_3)$  and  $\hat{G}_{n+1} := I(\hat{G}_n, \hat{A}_n)$ ,  $\hat{Y}_{n+1} := I(\hat{G}_n, \hat{A}_n, \hat{Y}_n)$ ,  $n \in [3,6]$ . Since  $\max(\mathcal{G}, \hat{G}_3, \hat{Y}_3) = 3$ ,  $\hat{G}_4 = C_5$ ,  $\hat{G}_5 = C_7$ ,  $\hat{G}_6 = C_{10}$ ,  $\hat{G}_7 = C_{15}$  and, by Lemma 15,  $\max(\mathcal{G}, \hat{G}_{n+1}, \hat{Y}_{n+1}) = \max(\mathcal{G}, \hat{G}_n, \hat{Y}_n) + 1$  for  $n \in [3,6]$ , the proof follows.

**Proposition 17.** If  $k \in [3, \infty)$ , then

1.  $f(k+1) \ge \min\{n \in [f(k)+1,\infty) : n - \lceil \lceil n/2 \rceil/2 \rceil \ge f(k)\};$ 

2.  $g(k+1) \ge \min\{n \in [g(k)+1,\infty) : n - \lceil \lceil n/2 \rceil/2 \rceil \ge g(k)\}.$ 

**Proof.** 1. Suppose that f(k+1) = n; by Proposition 9 then  $n \ge f(k) + 1$ . Take an input sequence  $Y \in \text{Is}(P_n)$  such that  $\max(\mathcal{G}, P_n, Y) = k + 1$ and put  $\dot{G} = R(P_n, Y)$ ,  $\dot{Y} = R(Y, P_n)$ . For the path  $\dot{G}$  we have, by Theorem 7,  $|V(\dot{G})| = n - f_1(\mathcal{G}, P_n, Y) \le n - \lceil \lceil n/2 \rceil/2 \rceil$ , and, by Lemma 3,  $\max(\mathcal{G}, \dot{G}, \dot{Y}) = \max(\mathcal{G}, P_n, Y) - 1 = k$ . Since  $|V(\dot{G})| < n = f(k+1)$ , due to Proposition 9 we obtain  $\max(\mathcal{G}, \dot{G}) = \max(\mathcal{G}, \dot{G}, \dot{Y}) = k$ . Thus,  $|V(\dot{G})| \ge f(k)$  and we see that  $n - \lceil \lceil n/2 \rceil/2 \rceil \ge f(k)$ .

2. The proof is completely analogous to that of 1.

**Theorem 18.**  $f(4) = g(4) = 5, f(5) = g(5) = 7, f(6) = 11, g(6) = 10, f(7) = 15 and <math>14 \le g(7) \le 15.$ 

**Proof.** Take  $k \in [4, 7]$ . The upper bounds for f(k) come from Corollary 14 and those for g(k) from Proposition 16. On the other hand, by Theorem 1 and Lemma 7 of [5],  $f(4) \ge 5$  and  $g(4) \ge 5$ , so that f(4) = g(4) = 5. Now, by Proposition 17,  $f(5) \ge 7$  and  $g(5) \ge 7$ , which implies f(5) = g(5) = 7. By Proposition 17 again, we get  $f(6) \ge 10$  and  $g(6) \ge 10$ , yielding g(6) = 10and, consequently,  $g(7) \ge 14$ .

Suppose that there is an input sequence  $Y \in \text{Is}(P_{10})$  such that  $\max(\mathcal{G}, P_{10}, Y) = 6$  and put  $\varphi = \operatorname{rank}(\mathcal{G}, P_{10}, Y)$ . Since f(4) = 5, from Lemma 3 (used twice) we see that  $\sum_{i=1}^{2} f_i(\mathcal{G}, P_{10}, Y) \leq 5$ . So, with help of Theorem 7,  $\sum_{i=1}^{2} f_i(\mathcal{G}, P_{10}, Y) = \sum_{i=3}^{6} f_i(\mathcal{G}, P_{10}, Y) = 5$ , and, by Lemma 5,  $f_1(\mathcal{G}, P_{10}, Y) = 3$ ,  $f_2(\mathcal{G}, P_{10}, Y) = 2$ . Consider the cycle  $\tilde{P}_{10} = C_{10}$  introduced before Lemma 6 and its high and low sections. First we show that there is no high section of  $\tilde{P}_{10}$  of length 3. Suppose there is one; by Lemmas 4.4 and 4.6, this section  $\prod_{i=q}^{q+2}(x_i)$  must also be a section of  $P_{10}$ . Then, by Lemma 4.7,  $\prod_{i=q-2}^{q+4}(x_i)$  is a section of  $P_{10}$  of type (2,1,3+,3+,3+,1,2). The remaining three vertices of  $P_{10}$  do not form a section of  $P_{10}$ , because two of them are high (otherwise we would obtain a contradiction with one of Lemmas 4.4, 4.6, 4.10 and 4.11). Thus, they form two nonempty end-sections of  $P_{10}$ . That containing only one vertex cannot be of type (3+)  $(P_{10}$  would have an endsection of type (3+,2) or (2,3+) in contradiction with Lemmas 4.4 and 4.6), hence that of length 2 is of type (3+,3+), which contradicts again Lemmas 4.4 and 4.6.

Denote the number of low sections of  $P_{10}$  and  $\tilde{P}_{10}$  by l and  $\tilde{l}$ , respectively. Clearly,  $\tilde{l} \geq 3$ , since for  $\tilde{l} = 2$  one of two high sections of  $\tilde{P}_{10}$  would be of length 3. By Lemmas 4.2, 4.3 and 4.5, any low section of  $P_{10}$  contains a vertex coloured with 1, hence  $l \leq 3$ . On the other hand,  $\tilde{l} \leq l$ , and we get  $l = \tilde{l} = 3$ . Thus,  $\tilde{P}_{10}$  has two low sections of type (1,2) or (2,1), one low section of type (1), two high sections of length 2 and one high section of length 1.

A high section of  $\tilde{P}_{10}$  of length 2 must be a section of  $P_{10}$ , too – otherwise, by Lemmas 4.4 and 4.6,  $\Pi_{i=1}^{3}(x_i)$  is of type (3+,1,2) and  $\Pi_{i=8}^{10}(x_i)$  is of type (2,1,3+), so that  $\Pi_{i=4}^{7}(x_i)$  is of type (3+,3+,1,3+) or (3+,1,3+,3+), which contradicts Lemma 4.8 or Lemma 4.9. Thus, two high sections of  $P_{10}$  of length 2 are, by Lemmas 4.8 and 4.9, separated by a low section of  $P_{10}$  of length 2; let  $\Pi_{i=q}^{q+5}(x_i)$  be the corresponding section of  $P_{10}$  with  $\min\{\varphi(x_i): i \in \{q, q+1, q+4, q+5\}\} \geq 3$ . Then q = 1 is impossible by

Lemma 4.4, q = 2 by Lemmas 4.3 and 4.10 and, symmetrically, q = 4 by Lemmas 4.5 and 4.11, q = 5 by Lemma 4.6. If q = 3, one endvertex of  $P_{10}$  is high, which contradicts Lemma 4.4 or Lemma 4.6.

So, we conclude that f(6) = 11, and then Proposition 17 yields f(7) = 15.

**Corollary 19.** For n = 5, 6,  $\chi_r^*(C_n) = \chi_r^*(P_n) = 4$ .

**Proof.** Those on-line ranking numbers must be at least 4, by Theorem 1 of [5]. On the other hand, due to Theorem 18,  $\max(\mathcal{G}, C_n) = \max(\mathcal{G}, P_n) = 4$ .

Note that, by Theorem 1 of [5], it holds  $\chi_r^*(C_4) = \chi_r^*(P_4) = 3$ . The value of on-line ranking number for simplest cycles and paths (with at most three vertices) is evidently equal to the corresponding number of vertices.

For an input sequence  $Y = \prod_{i=1}^{n} (y_i) \in \text{Is}(C_n)$  and  $j \in [0, n-1]$  let  $Y^{+j}$  be the input sequence for the graph  $C_n$  defined by  $Y^{+j} := \prod_{i=1}^{n} (y_{i+j})$ .

**Lemma 20.** If  $n \in [3, \infty)$ ,  $j \in [0, n-1]$  and  $Y \in \text{Is}(C_n)$ , then  $\max(\mathcal{G}, C_n, Y^{+j}) = \max(\mathcal{G}, C_n, Y)$ .

**Proof.** Evidently,  $V(C_n(Y^{+j}, x_i)) = \{x_{k+j} : x_k \in V(C_n(Y, x_i))\}$  for any  $i \in [1, n]$ . If  $i \in [1, n]$  and  $x_k \in V(C_n(Y, x_i))$ , the ranking rank $(\mathcal{G}, C_n, Y^{+j}, x_{i+j})$  attributes to the vertex  $x_{k+j}$  the same colour as the ranking rank $(\mathcal{G}, C_n, Y, x_i)$  does to the vertex  $x_k$ , hence the proof follows.

**Proposition 21.** If  $n \in [2,\infty)$ , then  $\max(\mathcal{G}, P_n) \leq \max(\mathcal{G}, C_{n+1}) \leq \max(\mathcal{G}, P_n) + 1$ .

**Proof.** The first inequality comes from Proposition 1, because  $P_n$  is an induced subgraph of  $C_{n+1}$ .

Take an input sequence  $Y = \prod_{i=1}^{n+1}(y_i) \in \operatorname{Is}(C_{n+1})$  such that  $\max(\mathcal{G}, C_{n+1}, Y) = \max(\mathcal{G}, C_{n+1})$ . Since  $C_{n+1}(Y, y_n)$  is a path with n vertices, with respect to Lemma 20 we may suppose that  $V(C_{n+1}(Y, y_n)) = \{x_i : i \in [1, n]\}$ . Then, for the input sequence  $Y^- = \prod_{i=1}^n (y_i) \in \operatorname{Is}(P_n)$ , we have  $\operatorname{rank}(\mathcal{G}, P_n, Y^-) = \operatorname{rank}(\mathcal{G}, C_{n+1}, Y, y_n)$ . That is why,  $\max(\mathcal{G}, P_n, Y^-) \geq \max(\mathcal{G}, C_{n+1}, Y) - 1 = \max(\mathcal{G}, C_{n+1}) - 1$  (the arrival of  $y_{n+1}$ , the last vertex of Y, can increase the number of used colours only by 1) and  $\max(\mathcal{G}, C_{n+1}) \leq \max(\mathcal{G}, P_n, Y^-) + 1 \leq \max(\mathcal{G}, P_n) + 1$ .

Corollary 22. If  $k \in [3, \infty)$ , then  $g(k) \leq f(k) + 1$ .

**Proof.** Suppose that f(k) = n. As  $n \ge k \ge 3$ , Proposition 21 implies  $\max(\mathcal{G}, C_{n+1}) \ge \max(\mathcal{G}, P_n) = k$ , and so, by Proposition 9,  $g(k) \le n+1 = f(k) + 1$ .

Theorem 23. Let i be a nonnegative integer. Then

- 1.  $11 \cdot 2^i \le f(2i+6) \le 12 \cdot 2^i 1.$
- 2.  $15 \cdot 2^i \le f(2i+7) \le 16 \cdot 2^i 1.$
- 3.  $10 \cdot 2^i \le g(2i+6) \le 12 \cdot 2^i$ .
- 4.  $14 \cdot 2^i \le g(2i+7) \le 16 \cdot 2^i$ .

**Proof.** Lower bounds come from Theorems 11 and 18. The upper bounds in 1 and 2 follow from Corollary 14, and then those in 3 and 4 from Corollary 22.

Theorem 24. Let i be a nonnegative integer.

- 1. If  $n \in [12 \cdot 2^i 1, 15 \cdot 2^i 1]$ , then  $\max(\mathcal{G}, P_n) = 2i + 6$ .
- 2. If  $n \in [15 \cdot 2^i, 16 \cdot 2^i 2]$ , then  $2i + 6 \le \max(\mathcal{G}, P_n) \le 2i + 7$ .
- 3. If  $n \in [16 \cdot 2^i 1, 22 \cdot 2^i 1]$ , then  $\max(\mathcal{G}, P_n) = 2i + 7$ .
- 4. If  $n \in [22 \cdot 2^i, 24 \cdot 2^i 2]$ , then  $2i + 7 \le \max(\mathcal{G}, P_n) \le 2i + 8$ .
- 5. If  $n \in [12 \cdot 2^i, 14 \cdot 2^i 1]$ , then  $\max(\mathcal{G}, C_n) = 2i + 6$ .
- 6. If  $n \in [14 \cdot 2^i, 16 \cdot 2^i 1]$ , then  $2i + 6 \le \max(\mathcal{G}, C_n) \le 2i + 7$ .
- 7. If  $n \in [16 \cdot 2^i, 20 \cdot 2^i 1]$ , then  $\max(\mathcal{G}, C_n) = 2i + 7$ .
- 8. If  $n \in [20 \cdot 2^i, 24 \cdot 2^i 1]$ , then  $2i + 7 \le \max(\mathcal{G}, C_n) \le 2i + 8$ .

**Proof.** Because of Proposition 1, the statements 1–4 follow from Theorems 23.1 and 23.2.

If  $n \in [12 \cdot 2^i, \infty)$ , then  $\max(\mathcal{G}, C_n) \geq 2i + 6$ , since otherwise, by Proposition 21,  $\max(\mathcal{G}, P_{n-1}) \leq \max(\mathcal{G}, C_n) \leq 2i + 5$ , which contradicts Theorem 23.1 (with respect to Proposition 1). Thus, 5 and 6 follow from Theorems 23.3 and 23.4. The remaining two statements are proved analogously.

**Theorem 25.** Let i be a nonnegative integer.

- 1. If  $n \in [12 \cdot 2^i 1, 15 \cdot 2^i 1]$ , then  $\chi^*_{\mathbf{r}}(P_n) \le 2\lfloor \log_2 n \rfloor$ .
- 2. If  $n \in [15 \cdot 2^i, 16 \cdot 2^i 1]$ , then  $\chi_{\mathbf{r}}^*(P_n) \le 2\lfloor \log_2 n \rfloor + 1$ .
- 3. If  $n \in [16 \cdot 2^i, 22 \cdot 2^i 1]$ , then  $\chi_r^*(P_n) \le 2\lfloor \log_2 n \rfloor 1$ .
- 4. If  $n \in [22 \cdot 2^i, 24 \cdot 2^i 2]$ , then  $\chi_r^*(P_n) \le 2\lfloor \log_2 n \rfloor$ .

- 5. If  $n \in [12 \cdot 2^i, 14 \cdot 2^i 1]$ , then  $\chi_{\mathbf{r}}^*(C_n) \le 2\lfloor \log_2 n \rfloor$ .
- 6. If  $n \in [14 \cdot 2^i, 16 \cdot 2^i 1]$ , then  $\chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor + 1$ .
- 7. If  $n \in [16 \cdot 2^i, 20 \cdot 2^i 1]$ , then  $\chi_r^*(C_n) \le 2\lfloor \log_2 n \rfloor 1$ .
- 8. If  $n \in [20 \cdot 2^i, 24 \cdot 2^i 1]$ , then  $\chi_{\mathbf{r}}^*(C_n) \le 2\lfloor \log_2 n \rfloor$ .

**Proof.** If  $n \in [12 \cdot 2^i - 1, 15 \cdot 2^i - 1]$ , then  $\lfloor \log_2 n \rfloor = i + 3$ , and, by Theorem 24.1,  $\chi_r^*(P_n) \leq \max(\mathcal{G}, P_n) = 2i + 6 = 2\lfloor \log_2 n \rfloor$ , which represents 1. The remaining assertions follow from Theorem 24, too.

**Theorem 26.** For any  $n \in [3, \infty)$ ,  $\chi_{r}(C_{n}) = \lfloor \log_{2}(n-1) \rfloor + 2$ .

**Proof.** First we show that  $\chi_{\mathbf{r}}(C_n) \geq 1 + \chi_{\mathbf{r}}(P_{n-1})$ . Suppose, on the contrary, that  $\chi_{\mathbf{r}}(C_n) = l \leq \chi_{\mathbf{r}}(P_{n-1})$ , and consider an *l*-ranking  $\varphi$  of  $C_n$ . If x is the (only) vertex of  $C_n$  coloured with l, then  $\varphi - \{(x, l)\}$  is an (l-1)-ranking of the path  $P_{n-1} = C_n - x$ , and so  $\chi_{\mathbf{r}}(P_{n-1}) \leq l-1$ , a contradiction. Thus, according to [1], we have  $\chi_{\mathbf{r}}(C_n) \geq 1 + \lfloor \log_2(n-1) \rfloor + 1 = \lfloor \log_2(n-1) \rfloor + 2$ .

Now, take  $k \in [1, \infty)$ ,  $m \in [1, 2^k - 1]$  and  $n = 2^k + m$ . From Lemma 2.1 of [1] it is easy to see that  $\chi_r(P_{2^k}) = k + 1$  and  $\chi_r(P_m) = \lfloor \log_2 m \rfloor + 1 = l(m) \leq k$ . Let  $\varphi_1$  be a (k+1)-ranking of  $P_{2^k}$  with  $V(P_{2^k}) = \{x_i : i \in [1, 2^k]\}$  and endvertices  $x_1, x_{2^k}$ , and let  $\varphi_2$  be an l(m)-ranking of  $P_m$  with  $V(P_m) = \{u_i : i \in [1,m]\}$ , with endvertices  $u_1, u_m$  and with  $V(P_{2^k}) \cap V(P_m) = \emptyset$ . Without loss of generality, by Proposition 2.1 of [1], we may suppose that  $\varphi_1(x_1) = k + 1$ . Let  $C_{2^k+m}$  be the cycle formed from  $P_{2^k} \cup P_m$  by adding the edges  $x_1u_m$  and  $x_{2^k}u_1$ . The colouring  $\varphi$  of  $C_{2^k+m}$  defined by  $\varphi(x_i) := \varphi_1(x_i), i \in [1, 2^k], \varphi(u_1) = k + 2$  and  $\varphi(u_i) = \varphi_2(u_i), i \in [2, m]$ , is easily seen to be a (k+2)-ranking. Thus,  $\chi_r(C_n) \leq k + 2 = \lfloor \log_2(n-1) \rfloor + 2$ .

For  $k \in [1, \infty)$  let  $\varphi'$  be such a (k+2)-ranking of  $P_{2^{k+1}}$  that the (unique) appearance of the colour k+2 is at an endvertex of  $P_{2^{k+1}}$ . Then,  $\varphi'$  is also a (k+2)-ranking of the cycle  $C_{2^{k+1}}$ , which is created from  $P_{2^{k+1}}$  by joining its endvertices by a new edge, and, for  $n = 2^k + 2^k = 2^{k+1}$ , we have  $\chi_{\mathbf{r}}(C_n) \leq k+2 = \lfloor \log_2(n-1) \rfloor + 2$ .

So,  $\chi_{\mathbf{r}}(C_n) \leq \lfloor \log_2(n-1) \rfloor + 2$  for any  $n \in [2^k + 1, 2^{k+1}]$  and any  $k \in [1, \infty)$ , and the desired result follows.

#### Theorem 27.

- 1. For any  $n \in [1, \infty)$ ,  $\lfloor \log_2 n \rfloor + 1 \leq \chi_{\mathrm{r}}^*(P_n) \leq 2 \lfloor \log_2 n \rfloor + 1$ .
- 2. For any  $n \in [3, \infty)$ ,  $\lfloor \log_2(n-1) \rfloor + 2 \leq \chi_r^*(C_n) \leq 2 \lfloor \log_2 n \rfloor + 1$ .

**Proof.** Lower bounds come from the values of  $\chi_r(P_n)$  and  $\chi_r(C_n)$  due to [1] and Theorem 26.

As concerns upper bounds, for  $n \in [12, \infty)$  see Theorem 25; for  $n \leq 11$  use Theorem 18 and the fact that f(i) = i, i = 1, 2, 3, and g(3) = 3.

First Fit Algorithm is not necessarily optimal when computing  $\chi_{\mathbf{r}}^*(P_n)$ , as shows our next statement.

**Theorem 28.**  $\chi_r^*(P_7) = 4 < 5 = \max(\mathcal{G}, P_7).$ 

**Proof.** According to Theorem 1 of [5], we have  $\chi_r^*(P_7) \ge 4$ . Consider the ranking algorithm  $\mathcal{G}'$  functioning just as  $\mathcal{G}$  does with the only exception: If  $G = P_5$ ,  $H = 2K_2$ ,  $\{x\} = V(G) - V(H)$  and  $\varphi$  is a ranking of H such that both neighbours of x (in G) are coloured with 2, then  $\mathcal{G}'(G, H, \varphi, x) = 4$  (and not 3, as required by  $\mathcal{G}$ ). We are going to show that  $m' = \max(\mathcal{G}', P_7, Y) \le 4$  for any  $Y \in \mathrm{Is}(P_7)$ .

First suppose that  $Y = \prod_{i=1}^{7} (y_i)$  is such that  $\varphi' = \operatorname{rank}(\mathcal{G}', P_7, Y) \neq \operatorname{rank}(\mathcal{G}, P_7, Y) = \varphi$ . Then  $P_7(Y, y_5) = P_5$  and it is easy to see that any neighbour of (a vertex of)  $P_7(Y, y_5)$  is coloured with 3 and any non-neighbour (at most one) of  $P_7(Y, y_5)$  is coloured with 1 by  $\varphi'$ ; thus, m' = 4.

Now, assume that  $\varphi' = \varphi$ . If  $y_7 \in \{x_3, x_4, x_5\}$ , then  $P_7(Y, y_6) = P_i \cup P_{6-i}, i \in \{2, 3\}$ . Clearly, the maximum colour used by  $\varphi'_6 = \operatorname{rank}(\mathcal{G}', P_7, Y, y_6)$  is not greater than  $\max\{\max(\mathcal{G}, P_i), \max(\mathcal{G}, P_{6-i})\}$ ; this maximum is equal to 3, by Proposition 1 and f(3) = 3, f(4) = 5 (Theorem 18), hence  $m' \leq 4$ .

If  $y_7 \in \{x_1, x_2\}$ , we may suppose that  $\varphi'_6$  uses colour 4 – otherwise we are done.

If  $y_7 = x_2$ , then  $P_7(Y, y_6) = P_1 \cup P_5$  and 4 is used by  $\varphi'_6$  for a vertex of  $P_5$ -component of  $P_7(Y, y_6)$ . If one of  $x_3, x_4$  is coloured with a colour  $\geq 3$ , then, using Lemma 4.3,  $\varphi'(x_2) = 2$ . On the other hand,  $\{\varphi'_6(x_3), \varphi'_6(x_4)\} \neq \{1, 2\}$ , because otherwise at least two vertices from among  $x_5, x_6, x_7$  would be coloured with a colour  $\geq 3$  (3 is used at least once by  $\varphi'_6$ ) in contradiction with one of Lemmas 4.2, 4.7, 4.12 and 4.13.

If  $y_7 = x_1$ , then  $P_7(Y, y_6) = P_6$ . We may assume that  $\varphi'_6(x_2) = 1$  and  $\varphi'_6(x_3) = 2$ , since if not, we would have  $\varphi'(x_1) \leq 2$ . Because of Lemmas 4.1, 4.7, 4.8 and 4.9, exactly two vertices from among  $x_4, x_5, x_6, x_7$  are coloured with a colour  $\geq 3$ . From Lemmas 4.2, 4.12, 4.13 and 4.15 it follows that these are  $x_4$  and  $x_7$ . If  $\varphi'_6(x_4) = 4$ , then  $\varphi'(x_1) = 3$ . Finally, suppose that  $\varphi'_6(x_4) = 3$  and  $\varphi'_6(x_7) = 4$ . Then  $\varphi'_6(x_6) = 1$  and  $\varphi'_6(x_5) = 2$  (by Lemma 4.6),  $x_4$  comes in Y before  $x_7$  (otherwise  $\varphi'_6(x_4) = 1$ ), which means that  $P_7(Y, y_4) = 2K_2$  and that the vertex  $y_5 = x_4$  has in  $P_7(Y, y_5)$  both

neighbours coloured with 2. This, however, is a contradiction, because in such a case  $4 = \varphi'(y_5) = \varphi'_6(y_5)$ .

The last possibility,  $y_7 \in \{x_6, x_7\}$ , leads to a situation which is symmetric with that of  $y_7 \in \{x_1, x_2\}$ .

Now, to conclude the proof, we use Theorem 18, from which it follows that  $\max(\mathcal{G}, P_7) = 5$ .

## **Theorem 29.** $\chi_r^*(C_7) = 5.$

**Proof.** By Theorem 1 of [5], we have  $\chi_r^*(C_7) \ge 4$ . We are going to show by the way of contradiction, that  $\chi_r^*(C_7) \ge 5$ ; this, together with  $\max(\mathcal{G}, C_7) = 5$  (Theorem 18), will mean that  $\chi_r^*(C_7) = 5$ .

We know from Theorem 26 that  $\chi_r(C_7) = 4$ . Let  $\varphi$  be a 4-ranking of  $C_7$ . It can be immediately seen that  $\varphi$  uses 3 and 4 exactly once, say, for vertices  $x_i$  and  $x_j$ . Since  $\chi_r(P_4) = 3 = \chi_r(P_5)$ , no component of  $H = C_7 - \{x_i, x_j\}$  can have more than 3 vertices, so that  $H = P_2 \cup P_3$ . Clearly,  $\varphi$  restricted to  $P_3$ -component of H uses 2 just once, for the internal vertex of that  $P_3$ . Also,  $\varphi$  restricted to  $P_2$ -component of H, uses 2 once. Thus,  $\varphi$  colours two vertices of  $C_7$  with 2 and two vertices with a colour  $\geq 3$ ; the mutual distance of vertices in those two pairs is 3.

Now, suppose that there is a ranking algorithm  $\mathcal{A}$  such that  $\max(\mathcal{A}, C_7) = 4$ . Consider an input sequence  $Y = \prod_{i=1}^7 (y_i) \in \operatorname{Is}(P_7)$  and the ranking  $\varphi = \operatorname{rank}(\mathcal{A}, C_7, Y)$ . As  $\chi_r(C_7) = 4$ ,  $\varphi$  is a 4-ranking of  $C_7$ . If  $C_7(Y, y_2) = P_2$ , the ranking  $\operatorname{rank}(\mathcal{A}, C_7, Y, y_2)$  must use colours 1 and 2. To see this suppose that a colour  $i \in \{3, 4\}$  is used for a vertex  $y_j$  of  $C_7(Y, y_2)$ . Assume, moreover, that  $C_7(Y, y_k) = P_2 \cup P_{k-2}, k = 3, 4, 5$  (we cannot avoid this situation). We have  $\varphi(y_k) \neq i, k = 3, 4, 5$ , hence it may happen that  $\varphi(y_k) = 7-i$  for some  $k \in [3, 5]$  and an endvertex  $y_k$  of  $C_7(Y, y_k) - \text{if } \{\varphi(y_3), \varphi(y_4)\} = \{1, 2\}, y_5$  may be an endvertex of  $C_7(Y, y_5)$  with the neighbour coloured with 1. Then, however, the distance between  $y_j$  and  $y_k$ , the vertices coloured with 3 and 4, may be 2 in contradiction with the structure of a 4-ranking of  $C_7$ .

If  $C_7(Y, y_2) = P_2$ ,  $C_7(Y, y_3) = P_3$  and the neighbour of  $y_3$  in  $C_7(Y, y_3)$  is coloured with 1, we have  $\varphi(y_3) = i \in \{3, 4\}$ . It may happen that  $C_7(Y, y_5) = P_3 \cup P_2$ . For vertices of  $P_2$ -component of  $C_7(Y, y_5)$  two from among colours 1,2 and 7-i are used. If 2 is used, it may happen that there are two vertices coloured with 2 by  $\varphi$ , whose distance is 2, a contradiction. On the other hand, the presence of 7-i could yield two vertices of distance 2, coloured with 3 and 4 by  $\varphi$ , a contradiction again.

196

## 5 Open Problems

There are several open problems which naturally arise from our analysis.

- 1. Find nontrivial lower bounds for  $\chi_{\mathbf{r}}^*(C_n)$  and  $\chi_{\mathbf{r}}^*(P_n)$ .
- 2. Which is the minimum n such that  $\chi_{\mathbf{r}}^*(P_n) = 5$ ?
- 3. Does there exist  $n \in [8, \infty)$  such that  $\chi_{\mathbf{r}}^*(C_n) < \max(\mathcal{G}, C_n)$ ? If so, which is the minimum n in such an inequality?
- 4. Determine g(7). (We conjecture that g(7) = 15.)
- 5. Prove or disprove that the sequence  $\{\max(\mathcal{G}, C_n)\}_{n=3}^{\infty}$  is non-decreasing.

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